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Div-curl type theorems on Lipschitz domains.
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DIV-CURL TYPE THEOREMS ON LIPSCHITZ DOMAINS

ZENGJIAN LOU

For Lipschitz domains of \( \mathbb{R}^n \) we prove div-curl type theorems, which are extensions to domains of the Div-Curl Theorem on \( \mathbb{R}^n \) by Coifman, Lions, Meyer and Semmes. Applying the div-curl type theorems we give decompositions of Hardy spaces on domains.

1. Introduction

In [4] two Hardy spaces are defined on domains \( \Omega \) of \( \mathbb{R}^n \), one which is reasonably speaking the largest, and the other which in a sense is the smallest. The largest, \( \mathcal{H}^1_r(\Omega) \), arises by restricting to \( \Omega \) arbitrary elements of \( \mathcal{H}^1(\mathbb{R}^n) \). The other, \( \mathcal{H}^1_z(\Omega) \), arises by restricting to \( \Omega \) elements of \( \mathcal{H}^1(\mathbb{R}^n) \) which are zero outside \( \bar{\Omega} \). Norms on these spaces are defined as following

\[
\| f \|_{\mathcal{H}^1_r(\Omega)} = \inf \| F \|_{\mathcal{H}^1(\mathbb{R}^n)},
\]

the infimum being taken over all functions \( F \in \mathcal{H}^1(\mathbb{R}^n) \) such that \( F|_{\Omega} = f \),

\[
\| f \|_{\mathcal{H}^1_z(\Omega)} = \| F \|_{\mathcal{H}^1(\mathbb{R}^n)},
\]

where \( F \) is the zero extension of \( f \) to \( \mathbb{R}^n \).

From [2], the dual of \( \mathcal{H}^1_z(\Omega) \) is \( \text{BMO}_r(\Omega) \), a space of locally integrable functions with

\[
\| f \|_{\text{BMO}_r(\Omega)} = \sup_{Q \subset \Omega} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2} < \infty,
\]

where \( f_Q = 1/|Q| \int_Q f(x) \, dx \), and the supremum is taken over all cubes \( Q \) in the domain \( \Omega \). The dual of \( \mathcal{H}^1_z(\Omega) \) is \( \text{BMO}_z(\Omega) \), the space of all functions in \( \text{BMO}(\mathbb{R}^n) \) supported in \( \bar{\Omega} \), equipped with the norm \( \| f \|_{\text{BMO}_z(\Omega)} = \| f \|_{\text{BMO}(\mathbb{R}^n)} \).

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Let $\Omega$ denote a Lipschitz domain — an assumption which is enough to ensure the existence of a bounded extension map from $\text{BMO}_r(\Omega)$ to $\text{BMO}(\mathbb{R}^n)$ ([6]). We use $H(\Omega)^n := H(\Omega, \mathbb{R}^n)$ to denote a space of functions $f : \Omega \to \mathbb{R}^n$ (when $n = 1$, write $H(\Omega)^1$ as $H(\Omega)$). For simplicity we introduce the following spaces

$$L^2_{\text{div}}(\Omega)^n = \left\{ f \in L^2(\Omega)^n : \text{div } f = 0, \ |f|_{L^2(\Omega)^n} \leq 1 \right\};$$

$$L^2_{\text{curl}}(\Omega)^n = \left\{ f \in L^2(\Omega)^n : \text{curl } f = 0, \ |f|_{L^2(\Omega)^n} \leq 1 \right\},$$

where $\nu$ denotes the outward unit normal vector. When $\Omega = \mathbb{R}^n$

$$L^2_{\text{div}}(\mathbb{R}^n)^n = \left\{ f \in L^2(\mathbb{R}^n)^n : \text{div } f = 0, \ |f|_{L^2(\mathbb{R}^n)^n} \leq 1 \right\};$$

$$L^2_{\text{curl}}(\mathbb{R}^n)^n = \left\{ f \in L^2(\mathbb{R}^n)^n : \text{curl } f = 0, \ |f|_{L^2(\mathbb{R}^n)^n} \leq 1 \right\}.$$

In [5, Theorems II.1 and III.2], among other results, Coifman, Lions, Meyer and Semmes established the following theorems.

