WEIGHTED COMPOSITION OPERATORS ON WEIGHTED DIRICHLET SPACES

Xiaosong LIU (刘小松)  Zengjian LOU (秦建敏)
Department of Mathematics, Shantou University, Guangdong Shantou 515063, China
E-mail: gdxsliu@163.com; zjlou@stu.edu.cn

Abstract  We characterize the boundedness and compactness of weighted composition operators on weighted Dirichlet spaces in terms of Nevanlinna counting functions and Carleson measure.

Key words  weighted composition operator; weighted Dirichlet space; s-Carleson measure; Nevanlinna counting function

2010 MR Subject Classification  47B33; 30H05

1 Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane and $\partial \mathbb{D}$ its boundary. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. Given $\psi, \varphi \in H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, the weighted composition operator $W_{\varphi, \psi}$ is defined by

$$W_{\varphi, \psi}f(z) = \psi(z)f(\varphi(z)) \quad (1.1)$$

for $f \in H(\mathbb{D})$. When $\psi \equiv 1$, we just have the composition operator $C_\varphi : f \to f \circ \varphi, f \in H(\mathbb{D})$.

For $-1 < \alpha < \infty$, the weighted Dirichlet space $D_\alpha^p (0 < p < \infty)$ consists of all $f \in H(\mathbb{D})$ with

$$\|f\|^p_{D_\alpha^p} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \left( \log \frac{1}{|z|} \right)^\alpha dA(z) < \infty.$$  

The weighted Bergman space $A_\alpha^p (0 < p < \infty)$ is the space of $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A_\alpha^p} = \int_{\mathbb{D}} |f(z)|^p \left( \log \frac{1}{|z|} \right)^\alpha dA(z) < \infty.$$  

Carleson measure for analytic function spaces is a very important notion in the theory of complex function spaces. For $s > 0$, a positive Borel measure $\mu$ on $\mathbb{D}$ is called an $s$-Carleson measure if there is a positive constant $C$ such that

$$\sup_{I \subset \partial \mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty,$$

Received April 3, 2012. This work was supported by NNSF of China (11171203, 11201280), New Teacher’s Fund for Doctor Stations, Ministry of Education (20114402120003) and NSF of Guangdong Province (1015150310100025, S201101004511, S2011040004131).

Corresponding author: Zengjian LOU.
\(\mu\) is called a vanishing \(s\)-Carleson measure, if

\[
\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^s} = 0,
\]

where \(|I|\) denotes the arc length of \(I\) and \(S(I)\) denotes the Carleson square base on \(I\):

\[
S(I)\left\{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I \right\}.
\]

From [1], \(\mu\) is an \(s\)-Carleson measure if and only if

\[
\sup_{a \in \mathbb{D}} \int_{D} \left( \frac{1 - |a|^2}{|1 - az|^2} \right)^s d\mu(z) < \infty,
\]

\(\mu\) is a vanishing \(s\)-Carleson measure if and only if

\[
\lim_{|a| \to 1} \int_{D} \left( \frac{1 - |a|^2}{|1 - az|^2} \right)^s d\mu(z) = 0.
\]

Given a Banach space \(X \subset H(\mathbb{D})\). A non-negative measure \(\mu\) on \(\mathbb{D}\) is called an \((X,p)\)-Carleson measure if there is a constant \(C > 0\) such that

\[
\int_{D} |f(z)|^p d\mu(z) \leq C\|f\|_X^p,
\]

\(\mu\) is called a vanishing \((X,p)\)-Carleson measure if for any sequence \(\{f_n\}\) in \(X\) with its norm \(\|f_n\|_X \leq 1\) and such that \(f_n(z) \to 0\) uniformly on compact subsets of \(\mathbb{D}\), we have that

\[
\lim_{n \to \infty} \int_{D} |f_n(z)|^p d\mu(z) = 0.
\]

The boundedness and compactness of weighted composition operators \(W_{\psi,\varphi}\) on Dirichlet space was studied in [8] and [6]. On Hardy spaces and Bergman spaces readers are referred to, for example, [4], [5] and [3]. In this paper, we study the problem on weighted Dirichlet spaces, different from [8] and [6]. We characterize the boundedness and compactness of \(W_{\varphi,\psi}\) by using Nevanlinna counting functions and Carleson measure.

