AN ATOMIC DECOMPOSITION FOR HARDY SPACES ASSOCIATED TO SCHröDINGER OPERATORS

LIANG SONG, CHAOQIANG TAN and LIXIN YAN

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Abstract

Let \( L = -\Delta + V \) be a Schrödinger operator on \( \mathbb{R}^n \) where \( V \) is a nonnegative function in the space \( L^1_{\text{loc}}(\mathbb{R}^n) \) of locally integrable functions on \( \mathbb{R}^n \). In this paper we provide an atomic decomposition for the Hardy space \( H^1_L(\mathbb{R}^n) \) associated to \( L \) in terms of the maximal function characterization. We then adapt our argument to give an atomic decomposition for the Hardy space \( H^1_L(\mathbb{R}^n \times \mathbb{R}^n) \) on product domains.

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1. Introduction

Let \( V \) be a locally integrable nonnegative function on \( \mathbb{R}^n \) (where \( n \geq 1 \)), which is not identically zero. We define the form \( Q \) by

\[
 Q(u, v) := \int_{\mathbb{R}^n} \nabla u \nabla v \, dx + \int_{\mathbb{R}^n} Vuv \, dx
\]

with domain

\[
 \mathcal{D}(Q) := \left\{ u \in W^{1,2}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} V|u|^2 \, dx < \infty \right\}
\]

The space \( W^{1,2}(\mathbb{R}^n) \) which appears in the formula above is the Sobolev space consisting of those \( L^2 \) functions on \( \mathbb{R}^n \) whose gradients are also square integrable. It is well known that this symmetric form is closed. We recall that it was shown by Simon [25] that this form coincides with the minimal closure of the form given by the same expression but defined on \( C^\infty_0(\mathbb{R}^n) \), the space of \( C^\infty \) functions with compact support.

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Let $L$ denote the self-adjoint operator associated with $Q$. The domain of $L$, written $\mathcal{D}(L)$, is defined to be the set of all $u \in \mathcal{D}(Q)$ for which there exists $v \in L^2$ such that

$$Q(u, \varphi) = \int_{\mathbb{R}^n} v \overline{\varphi} \, dx \quad \forall \varphi \in \mathcal{D}(Q).$$

Formally, we write $L = -\Delta + V$ as a Schrödinger operator with potential $V$. Since $V$ is a locally integrable nonnegative function on $\mathbb{R}^n$, the Feynman–Kac formula implies that the kernel $p_t(x, y)$ of the semigroup $e^{-tL}$ satisfies the estimate

$$0 \leq p_t(x, y) \leq (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$ (see [24, p. 195]).

Given a function $f \in L^2(\mathbb{R}^n)$, we consider the following nontangential maximal function associated with the Poisson semigroup generated by the operator $L$:

$$f^*_L(x) := \sup_{|y-x|<t} |e^{-t\sqrt{L}} f(y)| \quad \forall x \in \mathbb{R}^n.$$

The space $H^1_L(\mathbb{R}^n)$ is defined to be the completion of $L^2(\mathbb{R}^n)$ in the norm given by the $L^1$ norm of this maximal function, that is,

$$\|f\|_{H^1_L(\mathbb{R}^n)} := \|f^*_L\|_{L^1(\mathbb{R}^n)}.$$ 

See, for example, [1–3, 12, 17, 18] for the properties of $H^1_L(\mathbb{R}^n)$.

Note that if $L = -\Delta$, then the space $H^1_L(\mathbb{R}^n)$ is the classical Hardy space $H^1(\mathbb{R}^n)$ which is a natural substitute for $L^1(\mathbb{R}^n)$. Recall that the development of the theory of the classical Hardy spaces in $\mathbb{R}^n$ was initiated by Stein and Weiss [26] and was originally tied closely to the theory of harmonic functions. Real variable methods were introduced into this subject in the seminal paper of Fefferman and Stein [15], the evolution of whose ideas led eventually to a characterization of Hardy spaces via the so-called ‘atomic decomposition’ obtained by Coifman [7] when $n = 1$ and in higher dimensions by Latter [21].

An atomic decomposition for $H^1_L(\mathbb{R}^n)$ was given in [17] by combining the area $S$-function and the finite speed propagation property for the wave equation. Following [17], a function $a \in L^2(\mathbb{R}^n)$ is called a $(1, 2)$-atom associated to the operator $L$ if there exists a function $b \in \mathcal{D}(L)$, the domain of an operator $L$ and a ball $B$ of $\mathbb{R}^n$ such that

$$a = Lb;$$

$$\text{supp} \ L^k b \subseteq B;$$

$$\|(r_B^2 L)^k b\|_{L^2(\mathbb{R}^n)} \leq r_B^2 |B|^{-1/2},$$

where $k = 0, 1$ and $r_B$ denotes the radius of the ball $B$.

The aim of this paper is to get an atomic decomposition directly from the fact that $f^*_L \in L^1(\mathbb{R}^n)$ and then to provide a new proof of the atomic decomposition for $H^1_L(\mathbb{R}^n)$. Our first main result is the following theorem.
Theorem 1.1. Let $L = -\Delta + V$ where $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a nonnegative function on $\mathbb{R}^n$. Let $f \in H^1_L(\mathbb{R}^n)$. Then there exist $(1, 2)$-atoms $a_j$ and real numbers $\lambda_j$ for $j = 1, 2, 3, \ldots$ such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

in $H^1_L(\mathbb{R}^n)$. Furthermore, matters can be arranged so that the sequence $\lambda_j$ satisfies the inequality

$$\sum_{j=1}^{\infty} |\lambda_j| \leq C \|f\|_{H^1_L(\mathbb{R}^n)}$$

for some positive constant $C$, which may depend on $n$.

Conversely, any function $f$ which is written in the form of (1.2), where the $a_j$ are $(1, 2)$-atoms, satisfies the inequality

$$\|f\|_{H^1_L(\mathbb{R}^n)} \leq C \sum_{j=1}^{\infty} |\lambda_j|.$$

We mention that the localized version of the atomic decomposition for $H^1_L(\mathbb{R}^n)$ when $L = -\Delta + V$ was given in [13], by using the properties of local Hardy spaces (see [16]), under the assumption that $V$ was a fixed nonnegative function on $\mathbb{R}^n$ belonging to the reverse Hölder class $B_q$ for some $q > 1$. That is, there exists a positive constant $C$, possibly depending on $q$ and $V$, such that the reverse Hölder inequality

$$\left( \frac{1}{|B|} \int_B V^q \, dx \right)^{1/q} \leq C \left( \frac{1}{|B|} \int_B V \, dx \right)$$

holds for every ball $B$ in $\mathbb{R}^n$.

Let us now turn to the Hardy space on product domains. We note that the usual space $H^1(\mathbb{R}^n \times \mathbb{R}^n)$ on the product domain is now well understood (see, for instance, [4, 5, 14]). In this paper we shall be concerned with the space $H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$ associated to the Schrödinger operator $L$ (see [11] for more properties).

For any $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, define

$$f_L^\ast (x_1, x_2) = \sup_{|y_1 - x_1| < t_1} \sup_{|y_2 - x_2| < t_2} |e^{-t_1 \sqrt{L}} \otimes e^{-t_2 \sqrt{L}} f(y_1, y_2)|$$

where

$$e^{-t_1 \sqrt{L}} \otimes e^{-t_2 \sqrt{L}} f(y_1, y_2) := \int_{\mathbb{R}^n \times \mathbb{R}^n} p_{t_1}(y_1, z_1) p_{t_2}(y_2, z_2) f(z_1, z_2) \, dz_1 \, dz_2.$$

The space $H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$ is defined to be the completion of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ in the norm given by

$$\|f\|_{H^1_L(\mathbb{R}^n \times \mathbb{R}^n)} := \|f_L^\ast\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)}.$$
First, the function $a$ can be further decomposed into the form $a = \sum_{R \in m(\Omega)} a_R$ where for each $R \in m(\Omega)$ there exists a function $b_R$ such that $a_R = (L \otimes L)b_R$ and
\[ \text{supp}(L^i \otimes L^j)b_R \subseteq 10R, \quad i, j = 0, 1, \]
where $10R$ denotes the rectangle with the same center as $R$ and 10 times the side lengths.

