ISRAEL JOURNAL OF MATHEMATICS **227** (2018), 397–421
DOI: 10.1007/s11856-018-1738-5

ZERO DISTRIBUTION AND DIVISION RESULTS FOR EXPONENTIAL POLYNOMIALS

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** Supported by the discretionary budget (2016) of the President of the Open University of Japan.

† Supported by the JSPS KAKENHI Grant Numbers 16K05194 and 25400131.

†† Corresponding author. Supported by National Natural Science Foundation for the Youth of China (No. 11501402) and Shanxi Scholarship council of China (No. 2015-043).

Received December 22, 2016 and in revised form July 5, 2017

ABSTRACT

An exponential polynomial of order q is an entire function of the form

$$g(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)},$$

where the coefficients $P_j(z), Q_j(z)$ are polynomials in z such that

$$\max\{\deg(Q_j)\} = q.$$

It is known that the majority of the zeros of a given exponential polynomial are in domains surrounding finitely many critical rays. The shape of these domains is refined by showing that in many cases the domains can approach the critical rays asymptotically. Further, it is known that the zeros of an exponential polynomial are always of bounded multiplicity. A new sufficient condition for the majority of zeros to be simple is found. Finally, a division result for a quotient of two exponential polynomials is proved, generalizing a 1929 result by Ritt in the case $q = 1$ with constant coefficients. Ritt's result is closely related to Shapiro's conjecture that has remained open since 1958.

1. Introduction

In 1929 Ritt [12] proved the following division result for exponential sums.

Ritt's theorem: *Assume that $g(z)$ and $h(z)$ are exponential sums of the form*

$$g(z) = \sum_{j=1}^{m} a_j e^{\mu_j z} \quad \text{and} \quad h(z) = \sum_{j=1}^{n} b_j e^{\nu_j z},$$

where $a_j, \mu_j, b_j, \nu_j \in \mathbb{C}$. If $f(z) = g(z)/h(z)$ is entire, then f too is an exponential sum (or reduces to a constant).

An alternative proof for Ritt's theorem is due to Lax [7]. In 1960 Rahman [11] proved a generalization of Ritt's theorem in the case when the coefficients $a_j(z), b_j(z)$ are polynomials. In this generalization the coefficients of the quotient f are rational functions in such a way that the singularities are removable. Indeed, if $g(z) = e^z - 1$ and $h(z) = ze^z$, then $f(z) = g(z)/h(z)$ is an entire exponential polynomial but with rational coefficients.

In 1958, Shapiro [13] identified exponential sums as solutions of functional equations. His paper culminates in the following conjecture.

Shapiro's conjecture: *If two exponential sums have infinitely many zeros in common, they are both multiples of some third (entire transcendental) exponential sum.*

In the 1970's and thereafter the zeros of exponential sums were investigated by Moreno, van der Poorten, Tijdeman and Voorhoeve, to name a few. Finally, in 2014, D'Aquino–Macintyre–Terzo made some new progress by solving Shapiro's conjecture assuming Schanuel's (algebraic) conjecture [1]. Despite the considerable interest, Shapiro's conjecture remains unsolved in general.

Exponential sums are also related to the Riemann zeta function $\zeta(z)$ in terms of partial sums

$$\sum_{n\leq M}\frac{1}{n^z}=\sum_{n\leq M}e^{(-\log n)z}. \tag{1.1}$$

Apart from the negative even integers, the zeros of $\zeta(z)$ are located in the strip $0\leq \Re z\leq 1$. According to Riemann's conjecture, this strip should be replaceable with the line $\Re z=1/2$. In 1973, Moreno [10] proved that the partial sums of $\zeta(z)$ (summing over all prime numbers $n\leq M$ in (1.1)) have zeros near any vertical line in the strip $0\leq \Re z\leq 1$.

