A NOTE ON WEIGHTED NORM INEQUALITIES FOR FRACTIONAL MAXIMAL OPERATORS WITH NON-DOUBLING MEASURES

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Abstract. Let \( \mu \) be a non-negative Borel measure on \( \mathbb{R}^d \) which only satisfies some growth condition, we study two-weight norm inequalities for fractional maximal functions associated to such \( \mu \). A necessary and sufficient condition for the maximal operator to be bounded from \( L^p(v) \) into weak \( L^q(u) \) (\( 1 \leq p \leq q < \infty \)) is given. Furthermore, by using certain Orlicz norm, a strong type inequality is obtained.

1. INTRODUCTION

Let \( \mu \) be a non-negative “n-dimensional” Borel measure on \( \mathbb{R}^d \) which only satisfies the following growth condition: there exists \( n \in (0, d] \) such that

\[
\mu(Q) \leq \ell(Q)^n
\]

for any cube \( Q \subset \mathbb{R}^d \), where \( \ell(Q) \) stands for the side length of \( Q \). Throughout this paper, by a cube \( Q \subset \mathbb{R}^d \), we mean a closed cube whose sides are parallel to the coordinate axes and we shall always denote the side length as above. For \( \lambda > 0 \) and any cube \( Q \), \( \lambda Q \) is a cube concentric as \( Q \) and with \( \ell(\lambda Q) = \lambda \ell(Q) \). Moreover, \( Q(x, r) \) will be the cube centered at \( x \) with side length \( r \).

The classical theory of harmonic analysis for maximal functions and singular integrals on \( (\mathbb{R}^d, \mu) \) has been developed under the assumption that the underlying measure \( \mu \) satisfies the doubling property, i.e., there exists a constant \( C > 0 \) such that for \( x \in \mathbb{R}^d \) and \( r > 0 \),

\[
\mu(B(x, 2r)) \leq C \mu(B(x, r)),
\]

where \( B(x, r) \) stands for the open ball centered at \( x \) with radius \( r \) (see [1, 2, 4, 7, 12, 15]). However, it seems that this

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The doubling condition can be removed and many classical results of Calderón-Zygmund theory have been proved to continue to hold (see [5, 6, 8-11, 17-19]). The motivation for developing the analysis on non-homogeneous spaces and some examples of non-doubling measures can be found in [20]. For a complete account of this topic the reader is referred to [3] (Chapter 5, pp. 137-147).

For $0 \leq \alpha < 1$, define the non-centered fractional maximal function

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{\mu(5Q)^{1-\alpha}} \int_Q |f(y)| \, d\mu(y),$$

where the supremum is taken over all cubes that contain $x$. The purpose of the paper is to consider two-weight norm inequalities for the maximal function $M_\alpha$. We shall investigate that for which pairs of weights, $M_\alpha$ satisfies a weak or a strong type inequality. A weight $w$ will be a nonnegative and locally integrable function. For any measurable set $E$, we shall write $w(E) = \int_E w \, d\mu$ and $L^p(w) = L^p(\mu \circ \chi_E^{-1})$ for $0 < p < \infty$. If $1 \leq p \leq \infty$, as usual, $p'$ will be the exponent conjugate to $p$, that is, the one satisfying $p' = p/(p - 1)$.

García-Cuerva and Martell in [5] introduced the following radical fractional maximal functions: for $0 \leq \alpha < n$,

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{\ell(Q)^{n-\alpha}} \int_Q |f(y)| \, d\mu(y),$$

where the measure $\mu$ just satisfies the growth condition (1.1). As we can see in many papers (e.g. [13] and [16]), it is more natural to define the fractional maximal operators as showing in (1.2), which can be seen from the following example in the case of $\mathbb{R}$.

**Example 1.** Given a non-doubling measure $d\mu = \chi_{[0,1]} \, dx$, i.e. $\mu(Q) = \int_Q \chi_{[0,1]} \, dx$, for $Q \subset \mathbb{R}$. Obviously, the measure $\mu$ satisfies the growth condition (1.1), for $0 < n \leq 1$. In fact, let $\ell(Q) = r \geq 0$. When $r \geq 1$, then $\mu(Q) \leq 1 \leq r^n$. When $0 < r < 1$, then $\mu(Q) \leq r \leq r^n$. So, for any $n \in (0, 1]$, the measure $\mu$ satisfies the growth condition. As a conclusion, there are infinite maximal functions for a same measure $\mu$ according to the definition of $M_\alpha$ defined in (1.3). This fact causes difficulties in studying properties of the measure space. However, the maximal function $M_\alpha$ defined in (1.2) is unique in some sense.