Second,
\[ \|a\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq |\Omega|^{-1/2} \]
and
\[ \sum_{R \in m(\Omega)} \sum_{i,j=0}^1 \ell(I)^{4i-4} \ell(J)^{4j-4} \|L^i \otimes L^j b_R\|^2_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq |\Omega|^{-1} \]
where $R = I \times J$ denotes the dyadic rectangle of $\mathbb{R}^n \times \mathbb{R}^n$ whose side lengths are $\ell(I)$ and $\ell(J)$, and $10R$ denotes the set $\{10x \mid x \in R\}$ and $m(\Omega)$ denotes the set of maximal dyadic subrectangles of $\Omega$ (see Section 4 below).

The second main result of this paper is the following theorem.

**Theorem 1.2.** Let $L = -\Delta + V$ where $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a nonnegative function on $\mathbb{R}^n$. Let $f \in H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$. Then there exist product $(1, 2)$-atoms $a_j$ and numbers $\lambda_j$, where $j = 0, 1, 2, \ldots$, such that
\[ f = \sum_{j=1}^{\infty} \lambda_j a_j \quad (1.3) \]
in $H^1_L(\mathbb{R}^n \times \mathbb{R}^n)$, and the sequence $\lambda_j$ satisfies the condition that
\[ \sum_{j=1}^{\infty} |\lambda_j| \leq C\|f\|_{H^1_L(\mathbb{R}^n \times \mathbb{R}^n)}. \]

Conversely, for any decomposition of $f$ of the form in (1.3),
\[ \|f\|_{H^1_L(\mathbb{R}^n \times \mathbb{R}^n)} \leq C \sum_{j=1}^{\infty} |\lambda_j|. \]

The organisation of this paper is as follows. In Section 2 we introduce some notation and preliminary lemmas. Our main results, Theorems 1.1 and 1.2, are proved in Sections 3 and 4. The main contribution of this paper is to combine the Calderón reproducing formula, the finite propagation speed property and the methods of Wilson [27] to obtain an atomic decomposition of Hardy spaces and then to verify the required $L^2$ norm estimates of the atoms by using square function estimates.

Throughout this paper, the letters $C$ and $c$ denote (possibly different) constants that are independent of the essential variables.

## 2. Preliminaries

Recall that if $L$ is a nonnegative, self-adjoint operator on $L^2(\mathbb{R}^n)$ and $E_L(\lambda)$ denotes a spectral decomposition associated with $L$, then for every bounded Borel
function $F : [0, \infty) \to \mathbb{C}$ one defines the operator

$$F(L) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

by the formula

$$F(L) := \int_0^\infty F(\lambda) \, dE_L(\lambda). \quad (2.1)$$

In particular, the operator $\cos(t\sqrt{L})$ is well defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from [9, Theorem 3] (see also [6]) that the integral kernel $K_{\cos(t\sqrt{L})}$ of $\cos(t\sqrt{L})$ satisfies

$$\supp K_{\cos(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \leq t\}. \quad (2.2)$$

By the Fourier inversion formula, whenever $F$ is an even bounded Borel function with Fourier transform $\hat{F}$ in $L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. Specifically, using (2.1), we have

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^\infty \hat{F}(t) \cos(t\sqrt{L}) \, dt$$

which, when combined with (2.2), gives us that

$$K_{F(\sqrt{L})} = (2\pi)^{-1} \int_{|t| \geq |x - y|} \hat{F}(t)K_{\cos(t\sqrt{L})} \, dt.$$

**Lemma 2.1.** Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function such that $\supp \varphi \subseteq [-1, 1]$. Let $\Phi$ denote the Fourier transform of $\varphi$. Then for every $\kappa = 0, 1, 2, \ldots$ and for every $t > 0$ the kernel $K_{(t^2L)^\kappa \Phi(t\sqrt{L})}$ of $(t^2L)^\kappa \Phi(t\sqrt{L})$ satisfies the condition

$$\supp K_{(t^2L)^\kappa \Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \leq t\}.$$

**Proof.** We refer the reader to [17, Lemma 3.5] for the proof. $\square$

In this paper we use $\mathbb{R}^n_+$ to denote the upper half space of $\mathbb{R}^n$. In the following lemma we shall assume that $\varphi \in C_0^1(\mathbb{R}^n)$ is nonnegative, radial and nonincreasing. We also assume that $\varphi = 1$ on $B(0, 1/2)$, $\supp \varphi \subseteq B(0, 1)$ and $\int \varphi(x) \, dx = 1$. We sometimes use capital letters to denote points of $\mathbb{R}^n_+$ (for example, $X = (x, t)$), and set

$$u(x, t) = e^{-t\sqrt{L}} \varphi(x),$$

$$\nabla_X u(X) = (\nabla_x u, \partial_t u)$$

$$|\nabla_X u|^2 = |\nabla_x u|^2 + |\partial_t u|^2.$$

**Lemma 2.2.** For every $f, g \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n_+} \left| t|\nabla_X u(x, t)|^2 \varphi_t \ast g(x) \right|^2 \frac{dx \, dt}{t} \leq \int_{\mathbb{R}^n} |f(x)|^2 |g(x)|^2 \, dx + \int_{\mathbb{R}^n_+} |u(x, t)|^2 |\psi_t \ast g(x)|^2 \frac{dx \, dt}{t} \quad (2.3)$$

where $\psi$ is a vector-valued function with the same support as $\varphi$ and mean value 0.
Proof. The proof of Lemma 2.2 can be obtained by making minor modifications to the proof of [23, Lemma 3.1] in the case where \( L = -\Delta \) is the Laplace operator on \( \mathbb{R}^n \).

For the sake of completeness and for the reader’s convenience we give a brief sketch of the proof of this lemma.

Write \( \nabla_X^2 = \nabla_X \nabla_X \). Since \( u = e^{-t\sqrt{L}}f \), we have
\[
\nabla_X^2 u^2 = (\partial_t^2 + \Delta)u^2 = 2|\nabla_X u|^2 + 2Vu^2.
\]
This, together with the condition that \( V \geq 0 \), gives
\[
2 \int_{\mathbb{R}^{n+1}_+} |t \nabla_X u|^2 |\varphi_i * g|^2 \frac{dx dt}{t} = \int_{\mathbb{R}^{n+1}_+} \nabla_X^2 u^2 |\varphi_i * g|^2 t \, dx \, dt - 2 \int_{\mathbb{R}^{n+1}_+} Vu^2 |\varphi_i * g|^2 t \, dx \, dt \leq \int_{\mathbb{R}^{n+1}_+} \nabla_X^2 u^2 |\varphi_i * g|^2 t \, dx \, dt.
\]

After an integration by parts we obtain
\[
\int_{\mathbb{R}^{n+1}_+} \nabla_X^2 u^2 |\varphi_i * g|^2 t \, dx \, dt - \frac{2V}{2} \int_{\mathbb{R}^{n+1}_+} u^2 \nabla_X^2 (\varphi_i * g) \nabla_X (\varphi_i * g) \, dx \, dt - 2 \int_{\mathbb{R}^{n+1}_+} |u| \partial_t^2 |\varphi_i * g|^2 \, dx \, dt.
\]

Note that the conditions \( u(x, 0) \in L^2(\mathbb{R}^n) \) or \( f \in L^2(\mathbb{R}^n) \) are sufficient to ensure that the boundary terms ‘at \( \infty \)’ for this integration by parts vanish, as does the boundary term for \( t = 0 \).