Throughout the paper, we will denote \(C\) by a positive constant depending only on \(\alpha\) and \(C\) may differ from line to line. For any set \(E\) and any integrable function \(f\) respect to a measure \(\mu\), we will denote the integral mean of \(f\) over \(E\) by

\[
\oint_E f d\mu = \frac{1}{\mu(E)} \int_E f d\mu.
\]

We say two positive functions \(F\) and \(G\) are equivalent, denoted by \(F \approx G\), if there are two positive constants \(c\) and \(C\) such that \(cF \leq G \leq CF\).

\section{Boundedness of \(W_{\varphi,\psi}\) on \(D_{\alpha}\)}

In this section, we characterize the boundedness of \(W_{\psi,\varphi}\) on \(D_{\alpha}\) in terms of Nevanlinna counting functions. To prove the result, we need the following lemmas. Lemmas 2.1, 2.2 and 2.3 can be found respectively in [7, Theorem 2.25], [6, Lemma 2.2] and [3, Theorems 1 and 3] when \(p = 2\).
Lemma 2.1  Let $\alpha > -1$. If $\mu$ is a positive Borel measure on $\mathbb{D}$, then $\mu$ is a $(\mathcal{A}_2^2, 2)$-Carleson measure if and only if $\mu$ is a $(\alpha + 2)$-Carleson measure.

Suppose that $-1 < \alpha < \infty$. Let $\varphi \in \mathcal{D}_\alpha$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, $\psi \in \mathcal{D}_\alpha$ and $\psi' \in \mathcal{A}_2^2$. We define the measures $\mu_{\varphi,\psi^\prime,2}$ and $\nu_{\varphi,\psi^\prime,2}$ on $\mathbb{D}$ by

$$\mu_{\varphi,\psi^\prime,2}(E) = \int_{\varphi^{-1}(E)} |\psi(z)\psi'(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha dA(z)$$

and

$$\nu_{\varphi,\psi^\prime,2}(E) = \int_{\varphi^{-1}(E)} |\psi'(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha dA(z).$$

Lemma 2.2  Suppose $-1 < \alpha < \infty$. Let $\varphi \in \mathcal{D}_\alpha$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi \in \mathcal{D}_\alpha$. Then

$$\int_{\mathbb{D}} g(z) d\mu_{\varphi,\psi^\prime,2}(z) = \int_{\mathbb{D}} |\psi(z)\psi'(z)|^2 (g \circ \varphi)(z) \left( \log \frac{1}{|z|} \right)^\alpha dA(z)$$

and

$$\int_{\mathbb{D}} g(z) d\nu_{\varphi,\psi^\prime,2}(z) = \int_{\mathbb{D}} |\psi'(z)|^2 (g \circ \varphi)(z) \left( \log \frac{1}{|z|} \right)^\alpha dA(z),$$

where $g$ is an arbitrary measurable positive function on $\mathbb{D}$.

Lemma 2.3  Suppose $-1 < \alpha < \infty$. Let $\mu$ is a positive Borel measure on $\mathbb{D}$. Then $W_{\varphi,\psi}$ is bounded on $\mathcal{A}_2^2$ if and only if $\mu_{\varphi,\psi,2}$ is a $(\alpha + 2)$-Carleson measure. $W_{\varphi,\psi}$ is compact on $\mathcal{A}_2^2$ if and only if $\mu_{\varphi,\psi,2}$ is a vanishing $(\alpha + 2)$-Carleson measure.

Lemma 2.4  Suppose $-1 < \alpha < \infty$. Let $\varphi \in \mathcal{D}_\alpha$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, $\psi \in \mathcal{D}_\alpha$ and let $\nu_{\varphi,\psi^\prime,2}$ be a $(\mathcal{A}_\alpha, 2)$-Carleson measure. Then $W_{\varphi,\psi}$ is bounded on $\mathcal{D}_\alpha$ if and only if $W_{\varphi,\psi^\prime}$ is bounded on $\mathcal{A}_2^2$.