**Definition 1.1.** Let $1 \leq p \leq q < \infty$ and $0 \leq \alpha < 1$. We shall say that the pair of weights $(u, v) \in A_{p,q}^\alpha$, if for every cube $Q$

(i) $\frac{1}{\mu(5Q)^{1-\alpha}} \left( \int_Q u(x) \, d\mu(x) \right)^{\frac{1}{p'}} \left( \int_Q v(x)^{1-p'} \, d\mu(x) \right)^{\frac{1}{p}} \leq C$, when $1 < p < \infty$.

(ii) $\frac{1}{\mu(5Q)^{1-\alpha}} \left( \int_Q u(x) \, d\mu(x) \right)^{\frac{1}{p'}} \leq C v(x)$, for $\mu$-almost every $x \in Q$, when $p = 1$. 
Here and afterward, $C$ denotes a constant independent of functions, whose value may differ from line to line.

**Remark 1.1.** In Definition 1.1, we are implicitly assuming that $u, v^{1-p'} \in L^1_{\text{loc}}(\mu)$ and so $u < \infty, \ v > 0$ $\mu$-almost everywhere.

The main results of the paper can be stated as follows. Theorem 1.1 concerns with the problem of finding pairs of weights such that the maximal operator $M_\alpha$ satisfies a weak type inequality; Theorem 1.2 characterizes those pairs of weights for which $M_\alpha$ satisfies a strong type inequality, which can be achieved by certain Orlize norm localized in cubes. Their proofs are given respectively in Sections 2 and 3.

**Theorem 1.1.** Let $1 \leq p \leq q < \infty, 0 \leq \alpha < 1$ and $0 < \lambda < \infty, u$ and $v$ are two weights. Then the maximal operator $f \mapsto M_\alpha f$ is of weak-type $(L^p(v), L^q(u))$, i.e.,

$$u(\{x \in \mathbb{R}^d : M_\alpha f(x) > \lambda\}) \leq \frac{C}{\lambda^q} \left( \int_{\mathbb{R}^d} |f(x)|^p v(x) \, d\mu(x) \right)^{\frac{q}{p}}$$

if and only if the pair of weights $(u, v) \in A^\alpha_{p,q}$.

**Theorem 1.2.** Let $1 < p < q < \infty$ and $0 \leq \alpha < 1$. Let $(u, v)$ be a pair of weights such that for every cube $Q$

$$\ell(Q)^{\alpha(1 - \frac{1}{p})} \mu(Q)^{\alpha - 1} u(3Q)^{\frac{1}{q}} v^{-\frac{1}{p}} \|\Phi_Q\| \leq C,$$

where $\Phi$ is a Young function whose complementary function $\Phi \in B_p$. Then

$$\left( \int_{\mathbb{R}^d} (M_\alpha f(x))^q u(x) \, d\mu(x) \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^d} |f(x)|^p v(x) \, d\mu(x) \right)^{\frac{1}{p}}$$

for $f \in L^p(v)$ which is bounded with compact support.

The definitions of Young function $\Phi$, norm $\|\cdot\|_{\Phi,Q}$, and $B_p$ condition are given in Section 3.

2. PROOF OF THEOREM 1.1

To prove Theorem 1.1 we need the following lemma which provides an equivalence for the pair of weights in $A^\alpha_{p,q}$.

**Lemma 2.1.** Let $1 \leq p \leq q < \infty$ and $0 \leq \alpha < 1$. The pair of weights $(u, v) \in A^\alpha_{p,q}$ if and only if for every cube $Q$ and every $f \geq 0$,

$$\left( \frac{1}{\mu(5Q)^{1-\alpha}} \int_Q f(x) \, d\mu(x) \right)^q u(Q) \leq C \left( \int_Q f(x)^p v(x) \, d\mu(x) \right)^{\frac{q}{p}}.$$
**Proof.** When \( p = 1 \), for \((u, v) \in A^1_{\alpha, q}\),

\[
\left(\frac{1}{\mu(5Q)^{1-\alpha}} \int_Q f(x) \, d\mu(x)\right)^q u(Q)
= \left(\int_Q f(x) \frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(y) \, d\mu(y)\right)^{\frac{1}{q}} \, d\mu(x)\right)^q
\leq C \left(\int_Q f(x) v(x) \, d\mu(x)\right)^q.
\]

