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Ordering Extremes of Interdependent Random Variables*

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Abstract

For random variables with Archimedean copula or survival copula, we develop the reversed hazard rate order and the hazard rate order on sample extremes in the context of proportional reversed hazard models and proportional hazard models, respectively. The likelihood ratio order on sample maximum is also investigated for the proportional reversed hazard model. Several numerical examples are presented for illustrations as well.

Key words Archimedean copula; Hazard rate order; Proportional hazard model; Proportional reversed hazard model; Reversed hazard rate order

1 Introduction

Order statistics have been paid quite a lot of attention in statistics, reliability, operations research, risk management, auction theory and many other areas related to applied probability. The past three decades have witnessed a great many elegant and interesting results on probability and statistical properties of order statistics. In the context of reliability, random variables $X_1, \cdots, X_n$ usually represent component lifetimes, and thus the $k$th smallest observation, $X_{k:n}$ defines the lifetime of a $(n-k+1)$-out-of-$n$ system, for $k = 1, \cdots, n$. Specifically, sample extremes $X_{1:n}$ and $X_{n:n}$ denote the system lifetimes of series and parallel structure, respectively. If $X_1, \cdots, X_n$ are bids of bidders competing for the goods for sale, then $X_{1:n}$ and $X_{n:n}$ represent the winner’s price in the first-price procurement auction and the English auction, respectively. Therefore, it is of both theoretical and practical interest to study ordering properties of order

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statistics so that engineers can come up with better design in industry and auctioneers can earn more profit from carrying out auctions.

In tradition, most of the research comparing order statistics are mainly developed for independent and identically distributed (i.i.d.) random variables. Due to the in-depth development in the related areas, such as reliability, safety, risk and auction, order statistics based on i.i.d. random variables gets rather restrictive. As a result, in the past two decades many authors made their effort to study independent but non-identically distributed random variables. See the comprehensive reviews of Kochar (2012) and Balakrishnan and Zhao (2013) for examples. Although in practical situations, samples usually comprise of dependent and non-identically distributed random variables, fewer studies on order statistics of dependent random variables are found in the literature due to the complicated nature in probability. Recently, several researchers made some progress in comparing order statistics of random variables with Archimedean copulas. See, for example, Rezapour and Alamatsaz (2014), Li and Fang (2015), Li and Li (2015), Li et al. (2015) and Fang et al. (2016).

Among all references of stochastic comparisons on order statistics, many focus on ordering different order statistics from one sample and studying the impact of sample size on a given order statistic. For i.i.d. random variables, Boland et al. (1994) were among the first to prove that order statistics can be ordered in terms of the hazard rate order and even the likelihood ratio order, and Raqab and Amin (1996) further proved the existence of the likelihood ratio order between order statistics from i.i.d. samples of different sizes. In the context of $X_k \leq_{hr} X_{n+1}$, $k = 1, \ldots, n$, Boland et al. (1994) firstly obtained the hazard rate order $X_{i-1:n} \leq_{hr} X_{i:n+1}$ and further proved that $X_{n+1} \leq_{hr} X_k$ implies $X_{i:n} \geq_{hr} X_{i:n+1}$. Later, for a sample with observations arrayed in the likelihood ratio order, Bapat and Kochar (1994) obtained the likelihood ratio order between two general order statistics. Until recently, in the literature few discussion can be found on stochastic order between order statistics from either one dependent sample or two dependent samples of different sizes.

For homogeneous random variables with an Archimedean (survival) copula, Li and Fang (2015) took the first to derive the hazard rate order, the reversed hazard rate order and the likelihood ratio order among $X_{n-2:n}, X_{n-1:n}, X_{n:n}$ and $X_{n+1:n+1}$. Along this line, this paper makes an attempt to partially generalize theoretical results on sample extremes in Li and Fang (2015) to the interdependent and heterogeneous random variables.

To avoid getting involved too much with the complicated heterogeneity and dependence structure among marginal distributions, we mainly consider random variables with proportional reversed hazard (PRH) model and coupled by Archimedean copula or survival copula, and random variables with proportional hazard (PH) model and coupled by Archimedean copula or survival copula. In these scenarios, we present sufficient conditions for the reversed hazard rate order and the hazard rate order between extremes from samples with different number of observations. As for the PRH model, we show that for a specific dependence structure, adding
one more component to the sample may increase the sample maximum in the sense of likelihood ratio order. Moreover, we have a discussion on the effect of sample size on ordering properties of extremes for sample with general distributions and coupled by Archimedean copula or survival copula.

The rest of this paper is rolled out as follows: Section 2 recalls several concepts concerned with detailed discussions in the remaining sections. In Section 3, we investigate how the sample size plays a role in impacting sample extremes in the sense of the reversed hazard rate order and the hazard rate order in the context of random variables with PRH model or PH model and coupled by either Archimedean copula or survival copula. The likelihood ratio order is also investigated for the PRH model. We further discuss random variables without PRH model or PH model and coupled by Archimedean copula or survival copula in Section 4. Two simple applications are presented in the closing section as well.

For convenience, we set \( I_n = \{1, \ldots, n\} \) and denote \( \psi^{(k)} \) the \( k \)th-derivative of \( \psi \). Throughout this paper, all random variables are assumed to be absolutely continuous, and by ‘increasing’ and ‘decreasing’ we mean ‘non-decreasing’ and ‘non-increasing’, respectively.

2 Preliminaries

Denote \( F_X, f_X \) and \( F_X \) the distribution, density, and survival function of \( X \), respectively.

**Definition 2.1** A random variable \( X \) is said to be smaller than the other one \( Y \) in the

- (i) reversed hazard rate order (denoted by \( X \leq_{rh} Y \)) if \( F_Y(x)/F_X(x) \) is increasing in \( x \);
- (ii) hazard rate order (denoted by \( X \leq_{hr} Y \)) if \( F_Y(x)/F_X(x) \) is increasing in \( x \);
- (iii) likelihood ratio order (denoted by \( X \leq_{lr} Y \)) if \( f_Y(x)/f_X(x) \) is increasing in \( x \).

For more on such orders please refer to Shaked and Shanthikumar (2007) and Li and Li (2013).