**Theorem CLMS1.** Let $1 < p, q < \infty, 1/p + 1/q = 1, E \in L^p(\mathbb{R}^n)^n$, $\text{div } E = 0$, $F \in L^q(\mathbb{R}^n)^n$, $\text{curl } F = 0$. Then $E \cdot F \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\| E \cdot F \|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \| E \|_{L^p(\mathbb{R}^n)^n} \| F \|_{L^q(\mathbb{R}^n)^n}$$

for a constant $C$ depending only on the dimension $n$.

**Theorem CLMS2.** For $b \in \text{BMO}(\mathbb{R}^n)$

$$||b||_{\text{BMO}(\mathbb{R}^n)} \approx \sup_{E,F} \int_{\mathbb{R}^n} b \cdot E \cdot F \, dx,$$

where the supremum is taken over all $E \in L^2(\mathbb{R}^n)^n$, $F \in L^2(\mathbb{R}^n)^n$ with $\text{div } E = 0$, $\text{curl } F = 0$ and $||E||_{L^2(\mathbb{R}^n)^n} \leq 1$, $||F||_{L^2(\mathbb{R}^n)^n} \leq 1$, and the implicit constants in (1.2) depend only on $n$.

A natural question to ask is: under what conditions on domains $\Omega$ does the equivalence (1.2) hold on $\Omega$? As a main theorem of this paper, we solve this problem for Lipschitz domains in $\mathbb{R}^n$.

**Theorem 1.1.** Let $\Omega$ be a Lipschitz domain of $\mathbb{R}^n$.

(1) If $b \in \text{BMO}_r(\Omega)$, then

$$||b||_{\text{BMO}_r(\Omega)} \approx \sup_{e,f} \int_{\Omega} b \cdot e \cdot f \, dx,$$

the supremum being taken over all $e \in L^2_{\text{div}}(\Omega)^n$, $f \in L^2_{\text{curl}}(\Omega)^n$. 


(2) If \( b \in \text{BMO}_z(\Omega) \), then

\[
\|b\|_{\text{BMO}_z(\Omega)} \approx \sup_{e,f} \int_{\Omega} b \cdot e \, dx,
\]

the supremum being taken over all \( e = E|_{\Omega}, f = F|_{\Omega} \), \( E \in L^2_{\text{div}}(\mathbb{R}^n)^n \), \( F \in L^2_{\text{curl}}(\mathbb{R}^n)^n \).

The implicit constants in (1.3) and (1.4) depend only on the domain \( \Omega \) and on the dimension \( n \).

**Remark.** Results for other BMO-type spaces, such as dual of divergence-free Hardy spaces, can be found in [8] and [9].

**Corollary 1.2.**

(1) A function \( b \in \text{BMO}_r(\Omega) \) if and only if there exists a constant \( C \) such that

\[
\int_{\Omega} b \cdot e \, dx \leq C \text{ for all } e \in L^2_{\text{div}}(\Omega)^n \text{ and } f \in L^2_{\text{curl}}(\Omega)^n.
\]

(2) A function \( b \in \text{BMO}_z(\Omega) \) if and only if there exists a constant \( C \) such that

\[
\int_{\Omega} b \cdot f \, dx \leq C \text{ for all } e = E|_{\Omega} \text{ and } f = F|_{\Omega} \text{ with } E \in L^2_{\text{div}}(\mathbb{R}^n)^n, \quad F \in L^2_{\text{curl}}(\mathbb{R}^n)^n.
\]

Here and afterwards, unless otherwise specified, \( C \) denotes a constant depending only on the domain \( \Omega \) and the dimension \( n \). Such \( C \) may differ at different occurrences.