When \( 1 < p < \infty \), by Hölder’s inequality, we obtain

\[
\left(\frac{1}{\mu(5Q)^{1-\alpha}} \int_Q f(x) \, d\mu(x)\right)^q u(Q)
\leq \frac{1}{\mu(5Q)^{(1-\alpha)q}} \left(\int_Q f(x)^p v(x) \, d\mu(x)\right)^{\frac{q}{p}} \left(\int_Q v(x)^{1-p'} \, d\mu(x)\right)^{\frac{q}{p'}} \int_Q u(x) \, d\mu(x)
= \left(\int_Q f(x)^p v(x) \, d\mu(x)\right)^{\frac{q}{p}} \left(\frac{1}{\mu(5Q)^{1-\alpha}} \int_Q u(y) \, d\mu(y)\right)^{\frac{1}{q}} \left(\int_Q v(x)^{1-p'} \, d\mu(x)\right)^{\frac{1}{p'}}
\leq C \left(\int_Q f(x)^p v(x) \, d\mu(x)\right)^{\frac{q}{p}}.
\]

To prove the converse, for any \( S \subset Q \), apply (2.1) to \( f_{\chi_S} \),

\[
\left(\frac{1}{\mu(5Q)^{1-\alpha}} \int_S f(x) \, d\mu(x)\right)^q u(Q) \leq C \left(\int_S f(x)^p v(x) \, d\mu(x)\right)^{\frac{q}{p}}.
\]

Take \( f \equiv 1 \), (2.2) gives

\[
\left(\frac{1}{\mu(5Q)^{1-\alpha}} \int_S f(x) \, d\mu(x)\right)^q u(Q) \leq C \left(\int_S f(x)^p v(x) \, d\mu(x)\right)^{\frac{q}{p}}.
\]

(2.3)

\[
\left(\frac{\mu(S)}{\mu(5Q)^{1-\alpha}}\right)^q u(Q) \leq C v(S)^{\frac{q}{p}}.
\]

From the inequality above it follows that \( u \in L^1_{\text{loc}}(\mu) \) (unless \( v = \infty \) \( \mu \)-almost everywhere) and that \( v > 0 \) \( \mu \)-almost everywhere (unless \( u = 0 \) \( \mu \)-almost everywhere).

Now, we are going to show that \((u, v) \in A^\alpha_{p, q}\).

For \( p = 1 \), note that (2.3) can be written as

\[
\frac{1}{\mu(5Q)^{1-\alpha}} \left(\int_Q u(x) \, d\mu(x)\right)^{\frac{1}{q}} \leq C \frac{v(S)}{\mu(S)}, \quad \text{for any } S \subset Q \text{ with } \mu(S) > 0.
\]

Fix \( Q \) and consider

\[
a > \text{ess inf}_{Q} v = \inf \{t > 0 : \mu(S_t) > 0\},
\]

...
where \( S_t = \{ x \in Q : v(x) < t \} \subset Q \). Then \( \mu(S_a) > 0 \), and
\[
\frac{1}{\mu(5Q)^{1-\alpha}} \left( \int_Q u(x) \, d\mu(x) \right)^{ \frac{1}{q} } \leq \frac{C}{\mu(S_a)} \int_{S_a} v(x) \, d\mu(x) \leq Ca.
\]

Let \( a \to \text{ess inf}_Q v \), we get
\[
\frac{1}{\mu(5Q)^{1-\alpha}} \left( \int_Q u(x) \, d\mu(x) \right)^{ \frac{1}{q} } \leq C \text{ ess inf}_Q v \leq Cv(x) \quad \text{ for } \mu\text{-a.e. } x \in Q.
\]
That is \((u, v) \in A^a_{p, q}\).

For \( 1 < p < \infty \), take \( f(x) = f(x)^p v(x) \), that is \( f(x) = v(x)^{1-p'} \). Fix \( Q \) and define
\[ S_j = \{ x \in Q : v(x) > \frac{1}{j} \}, \quad j = 1, 2, \ldots. \]
Then, \( f \) is bounded in every \( S_j \) and \( \int_{S_j} f \, d\mu < \infty \) (fix \( j \) and \( Q \)). Applying (2.2) for \( S = S_j \) and \( f(x) = v(x)^{1-p'} \) gives
\[
\left( \frac{1}{\mu(5Q)^{1-\alpha}} \int_{S_j} v(x)^{1-p'} \, d\mu(x) \right)^{ \frac{q}{p} } \leq C \left( \frac{1}{\mu(S_j)^{1-\alpha}} \int_{S_j} v(x)^{1-p'} \, d\mu(x) \right)^{ \frac{q}{p} }.
\]
So
\[
\frac{1}{\mu(5Q)^{1-\alpha}} \left( \int_Q u(x) \, d\mu(x) \right)^{ \frac{1}{q} } \left( \int_{S_j} v(x)^{1-p'} \, d\mu(x) \right)^{ \frac{q}{p} } \leq C.
\]
Since \( S_1 \subset S_2 \subset \cdots \subset S_j \cdots \), and \( \bigcup_j S_j = \{ x \in Q : v(x) > 0 \} = Q \). Let \( j \to \infty \), we get
\[
\frac{1}{\mu(5Q)^{1-\alpha}} \left( \int_Q u(x) \, d\mu(x) \right)^{ \frac{1}{q} } \left( \int_Q v(x)^{1-p'} \, d\mu(x) \right)^{ \frac{q}{p} } \leq C.
\]
That is \((u, v) \in A^a_{p, q}\). The lemma is proved. \end{proof}