We use a further integration by parts to obtain
\[
2 \int_{\mathbb{R}^{n+1}_+} u \partial_t u (\varphi_i * g)^2 \, dx \, dt
\]
\[
= -\lim_{t \to 0} \int_{\mathbb{R}} u^2 (\varphi_i * g)^2 \, dx \, dt - 2 \int_{\mathbb{R}^{n+1}_+} u^2 (\varphi_i * g) \partial_t (\varphi_i * g) \, dx \, dt
\]
\[
= -\int_{\mathbb{R}} f^2 g^2 \, dx - 2 \int_{\mathbb{R}^{n+1}_+} u^2 (\varphi_i * g) \partial_t (\varphi_i * g) \, dx \, dt.
\]

When combined with (2.4), integration by parts and the Cauchy–Schwarz inequality, this gives (2.3) provided that
\[
|\psi_i * f|^2 = 9(|t \nabla_X \varphi_i * g)|^2 + 9|\check{\psi_i} * g|^2.
\]
Here \( \check{\psi} = (x_1 \varphi, \ldots, x_n \varphi) \). For the details, we refer the reader to [23, (3.8)]. This completes our proof. \( \square \)
Finally for \( s > 0 \) we define the set of measurable functions
\[
\mathcal{F}(s) := \left\{ \psi : \mathbb{C} \to \mathbb{C} \left| |\psi(z)| \leq C \frac{|z|^s}{(1 + |z|^{2s})} \right. \right\}.
\]
Then for any nonzero function \( \psi \in \mathcal{F}(s) \),
\[
\left\{ \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right\}^{1/2} < \infty.
\]
We write \( \psi_t(z) = \psi(tz) \). It follows from spectral theory (see [10]) that, if \( f \in L^2(\mathbb{R}^n) \), then
\[
\left\{ \int_0^\infty \|\psi(t\sqrt{L})f\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \right\}^{1/2} = \left\{ \left( \int_0^\infty \langle \tilde{\psi}(t\sqrt{L})\psi(t\sqrt{L})f, f \rangle \frac{dt}{t} \right)^{1/2} \right\}^{1/2}
= \left\{ \left( \int_0^\infty |\psi(t\sqrt{L})|^2 \frac{dt}{t} f, f \right) \right\}^{1/2}
= \kappa \|f\|_{L^2(\mathbb{R}^n)},
\]
where \( \kappa = \left\{ \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right\}^{1/2} \).

3. Proof of Theorem 1.1

We shall use \( M \) to denote the Hardy–Littlewood maximal function with respect to the balls of \( \mathbb{R}^n \). We use the notation
\[
\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1} \mid |x - y| < t\}
\]
to denote the standard cone (of aperture 1) with vertex \( x \in \mathbb{R}^n \).

For any closed subset \( F \) of \( \mathbb{R}^n \), we denote by \( \mathcal{R}(F) \) the union of all cones with vertices in \( F \), that is,
\[
\mathcal{R}(F) = \bigcup_{x \in F} \Gamma(x).
\]
If \( O \) is an open subset of \( \mathbb{R}^n \), then the ‘tent’ over \( O \), denoted by \( \hat{O} \), is defined to be
\[
\hat{O} = t[\mathcal{R}(\mathring{O})].
\]

**Proof of Theorem 1.1.** Let \( f \in H^1_L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \). We shall prove that \( f \) has an atomic decomposition as in (1.2). We start with a suitable version of the Calderón reproducing formula.

Let \( \varphi \) and \( \Phi \) be as in Lemma 2.1 and set \( \Psi(x) := x^4 \Phi(x) \) for all \( x \in \mathbb{R} \). By the \( L^2 \)-functional calculus (see [22]) for every \( f \in L^2(\mathbb{R}^n) \) we can write
\[
f = c_\varphi \int_0^\infty \Psi(t\sqrt{L})t\sqrt{L}e^{-t\sqrt{L}}f \frac{dt}{t}
= \lim_{N \to \infty} c_\varphi \int_{1/N}^N \Psi(t\sqrt{L})t\sqrt{L}e^{-t\sqrt{L}}f \frac{dt}{t}
\]
with the integral converging in \( L^2(\mathbb{R}^n) \).
For \( i \in \mathbb{Z} \) we define the sets
\[
O_i := \{ x \in \mathbb{R}^n \mid f_L^*(x) > 2^i \}
\]
and consider
\[
O_i^* = \{ x \in \mathbb{R}^n \mid M(\chi_{O_i})(x) > 2^{-(n+1)} \}.
\]
Then \( O_i \subseteq O_i^* \) and \( |O_i^*| \leq C|O_i| \) for every \( i \in \mathbb{Z} \).

Now let \( \{Q_j^i\}_j \) be a Whitney decomposition of \( O_i^* \) and let \( \hat{O}_i^* \) be a tent region, that is,
\[
\hat{O}_i^* := \{ (x, t) \in \mathbb{R}^n \times (0, \infty) \mid \text{dist}(x, cO_i^*) \geq t \}.
\]

For every \( i, j \in \mathbb{Z} \) we define
\[
T_j^i = (Q_j^i \times (0, +\infty)) \cap \hat{O}_i^* \setminus \hat{O}_{i+1}^*,
\]
and \( \lambda_j^i = 2^j |Q_j^i| \). Using formula (3.1), we write
\[
f = \sum_{j, i \in \mathbb{Z}} c_j \Psi(t\sqrt{L})(\chi_{T_j^i} t\sqrt{L}e^{-t\sqrt{L}}) f \frac{dt}{t} =: \sum_{j, i \in \mathbb{Z}} \lambda_j^i a_j^i
\]
where \( a_j^i = L b_j^i \) and
\[
b_j^i = (\lambda_j^i)^{-1} c_j \int_0^{\infty} t^4 L\Phi(t\sqrt{L})(\chi_{T_j^i} t\sqrt{L}e^{-t\sqrt{L}}) f \frac{dt}{t}.
\]
We claim that, up to normalization by a multiplicative constant, the \( a_j^i \) are \((1, 2)\)-atoms. Once this claim is established, we shall have
\[
\sum_{j, i} |\lambda_j^i| = \sum_{j, i} 2^j |Q_j^i| \leq C \sum_i 2^i |O_i^*| \leq C \sum_i 2^i |O_i| \leq C \|f\|_{H^1_{\ell}(\mathbb{R}^n)},
\]
as desired.