For \( X = (X_1, \ldots, X_n) \) with joint distribution function \( F \), joint survival function \( \bar{F} \), univariate marginal distribution functions \( F_1, \ldots, F_n \) and survival functions \( \bar{F}_1, \ldots, \bar{F}_n \), if there exist some \( C(u) : [0, 1]^n \mapsto [0, 1] \) and \( \bar{C}(u) : [0, 1]^n \mapsto [0, 1] \) such that, for all \( x_i, i \in I_n \),

\[
F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)),
\]

\[
\bar{F}(x_1, \ldots, x_n) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),
\]

then \( C(u) \) and \( \bar{C}(u) \) are called the copula and survival copula of \( X \), respectively.
Definition 2.2 For \((n-2)\)th differentiable \(\psi : [0, +\infty) \mapsto (0, 1]\) with \(\psi(0) = 1\) and \(\psi(+\infty) = 0\),

\[ C(u_1, \cdots, u_n) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_n)), \quad \text{for all } u_i \in (0, 1), \ i \in \mathcal{I}_n, \]

is said to be an Archimedean copula with generator \(\psi\) if \((-1)^{n-2} \psi^{(n-2)}(x)\) is decreasing and convex, and \((-1)^k \psi^{(k)}(x) \geq 0\) for all \(x \geq 0, k = 0, 1, \cdots, n - 2\).

For convenience, we denote \(\phi = \psi^{-1}\) as the right continuous inverse of \(\psi\), and the generator \(\psi\) is said to be strict if \(\psi(+\infty) = 0\). The Archimedean family is well-known to be flexible in the sense of comprising many useful copulas, including some very popular ones.

In the coming sections, we will discuss how the degree of dependence impacts sample extremes. Since the convexity of \((-1)^{n-2} \psi^{(n-2)}(t)\) plays a role in defining the Archimedean copula, we will consider the generator \(\psi\) with convex \(-\ln \psi(e^z)\) on \(z \in \mathbb{R}\) and with either log-convex or log-concave \(-\ln \psi(e^z)\) on \(z \in \mathbb{R}\). It is routine to verify the following two facts:

(i) the convexity of \(-\ln \psi(e^z)\) on \(\mathbb{R}\) is equivalent to the decreasing \(t \psi'(t)/\psi(t)\) on \(\mathbb{R}^+\), and

(ii) the log-concavity (log-convexity) \(-\ln \psi(e^z)\) on \(\mathbb{R}\) is equivalent to the increasing (decreasing) \(\frac{\psi(t) \ln \psi(t)}{\psi'(t)}\) on \(\mathbb{R}^+\).

On (i), Li and Fang (2015) provided several numerical examples of Archimedean copulas, which are positively or negatively dependent and have generator \(\psi\) with decreasing \(t \psi'(t)/\psi(t)\) or \(t[1 - \psi(t)]'/[1 - \psi(t)]\). As for (ii), observing that \(-\ln t\) is just the inverse \(\phi_0\) of the generator \(\psi_0 = e^{-t}\) of the independence copula, we have that the increasing property of \(\frac{\psi(t) \ln \psi(t)}{\psi'(t)}\) is equivalent to the log-concavity of \(\phi_0 \circ \psi(e^z)\). In view of the fact that concave \(\phi_0 \circ \psi(e^z)\) implies concave \(\ln \phi_0 \circ \psi(e^z)\), we reach one sufficient condition for the log-concavity of \(\phi_0 \circ \psi(e^z)\) with respect to \(z\): the degree of concavity of \(\phi_0 \circ \psi(t)\) with respect to \(t\) is strong enough to imply that \(\phi_0 \circ \psi(e^z)\) is also concave with respect to \(z\). In view of \(\phi_0 \circ \psi(0) = 0\) and by Proposition 21.A.11 of Marshall and Olkin (2007), we conclude that the concavity implies the sub-additivity of \(\phi_0 \circ \psi(t)\). As per Lemma 3.8 of Fang et al. (2016), the sub-additivity of \(\phi_0 \circ \psi(t)\) implies \(C_{\psi} \geq C_{\phi_0}\), meaning that \(C_{\psi}\) is positively lower orthant dependent.

Proportional hazard model

The random variables \((X_1, \cdots, X_{n+1})\) are said to follow a PH model if \(X_i\) has the survival function

\[ \tilde{F}_i(x) = \bar{F}^{\lambda_i}(x), \quad \text{for } \lambda_i > 0 \text{ and } i \in \mathcal{I}_{n+1}, \]

where \(\bar{F}\) is the baseline survival function and \(\lambda = (\lambda_1, \cdots, \lambda_{n+1})\) is the frailty vector. For example, let \(X_i \sim \mathcal{W}(\alpha, \lambda_i)\) be of Weibull distribution with survival function \(e^{-\lambda_i^\alpha x}\) for \(x > 0, \alpha > 0\) and \(\lambda_i > 0\), \(i \in \mathcal{I}_{n+1}\), then, \(X\) follows a PH model with baseline survival function \(e^{-x^\alpha}\) and frailty vector \((\lambda_1^\alpha, \cdots, \lambda_{n+1}^\alpha)\). Specifically, by \(X \sim \text{PH}(\bar{F}, \lambda, \psi)\) we denote the sample having Archimedean copula with generator \(\psi\) and following a PH model with baseline \(\bar{F}\) and frailty...
vector $\lambda$, and by $X \sim \text{PHS}(\hat{F}, \lambda, \psi)$ we denote the sample having *Archimedean survival copula* with generator $\psi$ and following a PH model with baseline $\hat{F}$ and frailty vector $\lambda$.

**Proportional reversed hazard model**

The random variables $X_1, \cdots, X_{n+1}$ are said to follow a PRH model with *baseline distribution* function $F$ and *resilience* vector $\lambda = (\lambda_1, \cdots, \lambda_{n+1})$ if $X_i$ has distribution function

$$F_i = F^{\lambda_i}, \quad \text{for } \lambda_i > 0 \text{ and } i \in \mathcal{I}_{n+1}.$$  

For example, if $X_i \sim F(\alpha, \lambda_i)$, the Fréchet distribution function $e^{-x^\alpha}$ for $x > 0$, $\alpha > 0$ and $\lambda > 0$, $i \in \mathcal{I}_{n+1}$, then $(X_1, \cdots, X_{n+1})$ follows a PRH model with baseline distribution function $e^{-x^\alpha}$ and resilience vector $(\lambda_1^\alpha, \cdots, \lambda_{n+1}^\alpha)$. Further, we denote $(X_1, \cdots, X_{n+1}) \sim \text{PRH}(F, \lambda, \psi)$ the sample having the *Archimedean copula* with generator $\psi$ and following a PRH model with $X_i \sim F^{\lambda_i}, \ i \in \mathcal{I}_{n+1}$. Specifically, by $X = (X_1, \cdots, X_{n+1}) \sim \text{PRHS}(F, \lambda, \psi)$ we denote the sample having the *Archimedean survival copula* with generator $\psi$ and following a PRH model with $X_i \sim F^{\lambda_i}, \ i \in \mathcal{I}_{n+1}$.