Proof of Theorem 1.1. Suppose that (1.4) holds. For \( f \geq 0 \) and a cube \( Q \), Take \( \lambda \) with
\[
0 < \lambda < m_{\alpha, Q}(f) =: \frac{1}{\mu(5Q)^{1-\alpha}} \int_Q f(x) \, d\mu(x).
\]
Note that
\[
m_{\alpha, Q}(f) \leq M_\alpha(f \chi_Q)(x), \quad x \in Q,
\]
we have \( Q \subset \{ x \in \mathbb{R}^d : M_{\alpha}(f \chi_Q)(x) > \lambda \} \). Using (1.4),

\[
    u(Q) \leq u \left( \{ x \in \mathbb{R}^d : M_{\alpha}(f \chi_Q)(x) > \lambda \} \right) \leq \frac{C}{\lambda^q} \left( \int_Q |f(x)|^p v(x) \, d\mu(x) \right)^{\frac{1}{p}}.
\]

That is,

\[
    \lambda^q u(Q) \leq C \left( \int_Q |f(x)|^p v(x) \, d\mu(x) \right)^{\frac{1}{p}} \quad \text{for} \quad 0 < \lambda < m_{\alpha,Q}(f).
\]

Let \( \lambda \to m_{\alpha,Q}(f) \), then

\[
    (m_{\alpha,Q}(f))^q u(Q) \leq C \left( \int_Q |f(x)|^p v(x) \, d\mu(x) \right)^{\frac{1}{p}}.
\]

Hence \((u, v) \in A_{p,q}^\alpha\) by Lemma 2.1.

In proving the converse, we still invoke Lemma 2.1. For fixed \( \lambda > 0 \) and \( A > 0 \) large enough, set

\[
    E_{A}^\lambda = \{ x \in \mathbb{R}^d : M_{\alpha} f(x) > \lambda, |x| \leq A \}.
\]

Then, for any \( x \in E_{A}^\lambda \), there is a cube \( Q_x \) containing \( x \) such that

\[
    \frac{1}{\mu(5Q_x)^{1-\alpha}} \int_{Q_x} |f(y)| \, d\mu(y) > \lambda.
\]

By Besicovitch covering lemma, there exists a countable collection of quasi-disjoint cubes \( \{Q_j\}_j = \{Q(x_j, r_j)\}_j \) with \( x_j \in E_{A}^\lambda \) and \( r_j = r_{x_j} \), such that

\[
    E_{A}^\lambda \subset \bigcup_j Q_j, \quad \chi_{Q_j}(x) \leq B(d),
\]

where \( B(d) > 1 \) is usually called the Besicovitch constant. Recall the equivalence of (2.1) and \((u, v) \in A_{p,q}^\alpha\) along with (2.4), we have

\[
    u(E_{A}^\lambda) \leq \sum_j u(Q_j)
\]

\[
    \leq C \sum_j \left( \frac{1}{\mu(5Q_j)^{1-\alpha}} \int_{Q_j} |f(x)| \, d\mu(x) \right)^{-q} \left( \int_{Q_j} |f(x)|^p v(x) \, d\mu(x) \right)^{\frac{q}{p}}
\]

\[
    \leq C \sum_j \left( \int_{Q_j} |f(x)|^p v(x) \, d\mu(x) \right)^{\frac{1}{p}}
\]

\[
    \leq C \left( \sum_j \int_{Q_j} |f(x)|^p v(x) \, d\mu(x) \right)^{\frac{1}{p}}
\]

\[
    \leq C \left( \int_{\mathbb{R}^d} |f(x)|^p v(x) \, d\mu(x) \right)^{\frac{1}{p}},
\]
where the constant $C$ is independent of $A$. Letting $A \to \infty$, the Monotone Convergence Theorem leads to the desired weak-type inequality (1.4). This completes the proof of Theorem 1.1.