Let us now prove the claim. We shall show that for every \( j, i \in \mathbb{Z} \), the function \( C^{-1} a_j^i \) is a \((1, 2)\)-atom associated with the cube \( 10\sqrt{n}Q_j^i \) for some constant \( C \) (independent of \( i \) and \( j \)). Observe that if \( (x, t) \in T_j^i \), then \( B(x, t) \in O_i^* \). This, together with the fact that \( Q_j^i \) is the Whitney cube of \( O_i^* \), allows us to deduce that
\[
t \leq 6 \sqrt{n} \ell(Q_j^i).
\]

By Lemma 2.1 the integral kernel \( K_{\Phi_i(\sqrt{L})} \) of the operator \( \Phi_i(\sqrt{L}) \) satisfies the condition that
\[
\text{supp} \ K_{\Phi_i(\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x - y| \leq t \}.
\]
This enables us to deduce that, whenever \( k = 0, 1 \),
\[
\text{supp}(L^k b^j) \subseteq 10\sqrt{n}Q_i^j.
\]
To continue, for each cube \( Q_i^j \) we consider some \( h \in L^2(Q_i^j) \) such that \( \|h\|_{L^2(Q_i^j)} = 1 \). Then for \( k = 0, 1 \),
\[
\left| \int_{\mathbb{R}^n} (\ell(Q_i^j)^2 L) b^j(x) h(x) \, dx \right| \\
= c_{\psi} \int_{\mathbb{R}^{n+1}} t^4 (\ell(Q_i^j)^2 L) L \Phi(t\sqrt{L}) (\chi_{T_i^j} \sqrt{Le^{-t\sqrt{L}}}) f(x) h(x) \frac{dx \, dt}{t} \\
\leq C \ell(Q_i^j)^2 \int_{\mathbb{R}^{n+1}} |(\chi_{T_i^j} \sqrt{Le^{-t\sqrt{L}}}) f(x) t^{2(k+1)} L^{k+1} \Phi(t\sqrt{L}) (h(x))| \frac{dx \, dt}{t} \\
\leq C \ell(Q_i^j)^2 \left( \int_{T_i^j} |t\sqrt{Le^{-t\sqrt{L}}} f(x)|^2 \frac{dx \, dt}{t} \right)^{1/2} \\
\times \left( \int_{\mathbb{R}^{n+1}} |(t^2 L)^{k+1} \Phi(t\sqrt{L}) (h(x))|^2 \frac{dx \, dt}{t} \right)^{1/2} \\
\leq C \ell(Q_i^j)^2 \left( \int_{T_i^j} |t\sqrt{Le^{-t\sqrt{L}}} f(x)|^2 \frac{dx \, dt}{t} \right)^{1/2}.
\]
Note that the first inequality is obtained from the fact that \( 0 < t < 6\sqrt{n} \ell(Q_i^j) \) and the third inequality follows from (2.5).
Therefore, in order to prove our claim, it suffices to show that
\[
\int_{T_i^j} |t\sqrt{Le^{-t\sqrt{L}}} f(y)|^2 \frac{dy \, dt}{t} \leq C 2^{2j} |Q_i^j|.
\]  
(3.2)
Let us show that (3.2) is satisfied. Let \( \varphi \in C_0^\infty(\mathbb{R}^n) \) be as in Lemma 2.2 and set
\[
F^j_i = 10\sqrt{n} Q_i^j \setminus O_{i+1}.
\]
We first show that, for all \((y, t) \in T_i^j\),
\[
|\varphi_t * \chi_{F^j_i}(y)| \geq C.
\]  
(3.3)
Indeed, for any \((y, t) \in T_i^j\), we can obtain
\[
B(y, t) \subseteq 10\sqrt{n} Q_i^j
\]
and
\[
B(y, t) \cap O_{i+1}^* \neq \emptyset.
\]
This shows that there exists \( x_0 \in B(y, t) \cap O_{i+1}^* \) such that
\[
\mathcal{M}(\chi_{O_{i+1}})(x_0) \leq 2^{-(n+1)}.
\]
It then follows that
\[
|B(y, t) \cap O_{i+1}| \leq 2^{-(n+1)} |B(y, t)|.
\]
This implies that
\[ |B(y, t/2) \cap F_i^j| \geq |B(y, t/2) \cap 10\sqrt{n}Q_i^k| - |B(y, t/2) \cap O_{i+1}| \]
\[ \geq |B(y, t/2)| - 2^{-(n+1)}|B(y, t)| \]
\[ = 2^{-(n+1)}|B(y, t)| \]
and then, for any \((y, t) \in T_i^j\),
\[ |\varphi_i \ast \chi_{F_i^j}(y)| = \left| \int \varphi_i(y-z)\chi_{F_i^j}(z) \, dz \right| \geq \tau^n|B(y, t/2) \cap F_i^j| \geq C \]
which proves estimate (3.3).

By Lemma 2.2, we have
\[
\int_{T_i^j} |t \sqrt{L} e^{-t \sqrt{L}} f(y)|^2 \frac{dy \, dt}{t} \\
\leq C \int_{\mathbb{R}^{n+1}} |\nabla x \cdot t \sqrt{L} e^{-t \sqrt{L}} f(y)|^2 |\varphi_i \ast \chi_{F_i^j}(y)|^2 \frac{dy \, dt}{t} \\
\leq C \left( \int_{\mathbb{R}^{n+1}} |e^{-t \sqrt{L}} f(y)|^2 |\varphi_i \ast \chi_{F_i^j}(y)|^2 \frac{dy \, dt}{t} + \int_{\mathbb{R}^n} |f(x)|^2 |\chi_{F_i^j}(x)|^2 \, dx \right) \\
=: T_1 + T_2.
\]
Observe that if \(\varphi_i \ast \chi_{F_i^j}(y) \neq 0\), then \(F_i^j \cap B(y, t) \neq \emptyset\) and there exists an element
\[ x_0 \in B(y, t) \cap (10\sqrt{n}Q_i^j) \cap cO_{i+1}. \]
This gives us that
\[ |e^{-t \sqrt{L}} f(y)| \leq f^j_L(x_0) \leq 2^{j+1}. \]
Hence
\[ T_1 \leq C 2^{2j+2} \int_{\mathbb{R}^{n+1}} |\varphi_i \ast \chi_{F_i^j}(y)|^2 \frac{dy \, dt}{t} \leq C 2^{2j}|Q_i^j|. \]
Also
\[ T_2 \leq C 2^{2j+2} \int_{\mathbb{R}^n} |\chi_{F_i^j}(x)|^2 \, dx \leq C 2^{2j}|Q_i^j| \]
and the estimate (3.2) follows readily.

We have shown that, up to normalization by a multiplicative constant, the \(a_j^i\) are \((1, 2)\)-atoms associated with the ball \(B(x_i^j, c_1 \ell(Q_i^j))\) for some constant \(c_1\) where \(x_i^j\) is the center of the cube \(Q_i^j\). This proves that \(f\) has an atomic decomposition as in (1.2).

To prove the converse we assume that \(f = \sum_j \lambda_j a_j\) where the \(a_j\) are \((1, 2)\)-atoms and \(\sum_j |\lambda_j| < \infty\). In this case, it was proved in \cite[Theorem 7.4]{17} that \(f \in H^1(I_{2n}(\mathbb{R}^n))\). We omit the details here. The proof of Theorem 1.1 is now complete. \(\square\)

4. Proof of Theorem 1.2

In this section we shall work exclusively with the domain \(\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+\) and its distinguished boundary \(\mathbb{R}^n \times \mathbb{R}^n\). If \(x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n\), then we shall denote
by $\Gamma(x)$ the product cone $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$, where

$$\Gamma(x_i) = \{(y_i, t_i) \in \mathbb{R}^{n+1}_+ \mid |x_i - y_i| < t_i\}$$

for $i = 1, 2$. If $(x, t) := ((x_1, t_1), (x_2, t_2)) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+$, then we shall write

$$B_{x,t} := B(x_1, t_1) \times B(x_2, t_2)$$

for the product ball.