### 3 Main results

For homogeneous observations $X_1, \cdots, X_n$ with Archimedean copula, Li and Fang (2015) recently developed sufficient and necessary conditions for the reversed hazard rate order and the hazard rate order among $X_{n-1:n}$, $X_{n:n}$ and $X_{n+1:n+1}$. Here we have a further study on how one more observation has an impact on the sample extremes in the context of observations with

(i) Archimedean (survival) copula and proportional reversed hazard models or
(ii) Archimedean (survival) copula and proportional hazard models. The present results are natural generalizations of concerned ones in Theorems 3.1 and 3.7 of Li and Fang (2015).

Since the degree of dependence is completely determined by the generator in Archimedean family, in this section we present various sufficient conditions for the hazard rate order or the reversed hazard rate order based on the log-concavity and log-convexity concerned with some functionals of the generator and the order between the proportional factor of the new observation and the extreme of former ones.

**Theorem 3.1** Suppose $-\ln \psi(e^z)$ is both convex and log-concave on $\mathbb{R}$. Then, $\lambda_{n+1} \geq \lambda_i$ for $i \in \mathcal{I}_n$ implies

(i) $X_{n:n} \leq_{\text{rh}} X_{n+1:n+1}$ for $(X_1, \cdots, X_{n+1}) \sim \text{PRH}(F, \lambda, \psi)$;

(ii) $X_{1:n} \geq_{\text{hr}} X_{1:n+1}$ for $(X_1, \cdots, X_{n+1}) \sim \text{PHS}(\hat{F}, \lambda, \psi)$.

**Proof:** Let us prove (i) only. (ii) can be verified in a very similar manner.
Note that, for all $x$,

$$
\left[ \frac{F_{X_{n+1:n+1}}(x)}{F_{X_{n:n}}(x)} \right]'
= \left[ \frac{\psi\left( \sum_{i=1}^{n+1} \phi(F_{\lambda_i}(x)) \right)}{\psi\left( \sum_{i=1}^{n} \phi(F_{\lambda_i}(x)) \right)} \right]'
$$

$$
given \quad \frac{\psi'\left( \sum_{i=1}^{n+1} \phi(F_{\lambda_i}(x)) \right)}{\psi\left( \sum_{i=1}^{n+1} \phi(F_{\lambda_i}(x)) \right)} \sum_{i=1}^{n+1} \frac{\lambda_i F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)} - \frac{\psi'\left( \sum_{i=1}^{n} \phi(F_{\lambda_i}(x)) \right)}{\psi\left( \sum_{i=1}^{n} \phi(F_{\lambda_i}(x)) \right)} \sum_{i=1}^{n} \frac{\lambda_i F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)}
$$

where $\text{sgn}$ means both sides have the same sign. Since the convexity of $-\ln \psi(e^z)$ implies that $t \psi'(t)/\psi(t)$ decreases in $t \geq 0$, it holds that, for all $x$,

$$
\frac{\psi'\left( \sum_{i=1}^{n+1} \phi(F_{\lambda_i}(x)) \right)}{\psi\left( \sum_{i=1}^{n+1} \phi(F_{\lambda_i}(x)) \right)} \sum_{i=1}^{n+1} \frac{\lambda_i F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)} \leq \frac{\psi'\left( \sum_{i=1}^{n} \phi(F_{\lambda_i}(x)) \right)}{\psi\left( \sum_{i=1}^{n} \phi(F_{\lambda_i}(x)) \right)} \sum_{i=1}^{n} \frac{\lambda_i F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)} \leq 0.
$$

Because of $\sum_{i=1}^{n+1} \frac{\lambda_i F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)} \leq 0$, to prove $\left[ \frac{F_{X_{n+1:n+1}}(x)}{F_{X_{n:n}}(x)} \right]' \geq 0$ it suffices to verify

$$
\sum_{i=1}^{n+1} \frac{\lambda_i F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)} \leq \sum_{i=1}^{n} \frac{\lambda_i F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)} + \frac{1}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)}
$$

which is equivalent to that, for all $x$,

$$
\sum_{i=1}^{n} \frac{\lambda_i F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)} \leq \sum_{i=1}^{n+1} \frac{\lambda_i F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)}
$$

$$
\iff \sum_{i=1}^{n} \frac{F_{\lambda_i}(x) \ln F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)} \geq \sum_{i=1}^{n+1} \frac{F_{\lambda_i}(x) \ln F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right)}
$$

$$
\iff \sum_{i=1}^{n+1} \phi(F_{\lambda_i}(x)) \phi(F_{\lambda_{i+1}}(x)) \left[ \frac{F_{\lambda_i}(x) \ln F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_{i+1}}(x)) \right) \phi(F_{\lambda_{i+1}}(x))} - \frac{F_{\lambda_i}(x) \ln F_{\lambda_i}(x)}{\psi'\left( \phi(F_{\lambda_i}(x)) \right) \phi(F_{\lambda_i}(x))} \right] \geq 0
$$

$$
\iff \sum_{i=1}^{n} \phi(F_{\lambda_i}(x)) \phi(F_{\lambda_{i+1}}(x)) \left[ L_{\psi} \left( \phi(F_{\lambda_{i+1}}(x)) \right) - L_{\psi} \left( \phi(F_{\lambda_i}(x)) \right) \right] \geq 0.
$$

(3.1)

where $L_{\psi}(t) = \frac{\psi(t) \ln \psi(t)}{t \psi(t)}$ is increasing due to the log-concavity of $-\ln \psi(e^z)$.

In light of $\lambda_{n+1} \geq \lambda_i$ for all $i \in I_n$ and the monotone $\phi$ we have

$$
\phi(F_{\lambda_{n+1}}(x)) \geq \phi(F_{\lambda_i}(x)) \quad \text{for all } x \text{ and } i \in I_n.
$$

Therefore, it follows that

$$
L_{\psi} \left( \phi(F_{\lambda_{n+1}}(x)) \right) \geq L_{\psi} \left( \phi(F_{\lambda_i}(x)) \right) \quad \text{for all } x \text{ and } i \in I_n,
$$
leading to the inequality (3.1). Now, we conclude that $F_{X_{n+1:n+1}}(x)/F_{X_{n:n}}(x)$ is increasing in $x$, and this invokes $X_{n:n} \leq_{rh} X_{n+1:n+1}$.}

According to Theorem 3.1(i), in the context of sample with some positively dependence, adding one more stochastically larger observation results in a larger sample maximum in the sense of the reversed hazard rate order.