In addition to exponential sums we consider more general exponential polynomials of order q, which are entire functions of the form

$$g(z)=P_1(z)e^{Q_1(z)}+\cdots+P_k(z)e^{Q_k(z)},$$

where P_j's and Q_j's are polynomials in z such that $\max\{\deg(Q_j)\}=q$. Following Steinmetz [14], such functions can be written in the form

$$g(z)=H_1(z)e^{w_1z^q}+\cdots+H_m(z)e^{w_mz^q}, \tag{1.2}$$

where $w_1,\ldots,w_m$ $(m\leq k)$ are pairwise different leading coefficients of the polynomials $Q_j(z)$, and the functions $H_j(z)$ are either exponential polynomials of order $\leq q-1$ or ordinary polynomials in z. We use the convention that an exponential polynomial of order zero is an ordinary polynomial.

We make a quick remark that the exponential polynomials $e^{2\pi iz^2}-1$ and $e^{2\pi iz}-1$ share infinitely many zeros, namely the integers, and yet their ratio is not an exponential polynomial. Hence a higher order analogue of Shapiro's conjecture would have to involve more than just "infinitely many zeros".

This paper is organized as follows. Zero distribution of exponential polynomials is discussed in Section 2. Our results will strengthen (and also clarify to some extent) the corresponding findings in [14]. In particular, if the coefficients $H_j(z)$ in (1.2) are of sub-maximal growth, then the majority of zeros of g are in finitely many domains, which approach the critical rays of g asymptotically. We will also see that, for an arbitrary $\varepsilon>0$, the zeros and poles of a quotient of two exponential polynomials of order $q \geq 1$ are (q, ε)-separated in the sense of Goldberg–Ostrovskii [3, p. 262]. In Section 3 we find sufficient conditions for exponential polynomials such that the majority of their zeros are simple. In general, it is known that the zeros of any exponential polynomial are of finite multiplicity. In Section 4 we will generalize the Ritt–Lax–Rahman result to a quotient of exponential polynomials of arbitrary order q. The proofs of our three main results are postponed to Sections 5–7.

2. Zeros of exponential polynomials

We begin with a motivational example on the fixed points of the exponential.

Example 2.1: The zeros of the exponential polynomial $g(z) = e^z - z$ are precisely the fixed points of the exponential function. The zeros of g attain its maximum exponent of convergence $\lambda(g) = 1$ from the asymptotic relation $N(r, 1/g) \sim r/\pi$ that follows from [14, Satz 2]. Another way to see this is to apply a variant of the second main theorem in [5, Theorem 2.5] to the function e^z with targets $0, \infty, z$.

We give an alternative proof for the fact that e^z has infinitely many fixed points, including the asymptotic location of these points. If $e^\zeta = \zeta = x + iy$, then $e^{2x} = |e^\zeta|^2 = |\zeta|^2 = x^2 + y^2$. Thus the fixed points of e^z are precisely on the curves

$$y = \pm\sqrt{e^{2x} - x^2} \sim \pm e^x.$$

If $\zeta = re^{i\theta}$ is a fixed point of e^z, then

$$\zeta = r\cos\theta + ir\sin\theta = \log\zeta = \log r + i(\theta + 2n\pi), \quad n \in \mathbb{Z}.$$

From $r\cos\theta = \log r$ we can get

$$|\theta| = \arccos\left(\frac{\log r}{r}\right) \sim \frac{\pi}{2} - \frac{\log r}{r}, \quad r \to \infty.$$

Now $r \sim r\sin\theta = \theta + 2n\pi \sim 2n\pi$ as $n \to \infty$ (or equivalently $r \to \infty$). That is, if $\{\zeta_n\} = \{r_n e^{i\theta_n}\}$, $n \in \mathbb{Z}$, is the sequence of fixed points of e^z, then

$$r_{\pm n} \sim 2n\pi \quad \text{and} \quad \theta_{\pm n} \sim \pm\left(\frac{\pi}{2} - \frac{\log n}{2n\pi}\right), \quad n \to \infty.$$

In particular, there are no fixed points of e^z in the left half-plane.