Applying Theorem 1 we have the following theorem which gives decompositions of \( \mathcal{H}^1_z(\Omega) \) and \( \mathcal{H}^1_r(\Omega) \) into quantities of forms “\( e \cdot f \)”.

**Theorem 1.3.**

(1) Any function \( u \in \mathcal{H}^1_z(\Omega) \) can be written as

\[
u = \sum_{k=1}^{\infty} \lambda_k \ e_k \cdot f_k,
\]

where \( e_k \in L^2_{\text{div}}(\Omega)^n \), \( f_k \in L^2_{\text{curl}}(\Omega)^n \) and \( \sum_{k=1}^{\infty} |\lambda_k| < \infty \).

(2) Any function \( u \in \mathcal{H}^1_r(\Omega) \) can be written as

\[
u = \sum_{k=1}^{\infty} \lambda_k \ e_k \cdot f_k,
\]

where \( e_k = E_k|_{\Omega}, f_k = F_k|_{\Omega} \), \( E_k \in L^2_{\text{div}}(\mathbb{R}^n)^n \), \( F_k \in L^2_{\text{curl}}(\mathbb{R}^n)^n \) and \( \sum_{k=1}^{\infty} |\lambda_k| < \infty \).
2. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemmas.

**Lemma 2.1. ([6, Theorem 1])** Let $b \in \text{BMO}_r(\Omega)$. Then there exists $B \in \text{BMO}(\mathbb{R}^n)$ such that

$$b = B|\Omega$$

and

(2.1) \[ \|B\|_{\text{BMO}(\mathbb{R}^n)} \leq C \|b\|_{\text{BMO}_r(\Omega)}. \]

**Lemma 2.2. ([7, Theorem 3.1])** Let $b$ be a locally integrable function on $\Omega$. Then

(2.2) \[ \|b\|_{\text{BMO}_r(\Omega)} \approx \|b\|_{\text{BMO}^h(\Omega)}, \]

where

\[ \|b\|_{\text{BMO}^h(\Omega)} = \sup_Q \left( \frac{1}{|Q|} \int_Q |b - b_Q|^2 \, dx \right)^{1/2}, \]

the supremum being taken over all cubes $Q$ with $2Q \subset \Omega$, the implicit constants in (2.2) depend only on $\Omega$ and $n$.

**Lemma 2.3.** For $b \in L^2_{\text{loc}}(\Omega)$

(2.3) \[ \|b\|_{\text{BMO}^h(\Omega)} \leq C \sup_{e,f} \int_{\Omega} b \, e \cdot \nu \, d\sigma, \]

the supremum being taken over all $e \in L^2_{\text{div}}(\Omega)^n$ and $f \in L^2_{\text{curl}}(\Omega)^n$.

The proof of Lemma 2.3 is given in the last section.

**Proof of Theorem 1.1:** (1) Let $B \in \text{BMO}(\mathbb{R}^n)$ be an extension of $b \in \text{BMO}_r(\Omega)$ such that $b = B|\Omega$ and (2.1) holds. For $e \in L^2_{\text{div}}(\Omega)^n$, $f \in L^2_{\text{curl}}(\Omega)^n$, define

\[ E = \begin{cases} e & \text{in } \Omega; \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \]

\[ F = \begin{cases} f & \text{in } \Omega; \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \]

Since $\text{div} \, e = 0$ on $\Omega$ and $e \cdot \nu|_{\partial \Omega} = 0$, it is easy to show that $\text{div} \, E = 0$ on $\mathbb{R}^n$. So $E \in L^2_{\text{div}}(\mathbb{R}^n)^n$. Similarly, $\text{curl} \, f = 0$ on $\Omega$ and $f \times \nu|_{\partial \Omega} = 0$ imply that $\text{curl} \, F = 0$.
on \( \mathbb{R}^n \). Therefore \( F \in L^2_{\text{curl}}(\mathbb{R}^n)^n \). By duality \( \mathcal{H}^1(\mathbb{R}^n)^* = \text{BMO}(\mathbb{R}^n) \), Lemma 2.1 and (1.1), we have