For any open set $\Omega \subseteq \mathbb{R}^{2n}$, the tent over $\Omega$, denoted by $\hat{\Omega}$, is the set

$$\{(x, t) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ \mid B_{x,t} \subseteq \Omega\}.$$

Let $m(\Omega)$ denote the set of maximal dyadic subrectangles of $\Omega$. Let $m_1(\Omega)$ denote the subset of those dyadic subrectangles $R = I \times J$ of $\Omega$ that are maximal in the $x_1$ direction. In other words, if $S = I' \times J \supseteq R$ is a dyadic subrectangle of $\Omega$, then $I = I'$. Similarly, define $m_2(\Omega)$ to be the collection of those dyadic subrectangles of $\Omega$ that are maximal in the $x_2$ direction. Let $M_s$ denote the strong maximal operator, that is, for any $x \in \mathbb{R}^{2n}$ let

$$M_s f(x) = \sup_{t_1 > 0, t_2 > 0} \frac{1}{|B_{x,t}|} \int_{B_{x,t}} |f(y_1, y_2)| \, dy_1 \, dy_2. \quad (4.1)$$

In order to prove Theorem 1.2 we need some auxiliary results. The first one is Journé’s covering lemma (see [20]).

**Lemma 4.1.** Let $\Omega$ be an open subset of $\mathbb{R}^n \times \mathbb{R}^n$ and let $R = I \times J \in m_2(\Omega)$ where $I, J$ are dyadic cubes of $\mathbb{R}^n$. Suppose that $\hat{I}$ is the biggest dyadic cube of $\mathbb{R}^n$ containing $I$ such that $\hat{I} \times J \subseteq \Omega^*$ where

$$\Omega^* = \{x \in \mathbb{R}^{2n} \mid M_s \chi_{\Omega}(x) > 1/2\}.$$

We set $\gamma_1(R) = |\hat{I}|/|I|$ and define $\gamma_2$ similarly. Then for any $\delta > 0$,

$$\sum_{R \in m_2(\Omega)} |R| \gamma_1^{-\delta}(R) \leq c_\delta |\Omega|$$

and

$$\sum_{R \in m_1(\Omega)} |R| \gamma_2^{-\delta}(R) \leq c_\delta |\Omega|$$

where $c_\delta$ is a constant depending only on $\delta$ and not on $\Omega$.

For every $i = 1, 2$ we let $\nabla_{X_i} = (\nabla_{x_i}, \partial_{t_i})$. In the following lemma we assume that $\varphi \in C_0^1(\mathbb{R}^n)$ is nonnegative, radial and nonincreasing. We also assume that $\varphi = 1$ on $B(0, 1/2)$, supp $\varphi \subseteq B(0, 1)$ and $\int \varphi(x) \, dx = 1$.

**Lemma 4.2.** For every $f, g \in L^2(\mathbb{R}^{2n})$ and $i = 1, 2$ there exist vector-valued functions $\psi^{(i)} \in C_0^\infty(\mathbb{R}^n)$ satisfying the conditions supp $\psi^{(i)} \subseteq B(0, 1)$, $\int_{\mathbb{R}^n} \psi^{(i)}(x) \, dx = 0$ and
such that
\[
\int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |t_1 \nabla_x e^{-t_1 \nabla L} \otimes t_2 \nabla_x e^{-t_2 \nabla L} f(y_1, y_2)|^2 |(\varphi_{t_1} \otimes \varphi_{t_2}) * g(y_1, y_2)|^2 \frac{dy \, dt}{t_1 t_2} \\
\leq \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |e^{-t_1 \nabla L} \otimes e^{-t_2 \nabla L} f(y_1, y_2)|^2 |(\psi_{t_1}^{(1)} \otimes \psi_{t_2}^{(2)}) * g(y_1, y_2)|^2 \frac{dy \, dt}{t_1 t_2} \\
+ \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} |e^{-t_1 \nabla L} f(x_1, x_2)|^2 |\psi_{t_1}^{(1)} * g(x_1, x_2)|^2 \frac{dy_1 \, dt_1}{t_1} \, dx_2 \\
+ \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} |e^{-t_2 \nabla L} f(x_1, x_2)|^2 |\psi_{t_2}^{(2)} * g(x_1, x_2)|^2 \frac{dy_2 \, dt_2}{t_2} \, dx_1 \\
+ \int_{\mathbb{R}^{n}} |f(x_1, x_2)|^2 |g(x_1, x_2)|^2 \, dx_1 \, dx_2.
\]

**Proof.** Repeated applications of Lemma 2.2 can be used to prove Lemma 4.2. \(\square\)

Finally we state the following lemma whose proof we omit since it is similar to that of [17, Lemma 4.3].

**Lemma 4.3.** Suppose that \(T\) is a bounded sublinear operator on \(L^2(\mathbb{R}^{2n})\) and that for every product \((1, 2)\)-atom \(a\) on product domains we have
\[
\|Ta\|_{L^1(\mathbb{R}^{2n})} \leq C
\]
where the constant \(C\) is independent of \(a\). Then for any decomposition of the form given in (1.3) of \(f\) we have
\[
\|Tf\|_{L^1(\mathbb{R}^{2n})} \leq C \sum_{j=1}^{\infty} |\lambda_j|.
\]

**Proof of Theorem 1.2** By condition (1.1) for every \(K = 0, 1, \ldots\) there exists a constant \(C_K\) such that the kernel \(p_{t,K}\) of the operator \((t \sqrt{L})^{2K} e^{-t \nabla L}\) satisfies the condition that
\[
|p_{t,K}(x, y)| \leq C_K \frac{\sqrt{t}}{(t + |x - y|)^{n+1}} \quad \forall t > 0 \quad (4.2)
\]
and almost every \(x, y \in \mathbb{R}^n\) (see, for instance, [17, Lemma 7.2]).

**Step 1.** Let \(f = \sum_j \lambda_j a_j\) where the \(a_j\) are product \((1, 2)\)-atoms and \(\sum_{j=1}^{\infty} |\lambda_j| < \infty\). Recall that the strong maximal operator \(\mathcal{M}\) defined in (4.1) is bounded on \(L^2(\mathbb{R}^{2n})\) (see [14]). This, together with condition (1.1), gives us that
\[
\|f^2\|_{L^2(\mathbb{R}^{2n})} \leq C \|\mathcal{M}f\|_{L^2(\mathbb{R}^{2n})} \leq C \|f\|_{L^2(\mathbb{R}^{2n})}.
\]
By Lemma 4.3, it is enough to show that \(\|a^*_L\|_{L^1(\mathbb{R}^{2n})} \leq C\) for every product \((1, 2)\)-atom \(a\), for some constant \(C\) which is independent of \(a\).
Suppose that
\[ a = \sum_{R \in m(\Omega)} a_R = \sum_{R \in m(\Omega)} (L \otimes L) b_R \]
is a product \((1,2)\)-atom supported on some open subset \(\Omega\) of \(\mathbb{R}^{2n}\). For any maximal dyadic subrectangle \(R = I \times J \in m(\Omega)\) let \(\ell(I), \ell(J)\) be the side-lengths of cubes \(I\) and \(J\) and let \(I'\) be the longest dyadic interval containing \(I\) so that
\[ I' \times J \subseteq \Omega^* = \{ x \in \mathbb{R}^{2n} \mid \mathcal{M}_s(\chi_{\Omega})(x) > 1/2 \}. \]

Then \(I' \times J\) is in \(m_1(\Omega^*)\). Let \(S\) be the longest dyadic interval so that \(S \supseteq J\) and \(I' \times S \subseteq \Omega^{**}\) where
\[ \Omega^{**} = \{ x \in \mathbb{R}^{2n} \mid \mathcal{M}_s(\chi_{\Omega^*})(x) > 1/2 \}. \]

Let \(\overline{R}\) be the 10-fold dilate of \(I' \times S\) concentric with \(I' \times S\). Clearly, an application of the strong maximal theorem (see [8, 19] for the proof) shows that
\[ \left| \bigcup \overline{R} \right| \leq c|\Omega^{**}| \leq c|\Omega^*| \leq c|\Omega|. \]