If g is the exponential polynomial in (1.2), then its Phragmén–Lindelöf indicator function

$$h_g(\theta) = \limsup_{r\to\infty} r^{-q} \log|g(re^{i\theta})|$$

takes the form

$$h_g(\theta) = \max_j\{\Re(w_j e^{iq\theta})\}; \tag{2.1}$$

see [6, p. 993]. The indicator $h_g(\theta)$ is 2π-periodic and differentiable everywhere except for cusps θ^* at which

$$h_g(\theta^*) = \Re(w_j e^{iq\theta^*}) = \Re(w_k e^{iq\theta^*}) \tag{2.2}$$

for a pair of indices j, k such that $j \neq k$. Thus, at least two terms $\Re(w_k e^{iq\theta^*})$ coincide, and the coinciding terms are also equal to the indicator $h_g(\theta^*)$. We make two simple observations:

(1) There are at most finitely many cusps θ^* on any interval of length 2π.
(2) If θ^* is a cusp, then the points $\theta^* + \frac{2\pi j}{q}$, $j = 0, \ldots, q-1$, are also cusps by periodicity and by (2.1).

Example 2.2: If $g(z) = e^{(1+i)z} - e^z - e^{iz} + 1$, then

$$h_g(\theta) = \max\{0, \cos\theta, -\sin\theta, \cos\theta - \sin\theta\}.$$

At least two (in fact precisely two) of the terms $0, \cos\theta, -\sin\theta, \cos\theta - \sin\theta$ coincide for the angles $\theta \in \{0, \pi, \pm\pi/4, \pm\pi/2, \pm 3\pi/4\}$ and for no other angles. If $\theta \in \{\pm\pi/4, \pm 3\pi/4\}$, we see that the coinciding terms do not determine the indicator $h_g(\theta)$, whereas for $\theta \in \{0, \pi, \pm\pi/2\}$ they do. Therefore, the cusps of h_g are $\theta^* \in \{0, \pi, \pm\pi/2\}$. Writing $g(z) = (e^z - 1)(e^{iz} - 1)$, we see that all zeros of g are on the coordinate axes, which supports this finding.

Based on earlier papers by various authors in the first-order case (i.e., $q = 1$ in (1.2)), Steinmetz proves in [14] that most of the zeros of g are inside the logarithmic strips

$$\Lambda(\theta^*, c) = \left\{ re^{i\theta} : r > 1,\ |\theta - \theta^*| < c\frac{\log r}{r} \right\}, \tag{2.3}$$

where $c > 0$ is large enough. More precisely, let $N_\Lambda(r)$ denote the integrated counting function of the zeros of g in $|z| \leq r$ lying outside the union of domains $\Lambda(\theta^*, c)$. Then [14, Satz 4] states that

$$N_\Lambda(r) = O(r^{q-1} + \log r). \tag{2.4}$$

Thus we call the rays $\arg(z) = \theta^*$ the **critical directions** for g.

Remark 2.1: (1) For clarity, we note that Steinmetz defines critical directions in terms of the support function $k(\theta)$ of the convex hull $\mathrm{co}(W)$, where $W = \{\overline{w}_1, \ldots, \overline{w}_m\}$. However, by formula (1.106) in [8, p. 75], we may write

$$k(\theta) = \max_j \Re(w_j e^{i\theta}). \tag{2.5}$$

Thus $h_g(\theta) = k(q\theta)$ by (2.1), so that either of the functions $h_g(\theta)$ or $k(\theta)$ can be used to define the critical directions in an equivalent way.