The next example finds that, for some generator satisfying the requirements in Theorem 3.1, introducing one more observation neither stochastically larger nor smaller than the others does not incur a change on the sample maximum in the sense of the reversed hazard rate order. However, the sample maximum gets larger in the sense of the hazard rate order. Similar phenomenon can be observed for the sample minimum.

**Example 3.2 (Clayton copula)** (i) Consider $(X_1, X_2, X_3, X_4) \sim \text{PRH}(F, \lambda, \psi)$, where the resilience vector $\lambda = (1, 3, 4, 2)$ such that $\lambda_4 > \lambda_1$ and $\lambda_4 < \lambda_2, \lambda_3$, the generator $\psi(t) = (\theta t + 1)^{-1/\theta}$ is for Clayton copula with parameter $\theta = 2$, the baseline distribution $F$ is uniform on $[0, 1]$. It is routine to check that $(-1)^k \psi(t) \geq 0$ for $k \in \mathcal{I}_4$, $\frac{d}{dt} \frac{\psi(t)}{\psi(t)} < 0$ and $\frac{d}{dt} \frac{\psi(t) \ln \psi(t)}{\psi(t)} \geq 0$. Then, $\psi$ satisfies the other requirement in Theorem 3.1(i).

![Figure 1: Ratios of distribution functions and of survival functions](image)

As is seen in Figure 1(a), the ratio $F_{X_{4:4}}/F_{X_{3:3}}$ is not monotonic in $[0, 1]$, and this refutes $X_{3:3} \leq_{rh} X_{4:4}$. However, the increasing ratio $\tilde{F}_{X_{4:4}}/\tilde{F}_{X_{3:3}}$ in Figure 1(b) invokes $X_{3:3} \leq_{hr} X_{4:4}$.

(ii) Consider $(X_1, X_2, X_3, X_4) \sim \text{PHS}(\tilde{F}, \lambda, \psi)$, where the frailty vector $\lambda = (1, 3, 4, 2)$ such that $\lambda_4 > \lambda_1$ and $\lambda_4 < \lambda_2, \lambda_3$, the generator is for the Clayton copula in case (i).

As is seen in Figure 2(a), the ratio $\tilde{F}_{X_{1:4}}/\tilde{F}_{X_{1:3}}$ is not monotonic in $[0, 1]$, and hence $X_{1:4} \not{\leq}_{hr} X_{1:3}$. However, the decreasing ratio $F_{X_{1:4}}/F_{X_{1:3}}$ in Figure 2(b) invokes $X_{1:4} \leq_{rh} X_{1:3}$.

Naturally, including one more observation should not decrease (increase) sample maximum (minimum). However, such an intuition is based on the sample of independent observations. In fact, according to the above example, in the presence of dependence, one more observation does not necessarily increase the sample maximum in the reversed hazard rate order (decrease the sample minimum in the hazard rate order). Besides, in this scenario, the effect of stochastic property of the extra observation should not be neglected. One insightful reviewer points out the
interest of checking whether the intuition on the independent situation could be asymptotically true in the dependent situation. Although we have no interesting result for now, it deserves a further study in the future.

**Theorem 3.3** Suppose $-\ln \psi(e^z)$ is log-convex on $\mathbb{R}$. Then, $\lambda_{n+1} \leq \lambda_i$ for $i \in \mathbb{I}_n$ implies

(i) $X_{n:n} \leq_{rh} X_{n+1:n+1}$ for $(X_1, \cdots, X_{n+1}) \sim \text{PRH}(F, \lambda, \psi);

(ii) $X_{1:n} \geq_{rh} X_{1:n+1}$ for $(X_1, \cdots, X_{n+1}) \sim \text{PHS}(\hat{F}, \lambda, \psi)$.

**Proof:** Since log-convex $-\ln \psi(e^z)$ implies convex $-\ln \psi(e^z)$ on $\mathbb{R}$, (i) can be proved in a similar manner to Theorem 3.1 (i), and correspondingly (ii) is just a similar practice.  

**Theorem 3.4** Suppose $-\ln(1-\psi(e^z))$ is convex on $\mathbb{R}$ and $-\ln \psi(e^z)$ is log-concave (log-convex) on $\mathbb{R}$. Then, $\lambda_{n+1} \leq (\geq) \lambda_i$ for $i \in \mathbb{I}_n$ implies

(i) $X_{n:n} \leq_{rh} X_{n+1:n+1}$ for $(X_1, \cdots, X_{n+1}) \sim \text{PRH}(F, \lambda, \psi);

(ii) $X_{1:n} \geq_{rh} X_{1:n+1}$ for $(X_1, \cdots, X_{n+1}) \sim \text{PHS}(\hat{F}, \lambda, \psi)$.

**Proof:** (i) We only consider the case of log-concave $-\ln \psi(e^z)$ on $\mathbb{R}$.

Note that, for all $x$,

$$
\begin{align*}
\begin{bmatrix}
\frac{F_{X_{n+1:n+1}}(x)}{F_{X_{n:n}}(x)}
\end{bmatrix}'
&= \begin{bmatrix}
\left(1 - \psi\left(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))\right)\right)'
\left(1 - \psi\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right)\right)
\end{bmatrix}'
\end{align*}
$$

$$
\begin{align*}
\text{sgn} \frac{\psi'\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right) \frac{\sum_{i=1}^{n} \lambda_i F^{\lambda_i}(x)}{\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))}}{1 - \psi\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right)} & - \frac{\psi'\left(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))\right) \frac{\sum_{i=1}^{n+1} \lambda_i F^{\lambda_i}(x)}{\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))}}{1 - \psi\left(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))\right)}
\end{align*}
$$

$$
= \frac{\psi'\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right) \frac{\sum_{i=1}^{n} \lambda_i F^{\lambda_i}(x)}{\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))} \frac{1}{\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))}}{1 - \psi\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right)} - \frac{\psi'\left(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))\right) \frac{\sum_{i=1}^{n+1} \lambda_i F^{\lambda_i}(x)}{\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))} \frac{1}{\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))}}{1 - \psi\left(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))\right)}.
\end{align*}
$$

Figure 2: Ratios of distribution functions and of survival functions.
Since \(-\ln(1-\psi(e^z))\) is convex and hence \(t\psi'(t)/(1-\psi(t))\) increases, it holds that, for all \(x\),

\[
\frac{\psi'(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x)) \sum_{i=1}^{n} \phi(F^{\lambda_i}(x))}{1 - \psi(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x)))} \leq \frac{\psi'(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))) \sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))}{1 - \psi(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x)))} \leq 0.
\]