Proof  Since $\nu_{\varphi,\psi^\prime,2}$ is a $(\mathcal{A}_\alpha, 2)$-Carleson measure, we have

$$\|W_{\varphi,\psi^\prime}f\|_{\mathcal{A}_2^2}^2 = \int_{\mathbb{D}} |\psi'(z)|^2 \|f(\varphi(z))\|^2 \left( \log \frac{1}{|z|} \right)^\alpha dA(z)$$

$$= \int_{\mathbb{D}} |f(z)|^2 d\nu_{\varphi,\psi^\prime,2}(z)$$

$$\leq C\|f\|_{\mathcal{D}_\alpha}^2,$$

for all $f \in \mathcal{D}_\alpha$. Suppose that $W_{\varphi,\psi}$ is bounded on $\mathcal{D}_\alpha$, that is

$$\|W_{\varphi,\psi}f\|_{\mathcal{D}_\alpha} \leq C\|f\|_{\mathcal{D}_\alpha}, \quad f \in \mathcal{D}_\alpha.$$  \hspace{1cm} (2.2)

Let $f \in \mathcal{A}_2^2$, then there exists $g \in \mathcal{D}_\alpha$ such that $g'(z) = f(z)$ and $g(0) = 0$. Then (2.1) and (2.2) give

$$\|W_{\varphi,\psi^\prime}f\|_{\mathcal{A}_2^2}^2 = \|\psi'(f \circ \varphi) + \psi'(g \circ \varphi) - \psi'(g \circ \varphi)|\|_{\mathcal{A}_2^2}^2$$

$$\leq 2\|\psi'(g \circ \varphi)|\|_{\mathcal{A}_2^2}^2 + \|\psi'(g \circ \varphi)|\|_{\mathcal{A}_2^2}^2$$

$$= 2\|W_{\varphi,\psi^\prime}g\|_{\mathcal{D}_\alpha}^2 + \|W_{\varphi,\psi^\prime}g\|_{\mathcal{A}_2^2}^2$$

$$\leq C\|g\|_{\mathcal{D}_\alpha}^2 = C\|f\|_{\mathcal{A}_2^2}^2.$$

So, $W_{\varphi,\psi^\prime}$ is bounded on $\mathcal{A}_2^2$. 

---

No. 4  X.S. LIU & Z.J. LOU: WEIGHTED COMPOSITION OPERATORS ON WEIGHTED
Conversely, suppose $W_{\varphi,\psi'}$ is bounded on $A^2_\alpha$. Let $f \in D_\alpha$ such that $f(0) = 0$, using (2.1) again, we have

$$
\|W_{\varphi,\psi}f\|_{D_\alpha}^2 = \|(\psi(f \circ \varphi))'\|_{A^2_\alpha}^2
= \|\psi'(f \circ \varphi) + \psi'(f \circ \varphi)\|_{A^2_\alpha}^2
\leq 2(\|W_{\varphi,\psi'}f'\|_{A^2_\alpha}^2 + \|W_{\varphi,\psi'}f\|_{A^2_\alpha}^2)
\leq C(\|f'\|_{A^2_\alpha}^2 + \|f\|_{D_\alpha}^2)
= C\|f\|_{D_\alpha}^2.
$$

Thus, $W_{\varphi,\psi}$ is bounded on $D_\alpha$.

**Theorem 2.5** Suppose $0 < \alpha < \infty$. Let $\varphi \in D_\alpha$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, $\psi \in D_\alpha$ and let the measure $\nu_{\varphi,\psi,2}$ be a $(D_\alpha,2)$-Carleson measure. Then the following assertions are equivalent:

(i) $W_{\varphi,\psi}$ is bounded on $D_\alpha$;

(ii) $\sup_{\zeta \in D} \int_{D(\zeta,r)} \sum_{\psi(z) = u} |\psi(z)|^2 \left( \frac{1}{|z|} \right)^\alpha dA(u) = O(r^\alpha)$,

where $D(\zeta,r) = \{z \in \mathbb{D} : |z - \zeta| < r, \zeta \in \partial \mathbb{D}\}$;

(iii) $\mu_{\varphi,\psi,2}$ is a $(\alpha + 2)$-Carleson measure.

**Proof** (i) $\Rightarrow$ (ii): Assume that $W_{\varphi,\psi}$ is bounded on $D_\alpha$. Fix a Carleson disc $D(\zeta,r)$ and $\omega = (1 - r)\zeta$, where $\zeta \in \partial \mathbb{D}$ and $0 < r < 1$. Consider

$$
\kappa_\omega(z) = \left( \frac{1 - |\omega|^2}{1 - \overline{z}\omega} \right)^{\alpha/2}, \quad \omega \in \mathbb{D}.
$$