(2) To get an idea on the shape of the domains $\Lambda(\theta^*, c)$, choose $\theta^* = 0$ for simplicity. On the boundary of $\Lambda(0, c)$, we have

$$re^{i\theta} = r\cos\left(c\frac{\log r}{r}\right) \pm ir\sin\left(c\frac{\log r}{r}\right) \sim r \pm ic\log r, \quad r \to \infty. \tag{2.6}$$

Hence $\Lambda(0, c)$ is asymptotically the domain between the curves $y = \pm c\log x$, $x \geq 1$. These curves do not approach asymptotically towards the critical ray $\arg(z) = 0$. In addition, Example 2.1 shows that the logarithmic strips in (2.3) have the correct order of magnitude when $\rho(g) = 1$. However, it is clear that for any $\varepsilon > 0$ we can find an $r_\varepsilon > 1$ such that $\Lambda(0, c) \setminus \{|z| < r_\varepsilon\}$ is strictly contained in the sector $|\arg(z)| < \varepsilon$. Thus, at least one of the critical directions is a Julia direction; see [14, Satz 2 & Satz 4].

Example 2.3: (1) As noted in [14], an exponential polynomial may have infinitely many zeros outside the aforementioned logarithmic strips. Indeed,

$$g_1(z) = (e^{2\pi i z^2} - 1)(e^{(1+i)\pi z} - 1)$$

has the indicator function

$$h_{g_1}(\theta) = \max\{0, -2\pi\sin(2\theta)\}$$

with cusps $\theta^* \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$. However, most of the zeros $z_n = n + ni$, $n \in \mathbb{Z}$, of g_1 are outside the logarithmic strips determined by these cusps. Note that $\{z_n\}$ has exponent of convergence equal to one, while $\rho(g_1) = 2$. This proves the sharpness of (2.4) in the case $q = 2$.

(2) The zeros of the exponential polynomial

$$g_2(z) = \sin z - z = \frac{1}{2i}(e^{iz} - e^{-iz}) - z$$

are precisely the fixed points of $\sin z$. By [14, Satz 2], we have the asymptotic relation $N(r, 1/g_2) \sim 2r/\pi$. Since $h_{g_2}(\theta) = \max\{0, \sin\theta, -\sin\theta\}$, the cusps are $\theta^* \in \{0, \pi\}$, so the fixed points should not be too far from the real axis. On the real axis there is precisely one fixed point, namely at the origin. The complex conjugate of a fixed point is also a fixed point. This follows from the Schwarz reflection principle, but a direct proof follows easily from the equations

$$(e^{-y} - e^{y})\cos x = -2y \quad \text{and} \quad (e^{-y} + e^{y})\sin x = 2x,$$

where $\zeta = x + iy$ satisfies $\sin\zeta = \zeta$. In particular, the zeros of g_2 are symmetrically on both sides of the critical rays $\arg(z) = 0, \pi$, not only on one side as in Example 2.1.

(3) Differing from Example 2.1, the zeros of an exponential polynomial can also be located at a fixed distance away from the critical lines (e.g., $g_3(z) = e^z - 2$), or they can be on the critical lines (e.g., $g_4(z) = e^z - e^{-z}$).

All of the above raises the concern whether the logarithmic strips have the correct shape in the higher order case, that is, when $q > 1$. We proceed to show that if the coefficients $H_j(z)$ in (1.2) are of sub-maximal growth, then the zero distribution result in (2.4) can be improved. To this end, we define the modified logarithmic strips

$$\Lambda_p(\theta^*, c) = \left\{ z = re^{i\theta} : r > 1, \, |\arg(z) - \theta^*| < c\frac{\log r}{r^p} \right\},$$

where $p \leq q$ is an integer and $\arg(z) = \theta^*$ is a critical direction. A calculation similar to the one in (2.6) shows that $\Lambda_p(0, c)$ is asymptotically the domain between the curves $y = \pm c \log x / x^{p-1}$, $x \geq 1$. Thus, if $p \geq 2$, the domains $\Lambda_p(\theta^*, c)$ actually curve asymptotically towards the critical rays $\arg(z) = \theta^*$, which is not the case in (2.3).