Likewise, to prove \([\hat{F}_{X_{n+1:n+1}}(x)/\hat{F}_{X_{n:n}}(x)]' \geq 0\) for all \(x\), it suffices to verify

\[
\frac{\sum_{i=1}^{n} \lambda_i F^{\lambda_i}(x)/\psi'(\phi(F^{\lambda_i}(x)))}{\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))} \leq \frac{\sum_{i=1}^{n+1} \lambda_i F^{\lambda_i}(x)/\psi'(\phi(F^{\lambda_i}(x)))}{\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))}, \quad \text{for all } x.
\]

Let \(L_{\psi}(t) = \frac{\psi(t) \ln \psi(t)}{t \psi'(t)}\). Then the above inequality is equivalent to

\[
\sum_{i=1}^{n} \frac{\lambda_i F^{\lambda_i}(x)/\phi(F^{\lambda_i}(x))}{\psi'(\phi(F^{\lambda_i}(x)))} \leq \sum_{i=1}^{n} \frac{\lambda_{n+1} F^{\lambda_{n+1}}(x)/\phi(F^{\lambda_{n+1}}(x))}{\psi'(\phi(F^{\lambda_{n+1}}(x)))}
\]
\[
\iff \sum_{i=1}^{n} \frac{F^{\lambda_i}(x)/\phi(F^{\lambda_i}(x)) \ln F^{\lambda_i}(x)}{\psi'(\phi(F^{\lambda_i}(x)))} \geq \sum_{i=1}^{n} \frac{F^{\lambda_{n+1}}(x)/\phi(F^{\lambda_{n+1}}(x)) \ln F^{\lambda_{n+1}}(x)}{\psi'(\phi(F^{\lambda_{n+1}}(x)))}
\]
\[
\iff \sum_{i=1}^{n} \phi(F^{\lambda_i}(x)) \phi(F^{\lambda_{n+1}}(x)) \left[ \frac{F^{\lambda_i}(x) \ln F^{\lambda_i}(x)}{\psi'(\phi(F^{\lambda_i}(x))) \phi(F^{\lambda_i}(x))} - \frac{F^{\lambda_{n+1}}(x) \ln F^{\lambda_{n+1}}(x)}{\psi'(\phi(F^{\lambda_{n+1}}(x))) \phi(F^{\lambda_{n+1}}(x))} \right] \geq 0
\]
\[
\iff \sum_{i=1}^{n} \phi(F^{\lambda_i}(x)) \phi(F^{\lambda_{n+1}}(x)) \left[ L_{\psi}(\phi(F^{\lambda_i}(x))) - L_{\psi}(\phi(F^{\lambda_{n+1}}(x))) \right] \geq 0. \tag{3.2}
\]

Owing to \(\lambda_{n+1} \leq \lambda_i\) for \(i \in I_n\) and the monotone \(\phi\) we have

\[
\phi(F^{\lambda_{n+1}}(x)) \leq \phi(F^{\lambda_i}(x)), \quad \text{for all } x \text{ and } i \in I_n.
\]

So, from the increasing \(L_{\psi}(t)\) it stems that

\[
L_{\psi}(\phi(F^{\lambda_{n+1}}(x))) \leq L_{\psi}(\phi(F^{\lambda_i}(x))), \quad \text{for } i \in I_n \text{ and all } x,
\]

which yields (3.2) and hence the increasing \(\hat{F}_{X_{n+1:n+1}}/\hat{F}_{X_{n:n}}\), leading to \(X_{n:n} \leq_{hr} X_{n+1:n+1}\).

The proof of (ii) can be similarly completed. \(\blacksquare\)

As per Theorem 3.4(i), for samples with some positively (negatively) dependence, including one more stochastically smaller (larger) observation increases the sample maximum in the sense of the hazard rate order.

Theorems 3.5 and 3.6 can be verified in a similar manner to Theorem 3.4 and hence are presented as follows without proof.

**Theorem 3.5** Suppose \(-\ln(1-\psi(e^z))\) is log-concave and \(-\ln(1-\psi(e^z))\) is convex. Then, \(\lambda_{n+1} \leq \lambda_i\) for \(i \in I_n\) implies

(i) \(X_{n:n} \leq_{hr} X_{n+1:n+1}\) for \((X_1, \ldots, X_{n+1}) \sim \text{PH}(\hat{F}, \lambda, \psi);\)
Suppose $\ln(1 - \psi(e^t))$ is log-concave and $-\ln \psi(e^t)$ is convex. Then $\lambda_{n+1} \geq \lambda_i$ for $i \in \mathcal{I}_n$ implies

(i) $X_{n:n} \leq_{hr} X_{n+1:n+1}$ for $(X_1, \ldots, X_{n+1}) \sim \text{PRH}(\tilde{F}, \lambda, \psi)$;

(ii) $X_{1:n} \geq_{hr} X_{1:n+1}$ for $(X_1, \ldots, X_{n+1}) \sim \text{PRHS}(F, \lambda, \psi)$.

In what follows, we present a sufficient condition for the sample maximum from PRH models increasing with respect to the sample size in the sense of the likelihood ratio order. Recall that a function $g(x, y)$ is said to be totally positive of order 2 (TP2) if for any $x_1 \leq x_2$ and $y_1 \leq y_2$, $g(x_1, y_1)g(x_2, y_2) \geq g(x_1, y_2)g(x_2, y_1)$. The next lemma will be useful in developing the sufficient condition.

**Lemma 3.7 (Theorem 11.2 in Karlin, 1968)** Assume that $g(x, y) > 0$ is TP2 on $\mathcal{X} \times \mathcal{Y}$ with $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}$ and $h(x, y)$ such that

(i) for each $x$, $h(x, y)$ changes sign at most once, and from $-\infty$ to $+\infty$, as $y$ traverses $\mathcal{Y}$,

(ii) for each $y$, $h(x, y)$ is increasing in $x$, and

(iii) for a sigma-finite measure $\mu$ the function $\omega(x) = \int_{\mathcal{Y}} h(x, y)g(x, y)\,d\mu(y)$ is continuous.

Then, $\omega(x)$ changes sign at most once and from negative to positive values.

**Theorem 3.8** Suppose $\ln(-\psi'(e^t))$ is convex on $\mathbb{R}$. Then, $X_{n:n} \leq_{hr} X_{n+1:n+1}$ for $(X_1, \ldots, X_{n+1}) \sim \text{PRH}(F, \lambda, \psi)$ if either one of the following two holds.