\[
\int_{\Omega} b \cdot f \, dx = \int_{\mathbb{R}^n} B \cdot E \cdot F \, dx \leq \| B \|_{\text{BMO}(\mathbb{R}^n)} \| E \cdot F \|_{\mathcal{H}^1(\mathbb{R}^n)} \\
\leq C \| b \|_{\text{BMO}(\Omega)} \| E \|_{L^2(\mathbb{R}^n)^n} \| F \|_{L^2(\mathbb{R}^n)^n} \\
= C \| b \|_{\text{BMO}(\Omega)} \| E \|_{L^2(\mathbb{R}^n)^n} \| f \|_{L^2(\mathbb{R}^n)^n} \leq C \| b \|_{\text{BMO}(\Omega)}.
\]

The proof of the reversed inequality in (1.3) follows from (2.2) and (2.3).

(2) Let \( b \in \text{BMO}_z(\Omega) \) and \( B \) be its zero extension to \( \mathbb{R}^n \). Then \( B \in \text{BMO}(\mathbb{R}^n) \) and \( \| B \|_{\text{BMO}(\mathbb{R}^n)} = \| b \|_{\text{BMO}_z(\Omega)} \). Using (1.1) again,

\[
\int_{\Omega} b \cdot f \, dx = \int_{\mathbb{R}^n} B \cdot E \cdot F \, dx \leq \| B \|_{\text{BMO}(\mathbb{R}^n)} \| E \cdot F \|_{\mathcal{H}^1(\mathbb{R}^n)} \\
\leq C \| b \|_{\text{BMO}_z(\Omega)} \| E \|_{L^2(\mathbb{R}^n)^n} \| F \|_{L^2(\mathbb{R}^n)^n} \\
\leq C \| b \|_{\text{BMO}_z(\Omega)}
\]

for all \( e = E|_{\Omega}, f = F|_{\Omega}, E \in L^2_{\text{div}}(\mathbb{R}^n)^n, F \in L^2_{\text{curl}}(\mathbb{R}^n)^n \).

For the converse, let \( b \in \text{BMO}_z(\Omega) \) and define \( B \) as above. Applying (1.2) yields

\[
\| b \|_{\text{BMO}_z(\Omega)} = \| B \|_{\text{BMO}(\mathbb{R}^n)} \leq C \sup_{E \in L^2_{\text{div}}, F \in L^2_{\text{curl}}} \int_{\mathbb{R}^n} B \cdot E \cdot F \, dx \\
= C \sup_{e = E|_{\Omega}, f = F|_{\Omega}, E \in L^2_{\text{div}}, F \in L^2_{\text{curl}}} \int_{\Omega} b \cdot f \, dx.
\]

Theorem 1.1 is proved.

3. Proof of Theorem 1.3

The proof of Theorem 1.3 relies on Theorem 1.1 and the following facts from functional analysis which can be found in [5, Lemmas III.1, III.2].

**Lemma 3.1.** Let \( V \) be a bounded subset of a normed vector space \( X \). We assume that \( \overline{V} \) (closure of \( V \) for the norm of \( X \)) contains the unit ball (centred at 0) of \( X \). Then, any \( x \) in that ball can be written as

\[
x = \sum_{j=0}^{\infty} \frac{1}{2^j} y_j,
\]

where \( y_j \in V \) for all \( j \geq 0 \).
Lemma 3.2. Let $V$ be a bounded symmetric $(x \in V \Rightarrow -x \in V)$ subset of a normed vector space $X$. Then, the closed convex hull $\overline{V}$ of $V$ (in $X$) contains a ball centred at 0 if and only if, for any $l \in X^*$,

$$\|l\|_{X^*} \approx \sup_{x \in V} \langle l, x \rangle.$$ 

Proof of Theorem 1.3: (1) Let $X = \mathcal{H}_2^1(\Omega)$ and

$$V = \{ e \cdot f : e \in L^2_{\text{div}}(\Omega)^n, f \in L^2_{\text{curl}}(\Omega)^n \}.$$ 