We then have
\[ \int_{\bigcup \overline{R}} a_L^*(x) \, dx \leq C \left| \bigcup \overline{R} \right|^{1/2} \| a_L^* \|_{L^2(\mathbb{R}^{2n})} \leq C \left| \bigcup \overline{R} \right|^{1/2} \| \mathcal{M}_s(a) \|_{L^2(\mathbb{R}^{2n})} \]
\[ \leq C|\Omega|^{1/2} \| a \|_{L^2(\mathbb{R}^{2n})} \leq C|\Omega|^{1/2}|\Omega|^{-1/2} \leq C. \]

We now find an estimate for
\[ \int_{\bigcup \overline{R}} a_L^*(x) \, dx \leq C. \]

We can write
\[ \int_{\bigcup \overline{R}'} a_L^*(x) \, dx \leq \sum_{R \in m(\Omega)} \int_{\overline{R}'} (a_R)_L^*(x) \, dx \]
\[ \leq \sum_{R \in m(\Omega)} \int_{x_1 \notin 10'} (a_R)_L^*(x) \, dx + \sum_{R \in m(\Omega)} \int_{x_1 \notin 10'} (a_R)_L^*(x) \, dx. \]

We only need to calculate the estimate for the first term above since the proof of the estimate for the second term is similar.

Observe that
\[ \sum_{R \in m(\Omega)} \int_{x_1 \notin 10'} (a_R)_L^*(x) \, dx = \sum_{R \in m(\Omega)} \left( \int_{x_1 \notin 10'} \int_{x_2 \notin 10} + \int_{x_1 \notin 10} \int_{x_2 \notin 10'} \right) (a_R)_L^*(x) \, dx \]
\[ =: E_1 + E_2. \]
By Hölder’s inequality, we have

\[ E_1 \leq \sum_{R \in \Omega(\text{Q})} |J|^{1/2} \int_{x_1 \in 100|I|} \left\| (a_R)^* (x_1, \cdot) \right\|_{L^2(dx_2)} \, dx_1 \]

\[ \leq C \sum_{R \in \Omega(\text{Q})} |J|^{1/2} \int_{x_1 \in 100|I|} \left\{ \int_{\mathbb{R}^n} \left( \sup_{t_1 < \mathcal{I}_I(t)} |e^{-t_1 \sqrt{T}} a_R(y_1, x_2)|^2 \, dx_2 \right) \right\}^{1/2} \, dx_1 \]

\[ \leq C \sum_{R \in \Omega(\text{Q})} |J|^{1/2} \int_{x_1 \in 100|I|} \left\{ \int_{\mathbb{R}^n} \left( \sup_{t_1 < \mathcal{I}_I(t)} |e^{-t_1 \sqrt{T}} a_R(y_1, x_2)|^2 \, dx_2 \right) \right\}^{1/2} \, dx_1 \]

\[ + C \sum_{R \in \Omega(\text{Q})} |J|^{1/2} \int_{x_1 \in 100|I|} \left\{ \int_{\mathbb{R}^n} \left( \sup_{t_1 \geq \mathcal{I}_I(t)} |e^{-t_1 \sqrt{T}} a_R(y_1, x_2)|^2 \, dx_2 \right) \right\}^{1/2} \, dx_1 \]

\[ =: E_{11} + E_{12}. \]

We consider the term \( E_{11} \) above. Let \( x_I \) denote the center of cube \( I \). Note that \( x_1 \in 100|I|' \) and \( |x_1 - y_1| < t_1 < \mathcal{I}_I(t) \). It follows from the estimate (4.2) that

\[ |e^{-t_1 \sqrt{T}} a_R(\cdot, x_2)(y_1)| \leq C \int_{\mathbb{R}^n} \frac{t_1}{|t_1 + |y_1 - z_1||^{n+1}} |a_R(z_1, x_2)| \, dz_1 \]

\[ \leq C \frac{\mathcal{I}_I(t)}{|x_1 - x_I|^{n+1}} \| a_R(\cdot, x_2) \|_{L^1(\mathbb{R}^n)} \]

\[ \leq C |J|^{1/2} \frac{\mathcal{I}_I(t)}{|x_1 - x_I|^{n+1}} \| a_R(\cdot, x_2) \|_{L^2(\mathbb{R}^n)} \]  

which, in combination with Lemma 4.1, gives us that

\[ E_{11} \leq C \sum_{R \in \Omega(\text{Q})} |J|^{1/2} |I|^{1/2} \left\{ \int_{x_1 \in 100|I|'} \frac{\mathcal{I}_I(t)}{|x_1 - x_I|^{n+1}} \, dx_1 \right\} \| a_R \|_{L^2(\mathbb{R}^n)} \]

\[ \leq C \sum_{R \in \Omega(\text{Q})} |R|^{1/2} \| a_R \|_{L^2(\mathbb{R}^n)} \mathcal{Y}_1(R)^{-1} \]

\[ \leq C \left\{ \sum_{R \in \Omega(\text{Q})} \| a_R \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/2} \left( \sum_{R \in \Omega(\text{Q})} |R| \mathcal{Y}_1(R)^{-2} \right)^{1/2} \]

\[ \leq C. \]

For the term \( E_{12} \) above, we apply the definition of the product \( (1, 2) \)-atom to obtain

\[ |e^{-t_1 \sqrt{T}} a_R(\cdot, x_2)(y_1)| \]

\[ \leq \left( \frac{\mathcal{I}_I(t)}{t_1} \right)^2 |y_1|^2 |y_1|^2 \mathcal{L}(-t_1 \sqrt{T}) (L^0 \otimes L^1) b_R(\cdot, x_2)(y_1)| \]

\[ \leq C \left( \frac{\mathcal{I}_I(t)}{t_1} \right)^2 \int_{\mathbb{R}^n} \frac{t_1}{(t_1 + |y_1 - z_1||^{n+1}} \mathcal{L}(-t_1 \sqrt{T}) (L^0 \otimes L^1) b_R(\cdot, x_2)(z_1) \, dz_1. \]
Note that
\[ x_1 \not\in 100I', |x_1 - y_1| < t_1, \ell(I) \leq t_1, z_1 \in I. \]
We can obtain the estimate
\[ t_1 + |y_1 - z_1| \geq |x_1 - x_I|/2 \]
and deduce that
\[ |e^{-t_1 \sqrt{r} a_R(\cdot, x_2)(y_1)}| \leq C \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \|\ell(I)^{-2}(L^0 \otimes L^1)b_R(\cdot, x_2)\|_{L^1(\mathbb{R}^n)}. \]  

(4.4)

It follows from (4.4) and Hölder’s inequality that
\[
E_{12} \leq C \sum_{R \in \Omega(\Omega)} |J|^{1/2}|J|^{1/2} \int_{x_1 \not\in 100I'} \frac{\ell(I)}{|x_1 - x_I|^{n+1}} d x_1 \\
\times \|\ell(I)^{-2}(L^0 \otimes L^1)b_R\|_{L^2(\mathbb{R}^n)} \\
\leq C \sum_{R \in \Omega(\Omega)} |J|^{1/2} |J|^{-1}\|\ell(I)^{-2}(L^0 \otimes L^1)b_R\|_{L^2(\mathbb{R}^n)} \\
\leq C \left( \sum_{R \in \Omega(\Omega)} |J|^{-1}\|\ell(I)^{-2}(L^0 \otimes L^1)b_R\|_{L^2(\mathbb{R}^n)} \right)^{1/2} \left( \sum_{R \in \Omega(\Omega)} \ell(I)^{-4}\|\ell(I)^{-2}(L^0 \otimes L^1)b_R\|_{L^2(\mathbb{R}^n)} \right)^{1/2} \\
\leq C
\]
which, together with the estimate of \( E_{11} \), gives us that \( E_1 \leq C \).