(i) $-\ln \psi(e^t)$ is log-concave, $\ln \psi(t)\left[\frac{\psi''(t)}{\psi'(t)} - 1\right]$ is increasing, and $\lambda_{n+1} \geq \lambda_i$ for $i \in \mathcal{I}_n$;

(ii) $-\ln \psi(e^t)$ is log-convex, $\ln \psi(t)\left[\frac{\psi''(t)}{\psi'(t)} - 1\right]$ is decreasing, and $\lambda_{n+1} \leq \lambda_i$ for $i \in \mathcal{I}_n$.

**Proof:** Let us prove (i) only, and (ii) can be verified in a similar manner. Without loss of generality, assume that $\lambda_i \leq \lambda_{i+1}$ for $i = 1, \cdots, n$. Denote the density function of the baseline distribution as $f$. $X_{n+1:n+1}$ and $X_{n:n}$ get density functions

$$f_{X_{n+1:n+1}}(x) = \psi\left(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))\right) \sum_{i=1}^{n+1} \lambda_i F^{\lambda_i-1}(x)f(x) \psi'(\phi(F^{\lambda_i}(x)))$$

and

$$f_{X_{n:n}}(x) = \psi\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right) \sum_{i=1}^{n} \lambda_i F^{\lambda_i-1}(x)f(x) \psi'(\phi(F^{\lambda_i}(x)))$$.
for all $x$, respectively. Note that, $X_{n;n} \leq_{lr} X_{n+1,n+1}$ if and only if, for any $c \in \mathbb{R}$, $\frac{f_{x_{n+1,n+1}}(x)}{f_{x_{n,n}}(x)} - c$ changes sign at most once, and when the sign changes, it is from $-\rightarrow +$. Moreover, one can verify that $\frac{f_{x_{n+1,n+1}}(x)}{f_{x_{n,n}}(x)} - c$ and

$$
\psi'\left(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))\right) \sum_{i=1}^{n+1} \frac{\lambda_i F^{\lambda_i}(x)}{\psi'(\phi(F^{\lambda_i}(x)))} - c \psi'\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right) \sum_{i=1}^{n} \frac{\lambda_i F^{\lambda_i}(x)}{\psi'(\phi(F^{\lambda_i}(x)))},
$$

(3.3)

have the same sign. Denote $\alpha_1 = \cdots = \alpha_n = 1$ and $\alpha_{n+1} = 0$. Then (3.3) can be rephrased as

$$
\psi'\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right) \sum_{i=1}^{n} \frac{\lambda_i F^{\lambda_i}(x)}{\psi'(\phi(F^{\lambda_i}(x)))} \left[\psi'\left(\sum_{j=1}^{n+1} \phi(F^{\lambda_j}(x))\right) - \psi'\left(\sum_{j=1}^{n} \phi(F^{\lambda_j}(x))\right) - c \alpha_i\right] = \sum_{i=1}^{n+1} \frac{-\lambda_i F^{\lambda_i}(x)}{\psi'(\phi(F^{\lambda_i}(x)))} \left[\psi'\left(\sum_{j=1}^{n+1} \phi(F^{\lambda_j}(x))\right) - \psi'\left(\sum_{j=1}^{n} \phi(F^{\lambda_j}(x))\right) - c \alpha_i\right].
$$

According to Lemma 3.7, it suffices to verify the following three requirements:

(i) The function $\frac{-\lambda_i F^{\lambda_i}(x)}{\psi'(\phi(F^{\lambda_i}(x)))}$ is TP$_2$ in $(i, x)$;

(ii) For any $x \geq 0$, $\frac{\psi'\left(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))\right)}{\psi'\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right)} - c \alpha_i$ changes sign at most once, and from negative to positive values as $i$ increases from 1 to $n+1$;

(iii) For each $i = 1, \cdots, n+1$, $\frac{\psi'\left(\sum_{i=1}^{n+1} \phi(F^{\lambda_i}(x))\right)}{\psi'\left(\sum_{i=1}^{n} \phi(F^{\lambda_i}(x))\right)} - c \alpha_i$ is increasing in $x$.

The requirement (ii) follows immediately from the definition of $\alpha_i$, $i = 1, \cdots, n$. For requirement (i), it suffices to show that, for $x \leq y$,

$$
\frac{-\lambda_i F^{\lambda_i}(x)}{\psi'(\phi(F^{\lambda_i}(x)))} - \lambda_{i+1} F^{\lambda_{i+1}}(y) \geq \frac{-\lambda_i F^{\lambda_i}(y)}{\psi'(\phi(F^{\lambda_i}(y)))} - \lambda_{i+1} F^{\lambda_{i+1}}(x) \iff \frac{F^{\lambda_i}(x)\psi'(\phi(F^{\lambda_{i+1}}(x)))}{F^{\lambda_{i+1}}(x)\psi'(\phi(F^{\lambda_i}(x)))} = \frac{F^{\lambda_i}(y)\psi'(\phi(F^{\lambda_{i+1}}(y)))}{F^{\lambda_{i+1}}(y)\psi'(\phi(F^{\lambda_i}(y)))} \leq 0 \iff \lambda_{i+1} \left[\frac{F^{\lambda_{i+1}}(x)\psi''(\phi(F^{\lambda_{i+1}}(x)))}{\psi'(\phi(F^{\lambda_i}(x)))} - 1\right] \leq \lambda_i \left[\frac{F^{\lambda_i}(x)\psi''(\phi(F^{\lambda_{i+1}}(x)))}{\psi'(\phi(F^{\lambda_i}(x)))} - 1\right] \iff N_\psi(\phi(F^{\lambda_{i+1}}(x))) \geq N_\psi(\phi(F^{\lambda_i}(x))),
$$

(3.4)

where $N_\psi(t) = \ln \psi(t) \left[\frac{\psi''(t)}{\psi'(t)} - 1\right]$ is increasing. In view of $\lambda_{i+1} \geq \lambda_i$ and the monotonicity of $\phi$ one has

$$
\phi(F^{\lambda_{i+1}}(x)) \geq \phi(F^{\lambda_i}(x)) \quad \text{for all } x,
$$