It is easy to check that $V$ is a bounded subset of $X$. In fact, for $e \in L^2_{\text{div}}(\Omega)^n$, $f \in L^2_{\text{curl}}(\Omega)^n$, let $E$ and $F$ be their zero extensions to $\mathbb{R}^n$ respectively. Then $E \in L^2_{\text{div}}(\mathbb{R}^n)^n, F \in L^2_{\text{curl}}(\mathbb{R}^n)^n$. From Theorem CLMS1, $E \cdot F \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|E \cdot F\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|E\|_{L^2(\mathbb{R}^n)^n} \|F\|_{L^2(\mathbb{R}^n)^n} \leq C.$$ 

Therefore $e \cdot f \in \mathcal{H}_2^1(\Omega)$ with $\|e \cdot f\|_{\mathcal{H}_2^1(\Omega)} \leq C$. Applying Theorem 1.1 (1) and Lemmas 3.1 and 3.2, we have the decomposition of Theorem 1.3 (1).

(2) Let $X = \mathcal{H}_1^1(\Omega)$ and

$$V = \{ e \cdot f : e = E|_{\Omega}, f = F|_{\Omega}, E \in L^2_{\text{div}}(\mathbb{R}^n)^n, F \in L^2_{\text{curl}}(\mathbb{R}^n)^n \}.$$ 

Similar to the case (1), we have $e \cdot f \in \mathcal{H}_1^1(\Omega)$ with

$$\|e \cdot f\|_{\mathcal{H}_1^1(\Omega)} = \inf_{e \cdot f = G|_{\Omega}, G \in \mathcal{H}_1^1(\mathbb{R}^n)} \|G\|_{\mathcal{H}_1^1(\mathbb{R}^n)} \leq \|E \cdot F\|_{\mathcal{H}_1^1(\mathbb{R}^n)} \leq C$$

for $e \cdot f \in V$. Using Theorem 1.1 (2) and those two lemmas again we finish the proof of Theorem 1.3.

\[\square\]

4. Proof of Lemma 2.3

To prove Lemma 2.3 we need the following result due to Nečas (see [10, Lemma 7.1, Chapter 3]). In Lemma 4.1, $W^{1,2}_0(\Omega)^n$ denotes the closure of $C_0^\infty(\Omega)^n$ in the Sobolev space $W^{1,2}(\Omega)^n$ and $\nabla \varphi = ((\partial \varphi_i)/(\partial x_j))_{n \times n}$ a $n \times n$ matrix (see [1] for Sobolev spaces).

Lemma 4.1. Let $\Omega$ be a Lipschitz domain in $\mathbb{R}^n$. If $f \in L^2(\Omega)$ has zero integral, then there exists $\varphi \in W^{1,2}_0(\Omega)^n$ such that

$$f = \text{div} \ \varphi$$

and

$$\|\nabla \varphi\|_{L^2(\Omega)^{n \times n}} \leq C \|f\|_{L^2(\Omega)}.$$ 

Corollary 4.2. Let $Q$ be a cube in $\mathbb{R}^n$. If $f \in L^2(Q)$ has zero integral, then there exists $\varphi \in W^{1,2}_0(Q)^n$ such that $f = \text{div} \ \varphi$ and

$$\|\nabla \varphi\|_{L^2(Q)^{n \times n}} \leq C_0 \|f\|_{L^2(Q)}.$$
for a constant $C_0$ independent of $Q$.