Consider the term \( E_2 \). We first estimate the maximal function \( (a_R)^*_L \). Now
\[
(a_R)^*_L(x) = \sup_{|y_1 - x_2| < t_2 \atop |y_1 - x_1| < t_1} \sup_{t_2 < \ell(J) \atop t_1 < \ell(I)} |e^{-t_1 \sqrt{r} a_R(\cdot, x_1)(y_1)}| \\
\leq \sup_{|y_1 - x_2| < t_2 \atop |y_1 - x_1| < t_1} \sup_{t_2 < \ell(J) \atop t_1 < \ell(I)} |e^{-t_1 \sqrt{r} a_R(\cdot, x_1)(y_1)}| \\
+ \sup_{|y_1 - x_2| < t_2 \atop |y_1 - x_1| < t_1} \sup_{t_2 < \ell(J) \atop t_1 < \ell(I)} |e^{-t_1 \sqrt{r} a_R(\cdot, x_1)(y_1)}| \\
+ \sup_{|y_1 - x_2| < t_2 \atop |y_1 - x_1| < t_1} \sup_{t_2 < \ell(J) \atop t_1 < \ell(I)} |e^{-t_1 \sqrt{r} a_R(\cdot, x_1)(y_1)}| \\
+ \sup_{|y_1 - x_2| < t_2 \atop |y_1 - x_1| < t_1} \sup_{t_2 < \ell(J) \atop t_1 < \ell(I)} |e^{-t_1 \sqrt{r} a_R(\cdot, x_1)(y_1)}| \\
=: E_{21} + E_{22} + E_{23} + E_{24}.
\]
We only need to estimate the term \( E_{22} \) since the estimates of the remaining terms are similar.
Applying (4.4) with $a_R(\cdot, x_2)$ replaced by $e^{-t_{2}^{\cdot}}a_R(\cdot, y_2)$, we obtain

$$
E_{22} \leq C \sup_{|y_2 - x_2| < t_2, t_2 \leq L} \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \left\| (\ell(I) - 2(L^0 \otimes e^{-t_{2}^{\cdot}}L^1) b_R(\cdot, y_2) \right\|_{L^1(\mathbb{R}^n)}
$$

$$
\leq C \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \sup_{|y_2 - x_2| < t_2, t_2 \leq L} \left\| (\ell(I) - 2L^0 \otimes e^{-t_{2}^{\cdot}}L^1) b_R(\cdot, y_2) \right\|_{L^1(\mathbb{R}^n)}.
$$

Applying (4.3) with $a_R(\cdot, x_2)$ replaced by $(\ell(I) - 2L^0 \otimes L^1) b_R(x_1, \cdot)$, together with Hölder’s inequality, we obtain

$$
E_{22} \leq C \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \left\| (\ell(I) - 2L^0 \otimes L^1) b_R \right\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)}
$$

$$
\leq C \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \left\| (\ell(I) - 2L^0 \otimes L^1) b_R \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}.
$$

A similar argument to that given for $E_{22}$ shows that

$$
(a_R)_{L}^*(x) \leq C \frac{\ell(J)}{|x_2 - x_J|^{n+1}} \frac{\ell(I)}{|x_1 - x_J|^{n+1}} |R|^{1/2}
$$

$$
\times \sum_{i,j=0} \ell(I)^{2i-2} \ell(J)^{2j-2} \left\| (L^i \otimes L^j) b_R \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}.
$$

Hence

$$
E_2 = \sum_{\text{Rem}(\Omega)} \int_{x_1 \not\in 10I} \int_{x_2 \not\in 10J} (a_R)_{L}^*(x) \, dx
$$

$$
\leq \sum_{\text{Rem}(\Omega)} \frac{\ell(I)/\ell(I')}{|R|^{1/2}} \sum_{i,j=0} \ell(I)^{2i-2} \ell(J)^{2j-2} \left\| (L^i \otimes L^j) b_R \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}
$$

$$
\leq C \sum_{\text{Rem}(\Omega)} |R|^{1/2} \gamma_1(R)^{-1} \sum_{i,j=0} \ell(I)^{2i-2} \ell(J)^{2j-2} \left\| (L^i \otimes L^j) b_R \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}.
$$

Applying Lemma 4.1 and the definition of product $(1, 2)$-atom, together with Hölder’s inequality, we obtain the estimate $E_2 \leq C$. We thus obtain the required estimate $\|a_R^*\|_{L^1(\mathbb{R}^{2n})} \leq C$ and can deduce that $f \in H_{L}^1(\mathbb{R}^n \times \mathbb{R}^n)$.

**Step 2.** Let

$$
f \in H_{L}^1(\mathbb{R}^n \times \mathbb{R}^n) \cap L^2(\mathbb{R}^n \times \mathbb{R}^n).
$$

We begin with a version of the Calderón reproducing formula. Let $\Psi(x) = x^4 \Phi(x)$ be the function in Lemma 2.1. Since $f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, applying the $L^2$-functional calculus gives us that

$$
f = c_{\Psi} \int_{0}^{\infty} \int_{0}^{\infty} \Psi_{t_1}(\sqrt{L})_{t_1} \sqrt{L} e^{-t_1^{\cdot}} \otimes \Psi_{t_2}(\sqrt{L})_{t_2} \sqrt{L} e^{-t_2^{\cdot}} \nabla f \frac{dt}{t_1 t_2}.
$$

(4.5)
For $k = 0, \pm 1, \ldots$ we set

$$
E_k = \{ x \mid f^*_k(x) > 2^k \},
$$

$$
E_k^* = \{ x \mid M_k \chi_{E_k}(x) > 2^{-(2n+1)} \}
$$

and

$$
E_k^{**} = \{ x \mid M_k \chi_{E_k}(x) > (4n)^{-n} \}.
$$

Then

$$
E_k \subseteq E_k^* \subseteq E_k^{**} \quad \text{and} \quad |E_k^{**}| \leq C |E_k^*| \leq C' |E_k|.
$$

We define $T_k := \hat{E}_k^{**} \setminus \hat{E}_{k+1}^{**}$ and apply formula (4.5) to write

$$
f = \sum_{k \in \mathbb{Z}} \lambda_k a_k
$$

where $\lambda_k = 2^k|E_k^*|$ and

$$
a_k = \lambda_k^{-1} \text{cyl} \int_0^\infty \int_0^\infty \Psi_{t_1}(\sqrt{L}) \Psi_{t_2}(\sqrt{L}) (\chi_T t_1 \sqrt{L} e^{-t_1} \otimes t_2 \sqrt{L} e^{-t_2}) f \frac{dt}{t_1 t_2}.
$$

It is clear that

$$
\sum_k |\lambda_k| \leq C \sum_k 2^k |E_k^*| \leq C \| f_L \|_{L^1(\mathbb{R}^n)}.
$$

We claim that for each $k \in \mathbb{Z}$ the term $a_k$ is a product $(1, 2)$-atom associated with the open set $E_k^{**}$ for some constant $C$.