this verifies (3.4) and hence requirement (i).
Note that requirement (iii) is equivalent to
\[
\begin{bmatrix}
\psi' \left( \sum_{j=1}^{n+1} \phi ( F^\lambda_j ( x ) ) \right) \\
\psi' \left( \sum_{j=1}^{n} \phi ( F^\lambda_j ( x ) ) \right)
\end{bmatrix}' \geq 0
\]
(3.5)
\[
\iff \frac{\psi'' \left( \sum_{j=1}^{n+1} \phi ( F^\lambda_j ( x ) ) \right)}{\psi' \left( \sum_{j=1}^{n+1} \phi ( F^\lambda_j ( x ) ) \right)} \sum_{j=1}^{n+1} -\lambda_j F^\lambda_j ( x ) \\
\geq \frac{\psi'' \left( \sum_{j=1}^{n} \phi ( F^\lambda_j ( x ) ) \right)}{\psi' \left( \sum_{j=1}^{n} \phi ( F^\lambda_j ( x ) ) \right)} \sum_{j=1}^{n} -\lambda_j F^\lambda_j ( x )
\]
\[
\iff Q_\psi \left( \sum_{j=1}^{n+1} \phi ( F^\lambda_j ( x ) ) \right) \frac{\sum_{j=1}^{n+1} -\lambda_j F^\lambda_j ( x )}{\sum_{j=1}^{n+1} \phi ( F^\lambda_j ( x ) )} \geq Q_\psi \left( \sum_{j=1}^{n} \phi ( F^\lambda_j ( x ) ) \right) \frac{\sum_{j=1}^{n} -\lambda_j F^\lambda_j ( x )}{\sum_{j=1}^{n} \phi ( F^\lambda_j ( x ) )},
\]
where \( Q_\psi (t) = \frac{\psi''(t)}{\psi'(t)} \) is decreasing due to the convexity of \( \ln(-\psi'(e^z)) \). Clearly,
\[
Q_\psi \left( \sum_{j=1}^{n+1} \phi ( F^\lambda_j ( x ) ) \right) \leq Q_\psi \left( \sum_{j=1}^{n} \phi ( F^\lambda_j ( x ) ) \right) \leq 0,
\]
and as is shown in the proof of Theorem 3.1, the log-concavity of \(-\ln(\psi(e^z))\) and \( \lambda_{n+1} \geq \lambda_i \) for \( i = 1, \ldots, n \) together imply that
\[
\frac{\sum_{j=1}^{n+1} -\lambda_j F^\lambda_j ( x )}{\sum_{j=1}^{n+1} \phi ( F^\lambda_j ( x ) )} \leq \frac{\sum_{j=1}^{n} -\lambda_j F^\lambda_j ( x )}{\sum_{j=1}^{n} \phi ( F^\lambda_j ( x ) )},
\]
verifying (3.5) and hence requirement (iii). Therefore, we complete the proof.

At first glance, in the same manner to the proof of Theorem 3.8 one can develop parallel conditions for PHS model, PH model and PRHS model, respectively. However, after checking all Archimedean copulas with strict generator in Table 4.1 of Nelson (2006), we find no generator satisfying the concerned conditions. In fact, even under the PRH model, the dependence structure depicted in Theorem 3.8 is somewhat specific, and in Table 4.1 of Nelson (2006) only Gumbel-Hougaard copula with generator \( e^{-1/\theta} \) is found to fulfill the assumption of Theorem 3.8. As a further illustration, Example 3.9 reveals that the likelihood ratio order may hold in the context other than the one of Theorem 3.8.

**Example 3.9** Let \((X_1, X_2, X_3, X_4) \sim \text{PRH}(F, \lambda, \psi_1)\) and \((Y_1, Y_2, Y_3, Y_4) \sim \text{PRH}(F, \lambda, \psi_2)\), where \( \lambda = (1, 3, 4, 2) \), \( \psi_1(t) = e^{-t/\theta} \) is for Gumbel-Hougaard copula with parameter \( \theta = 2 \), \( \psi_2(t) = e^{(1-t^2)/\theta} \) is for Gumbel-Barnett copula with parameter \( \theta = 0.2 \), and the baseline distribution \( F \) is uniform distribution on \([0, 1]\).

Since \( \lambda_4 = 2 > \lambda_1 \) but \( \lambda_1 < \lambda_2, \lambda_3 \), the vector \( \lambda \) violates the condition of Theorem 3.8. For \( k = 3, 4 \), \( X_{kk} \) and \( Y_{kk} \) get their respective probability densities
\[
f_{X_{kk}} (x) = \psi_1 \left( \sum_{i=1}^{k} \phi_1 (x^{\lambda_i}) \right) \prod_{i=1}^{k} \frac{\lambda_i x^{\lambda_i-1}}{\psi_1 (\phi_1 (x^{\lambda_i}))}, \quad f_{Y_{kk}} (x) = \psi_2 \left( \sum_{i=1}^{k} \phi_2 (x^{\lambda_i}) \right) \prod_{i=1}^{k} \frac{\lambda_i x^{\lambda_i-1}}{\psi_2 (\phi_2 (x^{\lambda_i}))}.
\]
As per Figure 3(a) and 3(b), both \( X_{3:3} \leq_{lr} X_{4:4} \) and \( Y_{3:3} \leq_{lr} Y_{4:4} \) are justified. ■
4 Two generalizations

So far, for samples with proportional hazard models or proportional reversed hazard models, we have a discussion on how the sample extremes vary in the sense of the hazard rate order and the reversed hazard rate order as the sample size grows. In this section, we have a further study on sample extremes of heterogeneous random variables coupled by Archimedean (survival) copulas.

Theorem 4.1 Suppose \( X_i \sim F_i \) for \( i \in \mathcal{I}_{n+1} \) have an Archimedean copula with generator \( \psi \). Then, \( X_{n:n} \leq_{rh} X_{n+1:n+1} \) whenever \( -\ln \psi(x) \) is convex.

**Proof:** Let \( M_\psi(x) = \frac{\psi'(x)}{\psi(x)} \) and denote \( f_i \) the density function of \( F_i \), \( i \in \mathcal{I}_{n+1} \). Then,

\[
\mathbb{P}(X_{n+1:n+1} \leq x) = \frac{\psi(\sum_{i=1}^{n+1} \phi(F_i(x)))}{\psi(\sum_{i=1}^{n+1} \phi(F_i(x)) - \sum_{i=1}^{n+1} f_i(x)} - \frac{\psi(\sum_{i=1}^{n} \phi(F_i(x)))}{\psi(\sum_{i=1}^{n} \phi(F_i(x)) - \sum_{i=1}^{n} f_i(x)}
\]

Consequently, from the convexity of \( -\ln \psi(x) \) it follows that \( M_\psi(x) = [\ln \psi(x)]' \) is decreasing and hence

\[
M_\psi(\sum_{i=1}^{n+1} \phi(F_i(x))) \leq M_\psi(\sum_{i=1}^{n} \phi(F_i(x))) \leq 0, \quad \text{for all } x.
\]

So, (4.1) is non-negative for all \( x \), and this implies \( X_{n:n} \leq_{rh} X_{n+1:n+1} \).