It is easy to check that $\kappa_\omega(z) \in D_\alpha$. We have

$$
\|W_{\varphi,\psi}\kappa_\omega\|_{D_\alpha}^2 = |\psi(0)|^2 |\kappa_\omega(\varphi(0))|^2 + \int_{D} |(\psi(z)\kappa_\omega(\varphi(z)))'|^2 \left( \frac{1}{|z|} \right)^\alpha dA(z)
= |\psi(0)|^2 |\kappa_\omega(\varphi(0))|^2
+ \int_{D} |\psi'(z)\kappa_\omega(\varphi(z)) + \psi(z)(\kappa_\omega(\varphi(z)))'|^2 \left( \frac{1}{|z|} \right)^\alpha dA(z)
\leq C\|W_{\varphi,\psi}\|_{D_\alpha}^2.
$$

(2.3)

Next, we need to divide the last integral in (2.3) to three parts:

$$
T_1 = \int_{D} |\psi'(z)\kappa_\omega(\varphi(z))|^2 \left( \frac{1}{|z|} \right)^\alpha dA(z),
$$

$$
T_2 = \int_{D} |\psi(z)\kappa_\omega'(\varphi(z))\varphi'(z)|^2 \left( \frac{1}{|z|} \right)^\alpha dA(z)
$$

and

$$
T_3 = 2\operatorname{Re} \int_{D} \psi'(z)\kappa_\omega(\varphi(z))\overline{\psi(\zeta)}\kappa_\omega'(\varphi(z))\overline{\varphi'(z)} \left( \frac{1}{|z|} \right)^\alpha dA(z).
$$

The Cauchy-Schwarz inequality implies $T_3^2 \leq 4T_1T_2$. 
Since $\nu_{\varphi,\varphi',2}$ is a $(D_\alpha,2)$-Carleson measure, we obtain
\[
T_1 = \int_D |\varphi'(z)\kappa_\omega(\varphi(z))|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(z)
\]
\[
= \int_D |\kappa_\omega(u)|^2 d\nu_{\varphi,\varphi',2}(u) 
\leq C.
\]
By the elementary inequality $\|x_2\| \leq \|x_1 + x_2\| + \|x_1\|$ for norms, we see that $T_2 \leq C$. Changing the variables in $T_2$, we have
\[
\int_D |\kappa_\omega'(u)|^2 \sum_{\varphi(z)=u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u) \leq C.
\]
Therefore, the inequality above holds when $D$ is replaced by the Carleson disc $D(\zeta, r)$, that is
\[
\int_{D(\zeta, r)} \frac{(1 - |\omega|^2)^\alpha}{|1 - \bar{\omega}u|^2} \sum_{\varphi(z)=u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u) \leq C.
\]
For $u \in D(\zeta, r)$, one has $|1 - \bar{\omega}u| \approx 1 - |\omega| = r$. Hence,
\[
\int_{D(\zeta, r)} \sum_{\varphi(z)=u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u) \leq Cr^\alpha.
\]
(ii) $\Rightarrow$ (iii): To prove $\mu_{\varphi,\varphi',2}$ is a $(\alpha + 2)$-Carleson measure. It is enough to show that
\[
\sup_{\omega \in D} \int_D \left( \frac{1 - |\omega|^2}{|1 - \omega z|^2} \right)^{\alpha+2} d\mu_{\varphi,\varphi',2}(z) < \infty.
\]
Let $\omega \in D$, $\zeta_\omega = \frac{1}{|\omega|}$ and $r_\omega = 1 - |\omega|$. For $n \geq 1$, consider the sets
\[
\Delta_n(\omega) = \{ u \in D : 2^{n-1}r_\omega \leq |\zeta_\omega - u| < 2^n r_\omega \}.
\]
Upon changing variable $u = \varphi(z)$, we deduce
\[
\int_D \left( \frac{1 - |\omega|^2}{|1 - \omega z|^2} \right)^{\alpha+2} d\mu_{\varphi,\varphi',2}(z)
\]
\[
= \int_D \left( \frac{1 - |\omega|^2}{|1 - \omega \varphi(z)|^2} \right)^{\alpha+2} \left| \psi(z)\varphi'(z) \right|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(z)
\]
\[
= \int_{\cup \Delta_n(\omega)} \left( \frac{1 - |\omega|^2}{|1 - \omega \varphi(z)|^2} \right)^{\alpha+2} \left| \psi(z)\varphi'(z) \right|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(z)
\]
\[
= \int_{\cup \Delta_n(\omega)} \left( \frac{1 - |\omega|^2}{|1 - \omega u|^2} \right)^{\alpha+2} \sum_{\varphi(z)=u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u)
\]
\[
= \sum_{n=1}^\infty \int_{\Delta_n(\omega)} \left( \frac{1 - |\omega|^2}{|1 - \omega u|^2} \right)^{\alpha+2} \sum_{\varphi(z)=u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u)
\]
\[
\approx \sum_{n=1}^\infty \frac{1}{(2^n r_\omega)^{\alpha+2}} \int_{\Delta_n(\omega)} \sum_{\varphi(z)=u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u), \tag{2.4}
\]
where in the last step we used the fact that for any \( u \in \Delta_n(\omega) \) one has \(|1 - \bar{\omega}u| \approx 2^n r_\omega\). Since \( \Delta_n(\omega) \subset D(\zeta_\omega, 2^n r_\omega) \), the last term in (2.4) is less than and equal to