3. PROOF OF THEOREM 1.2

We first recall some definitions and basic facts related to Orlicz spaces (see [14]). Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function, i.e. a continuous, convex, increasing function with $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. The Orlicz space $L_\Phi(\mathbb{R}^d, \mu)$ consists of measurable functions $f$ such that
\[
\int_{\mathbb{R}^d} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, d\mu(x) < \infty, \quad \text{for some } \lambda > 0.
\]
The space $L_\Phi(\mathbb{R}^d, \mu)$ is a Banach space if it is endowed with the Luxemburg norm
\[
\|f\|_\Phi = \inf \{ \lambda > 0 : \frac{1}{\ell(Q)^n} \int_Q \Phi \left( \frac{|f(x)|}{\lambda} \right) \, d\mu(x) \leq 1 \}.
\]
Each Young function $\Phi$ has associated to it a complementary Young function $\bar{\Phi}$ which satisfies
\[
t \leq \Phi^{-1}(t) \bar{\Phi}^{-1}(t) \leq 2t, \quad \text{for all } t > 0.
\]
Let us define the following localized version of the Orlicz norm: for every cube $Q \subset \mathbb{R}^d$ with $\mu(Q) < \infty$
\[
\|f\|_{\Phi, Q} = \inf \{ \lambda > 0 : \frac{1}{\ell(Q)^n} \int_Q \Phi \left( \frac{|f(x)|}{\lambda} \right) \, d\mu(x) \leq 1 \}.
\]
By the properties of Young function, it is easy to check that $\| \cdot \|_{\Phi, Q}$ provides a norm over $L_\Phi(Q, \mu)$: the space of all measurable functions on $Q$ such that there exists $\lambda > 0$ for which
\[
\frac{1}{\ell(Q)^n} \int_Q \Phi \left( \frac{|f(x)|}{\lambda} \right) \, d\mu(x) < \infty.
\]
From [14], the following generalization of Hölder inequality holds
\[
\frac{1}{\ell(Q)^n} \int_Q |f(x)h(x)| \, d\mu(x) \leq C \|f\|_{\Phi, Q} \|h\|_{\bar{\Phi}, Q}.
\]
For $1 < p < \infty$, it is said that a Young function $\Phi$ satisfies $B_p$ condition ($\Phi \in B_P$), if
\[
\int_c^\infty \Phi(t) \, \frac{dt}{t^{1/p}} < \infty, \quad \text{for some } c > 0.
\]
For the proof of Theorem 1.2, we also need the following lemma.
Lemma 3.1. Let $0 \leq \alpha < 1$ and $f \geq 0$ be a locally integrable function. If
\[
\frac{1}{\mu(5Q)^{1-\alpha}} \int_Q f(y) \, d\mu(y) > t,
\]
for some cube $Q$ and $t > 0$. Then there exists a dyadic cube $P$ such that $P \subset 5Q$, $Q \subset 3P$ and
\[
\frac{1}{\mu(P)^{1-\alpha}} \int_P f(y) \, d\mu(y) > 2^{-d}t.
\]

Proof. Take $s \in \mathbb{Z}$ such that $2^{s-1} \leq \ell(Q) < 2^s$, there exist dyadic cubes $P_1, \ldots, P_j, \ldots, P_N$ $(1 \leq N \leq 2^d)$ which intersect $Q$ with the side length $2^s$, and $P_j \subset 5Q, Q \subset 3P_j$, $j \in [1, N]$, and for at least one of them, say $P$, the following estimate holds
\[
\int_P f(y) \, d\mu(y) > t\mu(5Q)^{1-\alpha}.
\]
Otherwise,
\[
\int_Q f(y) \, d\mu(y) \leq \sum_{j=1}^N \int_{P_j} f(y) \, d\mu(y) \\
\leq \sum_{j=1}^N t\mu(5Q)^{1-\alpha} \\
\leq t\mu(5Q)^{1-\alpha},
\]
which contradicts the hypothesis. Note that $P \subset 5Q$, we get
\[
\frac{1}{\mu(P)^{1-\alpha}} \int_P f(y) \, d\mu(y) > \frac{t\mu(5Q)^{1-\alpha}}{2^d} \geq 2^{-d}t.
\]
Lemma 3.1 is proved.