**Proof of Lemma 2.3:** Suppose $b \in L^2_{\text{loc}}(\Omega)$. We shall show that for all cubes $Q$ with $2Q \subset \Omega$ there exists $e \in L^2_{\text{div}}(\Omega)^n$ and $f \in L^2_{\text{curl}}(\Omega)^n$ such that

(4.1) \[ \left( \frac{1}{|Q|} \int_Q |b - b_Q|^2 \, dx \right)^{1/2} \leq C \int_\Omega b \cdot e \, dx. \]

Let $h = b - b_Q$, then $h \in L^2(Q)$ with $\int_Q h \, dx = 0$. From Corollary 4, there exists $\varphi := (\varphi_1, \ldots, \varphi_n) \in W^{1,2}_0(Q)^n$ such that $h = \text{div} \varphi$ and

(4.2) \[ \| \nabla \varphi \|_{L^2(Q)^n} \leq C_0 \| h \|_{L^2(Q)}, \]

where $C_0$ is independent of $Q$. So

\[ \| h \|_{L^2(Q)}^2 = \int_Q h \sum_{i=1}^n \frac{\partial \varphi_i}{\partial x_i} \, dx \leq n \max_{1 \leq i \leq n} \left| \int_Q h \frac{\partial \varphi_i}{\partial x_i} \, dx \right| \]

(4.3) \[ = n \left| \int_Q h \frac{\partial \varphi_{i_0}}{\partial x_{i_0}} \, dx \right| \]

for some choice of $i_0$ ($i_0 = 1, \ldots, n$). Assuming without loss of generality that $i_0 = 1$ in (4.3). To prove (4.1), it is sufficient to show that

(4.4) \[ \left| \int_Q h \cdot \nabla \frac{1}{L^2(Q)} \frac{\partial \varphi_1}{\partial x_1} \, dx \right| \leq C |Q|^{1/2} \left| \int_\Omega b \cdot f \, dx \right|. \]

We next construct $e$ and $f$. Define

\[ f = \left( - \frac{\partial \varphi_1}{\partial x_i}, 0, \ldots, 0, \frac{\partial \varphi_1}{\partial x_1}, 0, \ldots, 0 \right) C_0^{-1} \| h \|_{L^2(Q)}^{-1}, \]

where $(\partial \varphi_1)/(\partial x_1)$ is the $i$-th component of $f$. Then $f \in L^2(Q)^n$ with $\text{div} \ f = 0$ and $\| f \|_{L^2(Q)^n} \leq 1$ by (4.2).

Let $\psi_0 \in C^\infty_0(\mathbb{R}^n)$ such that

\[ \psi_0 = \begin{cases} 1 & \text{on } [-1, 1]^n; \\ 0 & \text{outside } [-2, 2]^n. \end{cases} \]

Define

\[ e = \gamma C_0 |Q|^{-1/2} \nabla ((x_i - x_i^0) \psi_Q(x)), \quad 1 \leq i \leq n, \]
where $\psi_Q(x) = \psi_0 \left( (x - x^0) / (l(Q) / 2) \right)$, $x^0 = (x_1^0, \ldots, x_n^0)$ and $l(Q)$ denote the centre and the side-length of the cube $Q$, $\gamma > 0$ is a normalisation constant (independent of $x^0$ and $l(Q)$) so that $\|e\|_{L^2(Q)^n} \leq 1$. It is obvious that $e \in C_0^\infty(2Q)$ and $e = \gamma C_0 |Q|^{-1/2} \varepsilon_i$ on $Q$, where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, $1$ is the $i$-th component of $\varepsilon_i$. From the construction of $e$ and $f$, we get\[ e \cdot f = \gamma |Q|^{-1/2} \|h\|_{L^2(Q)}^{-1} \frac{\partial \varphi_1}{\partial x_1} \text{ on } Q \]and (4.4) is proved. \[ \square \]

**NOTE.** It should be added that at the time the paper was finished, the author was unfortunately unaware of a similar but unpublished work [3] (with different proof). Thanks go to Galia Dafni (Department of Mathematics & Statistics, Concordia University, Canada) for informing us her paper with Chang and Sadosky.

**REFERENCES**