\[
\sum_{n=1}^{\infty} \frac{1}{2^{n(1+\alpha)} r_\omega} \int_{D(\zeta_\omega, 2^n r_\omega)} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u).
\]

This along with the assumption (ii), we have

\[
\int_{\mathbb{D}} \left( \frac{1 - |\omega|^2}{|1 - \bar{\omega}z|^2} \right)^{\alpha + 2} \, d\mu_{\varphi, \psi, 2}(z) \leq C \sum_{n=1}^{\infty} \frac{1}{2^{n(2+\alpha)}},
\]

where the constant \( C \) is independent of \( \omega \). Hence \( \mu_{\varphi, \psi, 2} \) is a \((\alpha + 2)\)-Carleson measure.

(iii) \( \Rightarrow \) (i): Its proof follows from Lemmas 2.3 and 2.4. The proof of Theorem 2.5 is complete.

\[ \square \]

3 Compactness of \( W_{\varphi, \psi} \) on \( D_\alpha \)

In this section, we give a characterization of the compactness of \( W_{\psi, \varphi} \) on \( D_\alpha \). To prove the result, we need the following lemmas. Lemmas 3.1 and 3.2 can be obtained respectively by letting \( p = q = 2 \) in Lemma 2.9 and Theorem 2.10 in [6].

**Lemma 3.1** Suppose \(-1 < \alpha < \infty\). Let \( \varphi \in D_\alpha \) be such that \( \varphi(\mathbb{D}) \subset \mathbb{D} \). If \( \psi \in D_\alpha \) and \( W_{\varphi, \psi} \) is bounded on \( D_\alpha \), then \( W_{\varphi, \psi} \) is compact on \( D_\alpha \) if and only if \( \{f_n\} \subset D_\alpha \) with the norm (or seminorm) \( \|f_n\|_{D_\alpha} \leq 1 \) and \( f_n(z) \to 0 \) uniformly on compact subsets of \( D \) such that \( \|W_{\varphi, \psi} f_n\|_{D_\alpha} \to 0 \) as \( n \to \infty \).

**Lemma 3.2** Suppose \(-1 < \alpha < \infty\). Let \( \varphi \in D_\alpha \) be such that \( \varphi(\mathbb{D}) \subset \mathbb{D} \) and \( \psi \in A_\alpha^2 \). If \( \nu_{\varphi, \psi} \) is a vanishing \((D_\alpha, 2)\)-Carleson measure, then the operator \( W_{\varphi, \psi} \) is a bounded operator from \( D_\alpha \) to \( A_\alpha^2 \). Moreover, \( W_{\varphi, \psi} \) is a compact operator from \( D_\alpha \) to \( A_\alpha^2 \).

**Lemma 3.3** Suppose \(-1 < \alpha < \infty\). Let \( \varphi \in D_\alpha \) be such that \( \varphi(\mathbb{D}) \subset \mathbb{D} \) and \( \psi \in D_\alpha \). If \( \nu_{\varphi, \psi} \) is a vanishing \((D_\alpha, 2)\)-Carleson measure, then \( W_{\varphi, \psi} \) is compact on \( D_\alpha \) if and only if \( W_{\varphi, \psi, \varphi} \) is compact on \( A_\alpha^2 \).