In parallel, we also bring forth the following result concerning the hazard rate order between minimums from samples having Archimedean survival copulas. Due to the similarity in making it into shape, we omit the proof for the sake of briefness.
**Theorem 4.2** Suppose \((X_1, \cdots, X_{n+1})\) has marginal \(X_i \sim F_i\) for \(i \in I_{n+1}\) and Archimedean survival copula with generator \(\psi\). Then, \(X_{1:n+1} \preceq_{hr} X_{1:n}\) whenever \(-\ln \psi(x)\) is convex.

At first glance, Theorems 4.1 and 4.2 cover a wide range of situations; however, the convexity of \(-\ln \psi(x)\) is equivalent to the log-concave generator \(\psi(x)\), which is a bit restrictive. According to Li and Li (2015), the log-concavity of an Archimedean copula or Archimedean survival copula implies that the concerned marginals are negatively dependent; however a high-dimension Archimedean copula usually could not characterize a very strong negative dependence. One can check that most of Archimedean copulas in Table 4.1 of Nelsen (2006) have strict but not log-concave generators when the dimension is high. Intuitively, due to the generality of the marginal distributions, it is quite reasonable to require strict dependence structure so that the hazard rate order or the reversed hazard rate order between extremes of samples of different sizes is achieved. As an illustration on these two theorems, the next example presents some Archimedean copulas with log-concave generators.

**Example 4.3** (i) Consider Frank copula with generator \(\psi_1(t) = \theta^{-1} \ln[e^{-t}(e^{-\theta} - 1) + 1]\) for \(\theta \in (-\infty, +\infty)\). By routine calculus, one can check that \(\psi_1\) is log-concave for all \(\theta \leq 0\). Moreover, it is plain that not all real \(\theta\) is valid for \(\psi_1\) to be a genuine generator of a 3-dimension or 4-dimension Frank copula. In fact, for a 3-dimension Frank copula, \(\theta\) must be no smaller than \(-\ln 2\) and for a 4-dimension Frank copula, the admissible range of \(\theta\) is \([\ln 2 - \ln 3, +\infty)\). Suppose \((X_1, X_2, X_3, X_4)\) has Frank copula with parameter \(\theta = -0.4\), and \(X_i \sim Beta(3, i + 2)\) for \(i \in I_4\), where \(Beta(a, b)\) is the beta distribution with density

\[
\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1},
\]

where \(\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}dt\). As depicted in Figure 4(a), the increasing \(F_{X_{4:4}}/F_{X_{3:3}}\) confirms \(X_{4:4} \succeq_{hr} X_{3:3}\).

![Figure 4: Ratio of distribution functions and ratio of survival functions](image)

(ii) Consider Gumbel-Barnett copula with generator \(\psi_2(t) = \exp\{(1 - e^t)/\theta\}\) for \(\theta \in (0, 1]\). By routine calculus, one can check that \(\psi_2\) is log-concave for all \(\theta \in (0, 1]\) and not all \(\theta \in (0, 1]\) make \(\psi_2\) a genuine generator of a 3-dimension or 4-dimension Gumbel-Barnett copula. In
fact, for a 3-dimension Gumbel-Barnett copula, $\theta$ must be in the interval $(0, (6 - \sqrt{15})/7)$ and for a 4-dimension Gumbel-Barnett copula an admissible range of $\theta$ is $(0, 0.2226)$. Suppose $(X_1, X_2, X_3, X_4)$ has Gumbel-Barnett copula with parameter $\theta = 0.2$, and $Y_i \sim Beta(3, i + 2)$ for $i \in I_4$. As is seen in Figure 4(b), $\bar{F}_{X_{1:4}}/\bar{F}_{X_{1:3}}$ decreases in $[0, 1]$, justifying $X_{1:4} \leq_{hr} X_{1:3}$. 

As $\theta$ tends to 0 both Frank copula and Gumbel-Barnett copula converge to the independence copula. Also, as is observed in Example 4.3, for log-concave generators, the admissible range of the dependence parameter shrinks as the dimension grows, and this means that the dependence structure among the sample gets more and more close to the independence. In fact, it is well-known that for mutually independent observations the sample maximum increases in the sense of the reversed hazard rate order and the sample minimum decreases in the sense of the hazard rate order as sample size grows. Therefore, these two theorems slightly generalize the ordering results under the independence framework to the negative dependence structure not deviating too far from the independence.

5 Two simple applications

Weakest link theory

Due to the great flexibility, Weibull distribution is widely applied to describe random lifetimes and thus very commonly used in studying component reliability and material strength. In material science, it is a common sense to utilize Weibull distribution to characterize the tensile strength of brittle materials such as carbon fibres, glass fibres and bast fibres, see Pickering et al. (2007) for example. Usually, the fibre strength is described by a two-parameter Weibull distribution $F(x; \alpha, \lambda) = 1 - \exp\{-\lambda^\alpha x^\alpha\}$ for $x > 0$, where the shape parameter $\alpha > 0$ is the Weibull modulus, the scale parameter $\lambda > 0$ and hence the characteristic strength $\lambda$ is related to the fibre diameter and gauge length. With modern technologies, in practice textile fabrics are usually made of blended yarns, for example, jute/cotton and polyester.

Consider a yarn blended with fibres from a common producer, and assume that their strengths are of Weibull distributions with a common shape parameter $\alpha$ but possibly different scale parameters $\lambda_i, i = 1, 2, \cdots$. Consider a yarn made of $n + 1$ fibres. Let $X_i$’s be fibre strengths with scale parameter $\lambda_i$. Since the fibres are constructed by a common producer, it is reasonable to assume that their tensile strength are statistically dependent. Suppose the tensile strengths of fibres have some Archimedean survival copula. According to the well-known weakest-link theory and with the help of Theorem 3.3(ii) and Theorem 3.6(ii), one knows that for positively (negatively) dependent tensile strength, removing the fibre with the weakest (strongest) tensile strength will result in a yarn with a larger strength in the sense of the hazard rate order, and this solid theory helps fabric material engineers to adjust the amount of fibres and obtain a stronger blended yarn.
Engineering reliability

As two very important elementary reliability structures, the parallel and series systems are found everywhere in engineering and military industry. In practical situations, due to the pressure, temperature and other factors in the common working environment, system components typically have statistically dependent lifetimes. For parallel systems, in the presence of Archimedean copula for heterogeneous dependent components having PRH models, Theorem 3.1(i) and Theorem 3.3(i) state that for some positive (negative) dependence structure, appending one more component having stochastically larger (smaller) lifetime will lead to a more reliable system with respect to the reversed hazard rate order. In parallel, similar findings on series systems with components having PH model and Archimedean copulas can be derived based on Theorem 3.1(ii) and Theorem 3.3(ii).

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References