Let us prove the claim. First, it follows by Lemma 2.1 that the integral kernel $K_{\Psi, (\sqrt{L})}$ of the operator $\Psi, (\sqrt{L})$ has its support contained in

$$
\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x_1 - y_1| \leq t_1, |x_2 - y_2| \leq t_2 \}.
$$

This, together with the definition of $T_k$, shows that $\text{supp } a_k \subseteq E_k^{**}$. Second, for any dyadic rectangle $R = I \times J$ of $\mathbb{R}^n \times \mathbb{R}^n$, we define

$$
R^+ = \left\{ (y_1, y_2, t_1, t_2) \mid y_1 \in I, y_2 \in J, \frac{\ell(I)}{2} < t_1 \leq \ell(I), \frac{\ell(J)}{2} < t_2 \leq \ell(J) \right\}.
$$

It can be verified that if $T_k \cap R^+ \neq \emptyset$, then $R \subseteq E_k^{**}$. Applying the definition of $R^+$, we obtain $T_k = \bigcup_R (T_k \cap R^+)$ where the $R$ are all dyadic rectangles of $\mathbb{R}^n \times \mathbb{R}^n$. We can further decompose $a_k$ as follows:

$$
a_k = \sum_{R \in m(E_k^{**})} \sum_{R \subseteq \bar{R}} \lambda_k^{-1}
\times \int_0^\infty \int_0^\infty \Psi_{t_1}(\sqrt{L}) \Psi_{t_2}(\sqrt{L}) (\chi_T t_1 \sqrt{L} e^{-t_1} \otimes t_2 \sqrt{L} e^{-t_2}) f \frac{dt}{t_1 t_2}
\Rightarrow \sum_{R \in m(E_k^{**})} a_{k, \bar{R}} = \sum_{R \in m(E_k^{**})} (L \otimes L) b_{k, \bar{R}},
$$

where $m(E_k^{**})$ denotes the set of all maximal dyadic rectangles of $E_k^{**}$.\[17\] An atomic decomposition for Hardy spaces 141
By Lemma 2.1, if $i, j = 0, 1$, then

$$\text{supp}((L^i \otimes L^j)b_{k,R}) \subseteq 2\hat{R}.$$ 

To continue, for each $\hat{R}$ we consider some $h \in L^2(\hat{R})$ such that $\|h\|_{L^2(\hat{R})} = 1$. Then for every $k \in \mathbb{Z}$, we have

$$\|a_{k,\hat{R}}\|_{L^2} = \sup_{\|h\|_{L^2} \leq 1} |\langle a_{k,\hat{R}}, h \rangle| \leq C 2^{-k}|E^*_k|^{-1} \times \left( \sum_{R \subseteq \hat{R}} \int_{T_k \cap R^*} |t_1 \nabla_X e^{-t_1\sqrt{\xi}} \otimes t_2 \nabla_X e^{-t_2\sqrt{\xi}} f(y_1, y_2)|^2 \frac{dy dt}{t_1 t_2} \right)^{1/2}.$$ 

In order to verify that

$$\sum_{R \in \mathcal{E}^*_k} \|a_{k,\hat{R}}\|^2 \leq C |E^*_k|^{-1},$$

it is enough to prove that

$$\int_{T_k} |t_1 \nabla_X e^{-t_1\sqrt{\xi}} \otimes t_2 \nabla_X e^{-t_2\sqrt{\xi}} f(y_1, y_2)|^2 \frac{dy dt}{t_1 t_2} \leq C 2^{2k}|E^*_k|. \quad (4.6)$$

We now prove inequality (4.6). Let $\varphi \in C_0^\infty$ be the function in Lemma 4.2 and set $F_k = E^*_k \setminus E_{k+1}$. The argument given to prove formula (3.3) shows that

$$|(\varphi_{i_1} \otimes \varphi_{i_2}) \ast \chi f(y)| \geq C$$

for all $(y, t) \in T_k$. This, together with Lemma 4.2, gives us that

$$\int_{T_k} |t_1 \nabla_X e^{-t_1\sqrt{\xi}} \otimes t_2 \nabla_X e^{-t_2\sqrt{\xi}} f(y_1, y_2)|^2 \frac{dy dt}{t_1 t_2} \leq \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |t_1 \nabla_X e^{-t_1\sqrt{\xi}} \otimes t_2 \nabla_X e^{-t_2\sqrt{\xi}} f(y_1, y_2)|^2 |(\varphi_{i_1} \otimes \varphi_{i_2}) \ast \chi f(y)|^2 \frac{dy dt}{t_1 t_2}$$

$$\leq \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} |e^{-t_1\sqrt{\xi}} \otimes e^{-t_2\sqrt{\xi}} f(y_1, y_2)|^2 |(\varphi_{i_1}^{(1)} \otimes \varphi_{i_2}^{(2)}) \ast \chi f(y_1, y_2)|^2 \frac{dy dt}{t_1 t_2}$$

$$+ \int_{\mathbb{R}^{n+1} \times \mathbb{R}^n} |e^{-t_1\sqrt{\xi}} f(y_1, x_2)^2 |\varphi_{i_1}^{(1)} \ast \chi f(y_1, x_2)|^2 \frac{dy_1 dt_1}{t_1} \ dx_2$$

$$+ \int_{\mathbb{R}^{n+1} \times \mathbb{R}^n} |e^{-t_2\sqrt{\xi}} f(x_1, y_2)^2 |\varphi_{i_2}^{(2)} \ast \chi f(x_1, y_2)|^2 \frac{dy_2 dt_2}{t_2} \ dx_1$$

$$+ \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x_1, x_2)^2 |\chi f(x_1, x_2)|^2 \ dx_1 \ dx_2 =: I_1 + I_2 + I_3 + I_4.$$ 

In order to estimate the term $I_1$, we note that if

$$(\varphi_{i_1}^{(1)} \otimes \varphi_{i_2}^{(2)}) \ast \chi f(y_1, y_2) \neq 0,$$
then $F_k \cap B_{y,t} \neq \emptyset$. Moreover, there exists

$$x^0 = (x_1^0, x_2^0) \in B_{y,t} \cap E_k^* \cap E_{k+1}$$

and we have

$$|e^{-t_1 \sqrt{L}} \otimes e^{-t_2 \sqrt{L}} f(y_1, y_2)| \leq f_L^*(x^0) \leq 2^{k+1}$$

which gives us that $I_1 \leq C2^{2k} |E_k^*|$. We similarly have

$$I_2 + I_3 \leq C2^{2k} |E_k^*|.$$

We now obtain an estimate for the term $I_4$. It follows from the inequality $f(x_1, x_2) \leq f_L^*(x_1, x_2)$ that

$$I_4 \leq C2^{2k} |E_k^*|.$$

The desired estimate (4.6) follows easily and then

$$\sum_{\hat{R} \in \mathcal{M}(E_k^{**})} \|a_{k, \hat{R}}\|^2_{L^2(\mathbb{R}^{2n})} \leq C |E_k^{**}|^{-1}.$$

A similar argument to the one given above shows that

$$\sum_{\hat{R} \in \mathcal{M}(E_k^{**})} \sum_{i,j=0}^{1} \ell(I)^{4i-4} \ell(J)^{4j-4} \|\ell(L_j \otimes L_j) b_{k, \hat{R}}\|^2_{L^2(\mathbb{R}^{2n})} \leq C |E_k^{**}|^{-1}.$$

We have shown that for every $k \in \mathbb{Z}$ the expression $C^{-1}a_k$ is a product $(1, 2)$-atom associated with the open set $E_k^{**}$ for some constant $C$. This shows that $f$ has an decomposition of the form given in (1.3). The proof of Theorem 1.2 is complete. \qed

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**References**


LIANG SONG, Department of Mathematics, Sun Yat-sen (Zhongshan) University, Guangzhou 510275, PR China
e-mail: songl@mail.sysu.edu.cn

CHAOQIANG TAN, Department of Mathematics, Shantou University, Shantou Guangdong 515063, PR China
e-mail: cqtan@stu.edu.cn

LIXIN YAN, Department of Mathematics, Sun Yat-sen (Zhongshan) University, Guangzhou 510275, PR China
e-mail: mcsylyx@mail.sysu.edu.cn