**Proof** Suppose \( W_{\varphi, \psi} \) is compact on \( D_\alpha \). Let \( f_n \in A_\alpha^2 \) satisfy \( \|f_n\|_{A_\alpha^2} \leq 1 \) and \( f_n(z) \to 0 \) uniformly on compact subsets of \( D \) as \( n \to \infty \). Then there exist \( g_n \in D_\alpha \) such that \( g_n(0) = 0 \) and \( g_n \) converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \). We get

\[
\|W_{\varphi, \psi} f_n\|_{A_\alpha^2}^2 = \|\psi' (f_n \circ \varphi)\|_{A_\alpha^2}^2 = \|\psi' (f_n \circ \varphi) + \psi' (g_n \circ \varphi) - \psi' (g_n \circ \varphi)\|_{A_\alpha^2}^2 \\
\leq 2 \left( \|\psi' (g_n \circ \varphi)\|_{A_\alpha^2}^2 + \|\psi' (g_n \circ \varphi)\|_{A_\alpha^2}^2 \right) \\
= 2 \left( \|W_{\varphi, \varphi} g_n\|_{D_\alpha}^2 + \|W_{\varphi, \psi} g_n\|_{A_\alpha^2}^2 \right). \tag{3.1}
\]

Since \( W_{\varphi, \psi} \) is compact on \( D_\alpha \), then \( \|W_{\varphi, \psi} g_n\|_{D_\alpha} \to 0 \) as \( n \to \infty \) by Lemma 3.1. From Lemma 3.2, \( W_{\varphi, \psi} \) is compact from \( D_\alpha \) to \( A_\alpha^2 \), then \( \|W_{\varphi, \psi} g_n\|_{A_\alpha^2} \to 0 \) as \( n \to \infty \). Combining those with (3.1) implies that \( W_{\varphi, \psi, \varphi} \) is compact on \( A_\alpha^2 \).

Conversely, suppose \( W_{\varphi, \psi, \varphi} \) is compact on \( A_\alpha^2 \). Let \( f_n \in D_\alpha \) satisfy \( \|f_n\|_{D_\alpha} \leq 1 \) and \( f_n(z) \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \), without loss of generality, we assume
Take $g_n(z) = f_n(z)$, then $g_n \in A^2_\alpha$ with $\|g_n\|_{A^2_\alpha} \leq 1$ and $g_n$ converges to zero uniformly on compact subsets of $D$ as $n \to \infty$. Using Lemma 3.2 again, we have

$$\|W_{\varphi, \psi}f_n\|_{D_\alpha}^2 = \| (\varphi(f_n \circ \varphi))' \|^2_{A^2_\alpha} = \| \varphi'(f_n \circ \varphi) + \psi'(f_n \circ \varphi) \|^2_{A^2_\alpha} \leq 2 (\|W_{\varphi, \psi'}g_n\|^2_{A^2_\alpha} + \|W_{\varphi, \psi'}f_n\|^2_{A^2_\alpha}) \to 0 \text{ as } n \to \infty.$$ 

So, $W_{\varphi, \psi}$ is compact on $D_\alpha$. \hfill $\square$

**Theorem 3.4** Suppose $0 < \alpha < \infty$. Let $\varphi \in D_\alpha$ be such that $\varphi(D) \subseteq \mathbb{D}$, $\psi \in D_\alpha$. If $\nu_{\varphi, \psi', 2}$ is a vanishing $(D_\alpha, 2)$-Carleson measure, then the following assertions are equivalent:

(i) $W_{\varphi, \psi}$ is compact on $D_\alpha$;

(ii) $\sup_{\zeta \in \partial D} \int_{D(\zeta, r)} \sum_{\varphi(z) = u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha dA(u) = o(r^{\alpha})$ as $r \to 0$;

(iii) $\mu_{\varphi, \psi', 2}$ is a vanishing $(\alpha + 2)$-Carleson measure.

**Proof** The proof is a modification of that of Theorem 2.5, for the completeness of the paper, we give the sketch of it.