Proof of Theorem 1.2. We will employ the ideas in [5] with modifications. Let
\[
M_R^\alpha f(x) = \sup_{Q \ni x, \ell(Q) < R} \frac{1}{\mu(5Q)^{1-\alpha}} \int_Q |f(y)| \, d\mu(y).
\]
First, we will prove (1.6) with $M^\alpha f(x)$ replaced by $M_R^\alpha f(x)$. Decompose $\mathbb{R}^d$ in the following way
\[
\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}} \Omega_k, \quad \text{with} \quad \Omega_k = \{x \in \mathbb{R}^d : 2^k < M_R^\alpha f(x) \leq 2^{k+1}\}. 
\]
Then, for $k \in \mathbb{Z}$ and $x \in \Omega_k$, there exists a cube $Q^k_x$ containing $x$, such that
\[
\frac{1}{\mu(5Q^k_x)^{1-\alpha}} \int_{Q^k_x} |f(y)| \, d\mu(y) > 2^k.
\]
By Lemma 3.1, there exists a dyadic cube $P^k_x$ with $Q^k_x \subset 3P^k_x$, $P^k_x \subset 5Q^k_x$, such that
\[
(3.1) \quad \frac{1}{\mu(P^k_x)^{1-\alpha}} \int_{P^k_x} |f(y)| \, d\mu(y) > 2^{-d} 2^k.
\]
From the definition of $M^R_\alpha$, every cube $Q^k_x$ has bounded size, and so has every dyadic cube $P^k_x$. Then, for fixed $k$, there is a sub-collection of maximal dyadic cubes $\{P^k_j\}_{j \in \mathbb{N}}$ satisfying that every $Q^k_x$ is contained in $3P^k_j$ for some $j$. We can write
\[
\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}} \Omega_k \cup \bigcup_{j,k} 3P^k_j.
\]
Next, notice that these $P^k_j$ are the maximal cubes for which (3.1) holds. However, for $P^k_{i+1}$,
\[
\frac{1}{\mu(P^k_{i+1})^{1-\alpha}} \int_{P^k_{i+1}} |f(y)| \, d\mu(y) > 2^{-d} 2^{k+1} > 2^{-d} 2^k.
\]
Hence, for every $i$, there exists $j = j(i, k)$ such that $P^k_{i+1} \subset P^k_j$. In short, for a fixed $k$, the cubes $\{P^k_j\}_{j \in \mathbb{N}, k \in \mathbb{Z}}$ are pairwise disjoint and they are strictly nested for different $k$.

Let $K > 0$ be a large integer, $\Lambda_K = \{(j, k) \in \mathbb{N} \times \mathbb{Z} : |k| \leq K\}$. Then
\[
\mathcal{I}_K = \int_{\bigcup_{k=-K}^K \Omega_k} (M^R_\alpha f(x))^q u(x) \, d\mu(x)
\]
\[
\leq \sum_{(j, k) \in \Lambda_K} \int_{3P^k_j} (M^R_\alpha f(x))^q u(x) \, d\mu(x)
\]
\[
\leq \sum_{(j, k) \in \Lambda_K} u(3P^k_j) \left(2^{k+1}\right)^q
\]
\[
\leq C \sum_{(j, k) \in \Lambda_K} u(3P^k_j) \left(\frac{1}{\mu(P^k_j)^{1-\alpha}} \int_{P^k_j} |f(y)| \, d\mu(y)\right)^q
\]
\[
\leq C \sum_{(j, k) \in \Lambda_K} u(3P^k_j) \left(P^k_j\right)^{(a-1)q} \left(P^k_j\right)^{nq} \left\|fu^\frac{1}{p}\right\|_{\Phi, P^k_j} \left\|u^{-\frac{1}{q}}\right\|_{\Phi, P^k_j},
\]
where we used the generalization of Hölder inequality. From (1.5), we have
\[
(3.2) \quad \mathcal{I}_K \leq C \sum_{(j, k) \in \Lambda_K} \ell_p \left(P^k_j\right)^{\frac{2n}{p}} \left\|fu^\frac{1}{p}\right\|_{\Phi, P^k_j}^q =: C \int_\gamma \left(T_K(fu^\frac{1}{p})\right)^q d\nu,
\]
\[ \mathcal{Y} = \mathbb{N} \times \mathbb{Z}, \nu \text{ is a measure on } \mathcal{Y} \text{ given by } \nu(j, k) = \ell(P_j^{k_j})^{\frac{n}{p}} \text{ and the operator } T_K \text{ is defined by} \]

\[ T_Kg(j, k) = \|g\|_{\bar{\Phi}, P_j^k} \chi_{\Lambda_K}(j, k). \]

We next to show that \( T_K : L^p(\mathbb{R}^d, \mu) \rightarrow L^q(\mathcal{Y}, \nu) \) is bounded independently of \( K \). For a bounded function \( g \) with compact support and \( \lambda \geq 0 \), take

\[ F_{\lambda} = \{(j, k) \in \mathcal{Y} : T_Kg(j, k) > \lambda\} = \{(j, k) \in \Lambda_K : \|g\|_{\bar{\Phi}, P_j^k} > \lambda\}. \]