(i) $\Rightarrow$ (ii): Assume that $\nu_{\varphi, \psi', 2}$ is a vanishing $(D_\alpha, 2)$-Carleson measure and $W_{\psi, \varphi}$ is compact on $D_\alpha$. Since $\kappa_\varphi(z) = (\frac{1 - |z|^2}{1 - \psi^2(z)})^{\alpha/2} \to 0$ uniformly on compact subsets of $D$ as $|\varphi| \to 1$, then $\|W_{\varphi, \psi} \kappa_\varphi\|_{D_\alpha} \to 0$ as $|\varphi| \to 1$ by Lemma 3.1. That is

$$\lim_{|\varphi| \to 1} \int_D |(\psi(z) \kappa_\varphi(\varphi(z)))'|^2 \left( \log \frac{1}{|z|} \right)^\alpha dA(z) = 0. \quad (3.2)$$

Again, we divide the integral in (3.2) into three parts: $T_1$, $T_2$ and $T_3$ as in the proof of Theorem 2.5. Since $\nu_{\varphi, \psi', 2}$ is a vanishing $(D_\alpha, 2)$-Carleson measure, we obtain

$$\lim_{|\varphi| \to 1} T_1 = \lim_{|\varphi| \to 1} \int_D |\psi'(z) \kappa_\varphi(\varphi(z))|^2 \left( \log \frac{1}{|z|} \right)^\alpha dA(z) = \lim_{|\varphi| \to 1} \int_D |\kappa_\varphi(u)|^2 d\nu_{\psi, \psi', 2}(u) = 0.$$

Using $\|x_2\| \leq \|x_1 + x_2\| + \|x_1\|$ again, we see that $\lim_{|\varphi| \to 1} T_2 = 0$, i.e.,

$$\lim_{|\varphi| \to 1} \int_D |\psi(z) \kappa_\varphi'(\varphi(z)) \psi'(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha dA(z) = 0.$$

Replacing $D$ by the Carleson disc $D(\zeta, r)$ in the above equation, we get

$$\lim_{|\varphi| \to 1} \int_{D(\zeta, r)} \frac{(1 - |\varphi|^2)^\alpha}{|1 - \varphi|^2 + 2} \sum_{\varphi(z) = u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha dA(u) = 0.$$ 

Note that for $u \in D(\zeta, r)$, one has $|1 - \varphi u| \approx 1 - |\varphi| = r$. We have

$$\frac{1}{r^\alpha} \int_{D(\zeta, r)} \sum_{\varphi(z) = u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha dA(u) \to 0, \quad r \to 0. \quad (3.3)$$
Combining (3.3) with the proof of Theorem 2.5 yields

\[
\sup_{\zeta \in \partial D} \int_{D(\zeta, r)} \sum_{\varphi(z) = u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u) = o(r^\alpha) \text{ as } r \to 0.
\]

(ii) \(\Rightarrow\) (iii): As in the proof of Theorem 1. Let \(\omega \in \mathbb{D}, \frac{\omega}{|\omega|}, r_\omega = 1 - |\omega|\) and \(\Delta_n(\omega) = \{u \in \mathbb{D} : 2^n - 1 \leq |\omega - u| < 2^n r_\omega \} \ (n \geq 1)\). We have

\[
\lim_{|\omega| \to 1} \int_{\mathbb{D}} \left( \frac{1 - |\omega|^2}{|1 - \omega z|^2} \right)^{\alpha + 2} \mu_{\varphi, \psi \varphi'}(z) = \lim_{|\omega| \to 1} \int_{\cup \Delta_n(\omega)} \left( \frac{1 - |\omega|^2}{|1 - \omega \varphi(z)|^2} \right)^{\alpha + 2} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u)
\]

\[
= \lim_{|\omega| \to 1} \sum_{n=1}^{\infty} \int_{\Delta_n(\omega)} \left( \frac{1 - |\omega|^2}{|1 - \omega u|^2} \right)^{\alpha + 2} \sum_{\varphi(z) = u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u)
\]

\[
\approx \lim_{r_\omega \to 0} \sum_{n=1}^{\infty} \frac{1}{(2^n r_\omega)^{\alpha + 2}} \int_{\Delta_n(\omega)} \sum_{\varphi(z) = u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u)
\]

\[
= \lim_{r_\omega \to 0} \sum_{n=1}^{\infty} \frac{1}{2^{n(1 + \alpha)}(2^n r_\omega)^{\alpha}} \int_{D(\zeta, 2^n r_\omega)} \sum_{\varphi(z) = u} |\psi(z)|^2 \left( \log \frac{1}{|z|} \right)^\alpha \, dA(u)
\]

\[
= 0.
\]

We proved that \(\mu_{\varphi, \psi \varphi'}\) is a vanishing \((\alpha + 2)\)-Carleson measure.

(iii) \(\Rightarrow\) (i): Its proof follows from Lemmas 2.3 and 3.3. The proof of Theorem 3.4 is finished.

\[\square\]

References