Without loss of generality, we may assume that \( \bar{\Phi}(1) = 1 \). Write

\[ g(x) = g(x)\chi_{\{x : |g(x)| > \frac{\lambda}{2}\}}(x) + g(x)\chi_{\{x : |g(x)| \leq \frac{\lambda}{2}\}}(x) = g_1(x) + g_2(x). \]

Then

\[ \frac{1}{\ell(Q)^n} \int_Q \Phi \left( \frac{2|g_2(x)|}{\lambda} \right) d\mu(x) \leq \frac{1}{\ell(Q)^n} \int_Q \Phi(1) d\mu(x) = \frac{\mu(Q)}{\ell(Q)^n} \leq 1. \]

So, for every \( Q \), \( \|g_2\|_{\bar{\Phi}, Q} \leq \lambda/2 \). For \( Q \) such that \( \|g\|_{\bar{\Phi}, Q} > \lambda \), the triangle inequality gives

\[ \lambda < \|g\|_{\bar{\Phi}, Q} = \|g_1 + g_2\|_{\bar{\Phi}, Q} \leq \|g_1\|_{\bar{\Phi}, Q} + \|g_2\|_{\bar{\Phi}, Q} \leq \|g_1\|_{\bar{\Phi}, Q} + \frac{\lambda}{2}, \]

i.e. \( \|g_1\|_{\bar{\Phi}, Q} > \lambda/2 \). Thus

\[ F_{\lambda} = \{(j, k) \in \Lambda_K : \|g\|_{\bar{\Phi}, P_j^k} > \lambda\} \subset \{(j, k) \in \Lambda_K : \|g_1\|_{\bar{\Phi}, P_j^k} > \frac{\lambda}{2}\} = \tilde{F}_{\lambda}. \]

If \((j, k) \in \tilde{F}_{\lambda}\), then

\[ \frac{1}{\ell(P_j^{k_j})^n} \int_{P_j^{k_j}} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu > 1. \]

So

\[ \ell(P_j^{k_j})^n < \int_{P_j^{k_j}} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu. \]

It follows that

\[ \nu(F_{\lambda}) \leq \nu(\tilde{F}_{\lambda}) = \sum_{(j, k) \in \tilde{F}_{\lambda}} \ell(P_j^{k_j})^{\frac{n}{p}} \leq \sum_{(j, k) \in \tilde{F}_{\lambda}} \ell(P_j^{k_j})^{n \left( \frac{n}{p} - 1 \right)} \int_{P_j^{k_j}} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu. \]
Notice that the dyadic cubes \( \{ P^k_j \}_{j \in \mathbb{N}, k \in \mathbb{Z}} \) have bounded size, so we can extract a maximal sub-collection \( \{ P_i \} \) to get
\[
\nu(F_\lambda) \leq \sum_i \sum_{m=0}^{\infty} \sum_{P^k_j \subset P_i} \ell(P^k_j)^n \left( \frac{2|g_1|}{\lambda} \right)^{\frac{n}{p}-1} \int_{P^k_j} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu
\]
\[
= \sum_i \sum_{m=0}^{\infty} \sum_{P^k_j \subset P_i} \ell(P^k_j)^n \left( \frac{2|g_1|}{\lambda} \right)^{\frac{n}{p}-1} \int_{P^k_j} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu
\]
\[
\leq \sum_i \ell(P_i)^n \left( \frac{2|g_1|}{\lambda} \right)^{\frac{n}{p}-1} \int_{P_i} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu \sum_{m=0}^{\infty} 2^{-mn} \left( \frac{2|g_1|}{\lambda} \right)^{\frac{n}{p}-1}
\]
\[
\leq C \sum_i \ell(P_i)^n \left( \frac{2|g_1|}{\lambda} \right)^{\frac{n}{p}-1} \int_{P_i} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu.
\]
Furthermore, for every \( i \), there exists \((j, k) \in \tilde{F}_\lambda\) such that \( P_i = P^k_j \). Consequently
\[
\nu(F_\lambda) \leq C \sum_i \left( \int_{P_i} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu \right)^{\frac{n}{p}}
\]
\[
\leq C \left( \sum_i \int_{P_i} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu \right)^{\frac{n}{p}}
\]
\[
\leq C \left( \int_{\mathbb{R}^d} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu \right)^{\frac{n}{p}}
\]
\[
= C \left( \int_{\{x \in \mathbb{R}^d : |g(x)| > \frac{\lambda}{2} \}} \Phi \left( \frac{2|g_1|}{\lambda} \right) d\mu \right)^{\frac{n}{p}}.
\]
Now we can verify
\[
(3.3) \quad \int_{\mathcal{Y}} T_K g(j, k)^q \, d\nu(j, k) \leq C \left( \int_{\mathbb{R}^d} |g(x)|^p \, d\mu(x) \right)^{\frac{q}{p}}
\]
where the constant \( C \) is independent of \( K \).

To see this, we begin by using the formula of distribution function
\[
\int_{\mathcal{Y}} T_K g(j, k)^q \, d\nu(j, k) = q \int_0^{\infty} \lambda^q \nu \left( \{ (j, k) \in \mathcal{Y} : T_K g(j, k) > \lambda \} \right) \frac{d\lambda}{\lambda}
\]
\[
= q \int_0^{\infty} \lambda^q \nu(F_\lambda) \frac{d\lambda}{\lambda}.
\]
By the calculation of $\nu(F_\lambda)$, we have
\[
\int_{Y} T_K g(j, k)^q \, d\nu(j, k) \leq C \sum_{s \in \mathbb{Z}} \int_{2^s}^{2^{s+1}} \lambda^p \int_{\{x \in \mathbb{R}^d : |g(x)| > \frac{2s}{\lambda} \}} \Phi \left( \frac{2|g(x)|}{\lambda^2} \right) \, d\mu(x) \frac{2^p}{\lambda} \, d\lambda
\]
\[
\leq C \sum_{s \in \mathbb{Z}} \left( (2^{s+1})^p \int_{\{x \in \mathbb{R}^d : |g(x)| > \frac{2s}{\lambda} \}} \Phi \left( \frac{2|g(x)|}{\lambda^2} \right) \, d\mu(x) \right) \frac{2^p}{\lambda} \, d\lambda.
\]
Note that
\[
\sum_{s \in \mathbb{Z}} (2^{s+1})^p \int_{\{x \in \mathbb{R}^d : |g(x)| > \frac{2s}{\lambda} \}} \Phi \left( \frac{2|g(x)|}{\lambda^2} \right) \, d\mu(x) \frac{2^p}{\lambda} \, d\lambda
\]
\[
= \sum_{s \in \mathbb{Z}} \int_{2^s}^{2^{s+1}} (2^{s+1})^p \int_{\{x \in \mathbb{R}^d : |g(x)| > \frac{2s}{\lambda} \}} \Phi \left( \frac{2|g(x)|}{\lambda^2} \right) \, d\mu(x) \frac{2^p}{\lambda} \, d\lambda
\]
\[
\leq \int_0^\infty (2\lambda)^p \int_{\{x \in \mathbb{R}^d : |g(x)| > \frac{2s}{\lambda} \}} \Phi \left( \frac{4|g(x)|}{\lambda^2} \right) \, d\mu(x) \frac{2d\lambda}{\lambda}
\]
\[
= 2^{p+1} \int_{\mathbb{R}^d} \int_0^\infty \Phi \left( \frac{4|g(x)|}{\lambda^2} \right) \, d\lambda \frac{dt}{\lambda} \, d\mu(x)
\]
\[
= 2^{p+1} \int_{\mathbb{R}^d} \int_1^\infty \left( \frac{4|g(x)|}{t} \right)^p \Phi(t) \, dt \, d\mu(x)
\]
\[
= 2^{2p+1} \left( \int_{\mathbb{R}^d} |g(x)|^p \, d\mu(x) \right) \left( \int_1^\infty \Phi(t) \, dt \right) \frac{2^p}{\lambda}
\]
\[
\leq C \int_{\mathbb{R}^d} |g(x)|^p \, d\mu(x),
\]
where we used the fact that $\Phi \in B_p$. We have proved the estimate (3.3) with the constant $C$ independent of $K$. Combining (3.2) and (3.3) leads to that
\[
I_K \leq C \int_{Y} T_K \left( f v^{\frac{1}{2}} \right)^q \, d\nu \leq C \left( \int_{\mathbb{R}^d} |f|^p v \, d\mu \right)^{\frac{q}{p}}.
\]
The uniformity in $K$ of this estimate and the Monotone Convergence Theorem imply that
\[
\int_{\mathbb{R}^d} (M^R f(x))^q u(x) \, d\mu(x) \leq C \left( \int_{\mathbb{R}^d} |f|^p v(x) \, d\mu(x) \right)^{\frac{q}{p}},
\]
where the constant $C$ is independent of $R$. Letting $R \to \infty$, using the Monotone Convergence Theorem again, we complete the proof of Theorem 1.2.
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REFERENCES