Let $N_{\Lambda_p}(r)$ denote the integrated counting function of the zeros of g in $|z| \leq r$ lying outside the finite union of domains $\Lambda_p(\theta^*, c)$.

Theorem 2.4: *Let g be an exponential polynomial of the form* (1.2) *for $q \geq 1$, where the coefficients $H_j(z)$ are of the growth $\rho(H_j) \leq q - p$ for some integer $1 \leq p \leq q$. Then $N_{\Lambda_p}(r) = O(r^{q-p} + \log r)$.*

The assumption $\rho(H_j) \leq q - p$ for $p = 1$ holds true by the definition of exponential polynomials. In the case $p = q$ the coefficients $H_j(z)$ reduce to polynomials, in which case g has at most finitely many zeros outside the domains Λ_p. The big "O" in Theorem 2.4 does not depend on the leading coefficients w_j of g, but depends on the coefficient functions H_j. The same conclusion would hold if g would be replaced by $G = Kg$, where K is any exponential polynomial with zeros satisfying

$$N(r,1/K)=O(r^{q-p}+\log r).$$

The proof of Theorem 2.4 is postponed to Section 5.

As a trivial consequence of Theorem 2.4 (or of (2.4)), we observe that the partial sums of the Riemann zeta function in (1.1) have at most finitely many zeros in the (slightly squeezed) left half-plane, no matter how large M.

Remark 2.2: Theorem 2.4 shows that, for an arbitrary $\varepsilon>0$, the zeros and poles of a quotient of two exponential polynomials of order $q\geq 1$ are (q,ε)-separated in the sense of Goldberg–Ostrovskii [3, p. 262]. In general, for such a function f the limit

$$\lim_{r\to\infty} r^{-q}T(r,f)$$

exists (but may be infinite), see Theorem 1.1 in [3, p. 262]. By appealing to [14, Satz 1], we conclude that for any given finite constant $C>0$ there exists an exponential polynomial f for which the above limit equals C.

3. A geometric condition for simple zeros

Any exponential polynomial g is a solution of some linear differential equation of order k with polynomial coefficients [17]. Thus the zeros of g are of multiplicity $\leq k-1$. In this section we will give a sufficient condition for the majority of the zeros (and thus the a-points) of g to be simple. Note, in particular, that the integrated quantity $N(r,1/g)$ for the zeros of g given in [14] counts multiplicities.

THEOREM 3.1: *Let g be an exponential polynomial of the form* (1.2)*, where the coefficients $H_j(z)$ are exponential polynomials of growth $\rho(H_j)\leq q-p$ for some integer $1\leq p\leq q$. Suppose that the points $\overline{w}_1,\dots,\overline{w}_m$ are either corner points or interior points of the convex hull* co(W)*. (In other words, all points $\overline{w}_1,\dots,\overline{w}_m$ that are on the boundary of* co(W) *are corner points.) Then the majority of the zeros of g are simple in the sense that*

$$N_{\geq 2}(r,1/g)=O(r^{q-p}+\log r).$$

The proof of Theorem 3.1 is based on Theorem 2.4 as well as on the proof of [2, Satz 1], and it is postponed to Section 6. We make three observations.

(1) There are some determinant conditions as assumptions of [2, Satz 1]. In our case these assumptions are not needed as they will be proven to hold in the situation where the method is applied.
(2) Differing from the situation in [2], the coefficients of exponential polynomials are of sub-maximal growth order $\leq q-p$, as in Theorem 2.4.
(3) The critical directions in [2] are called "gehöringen starken Wechselrichtungen", and they are defined as rays which are outer normals to the lines through two consecutive corner points of the convex polygon $\mathrm{co}(W)$. To distinguish our notation and terminology from [14], the critical directions in [2] are called **orthogonal directions** for $\mathrm{co}(W)$, and they are denoted by $\arg(z)=\theta^{\perp}$. There exist s orthogonal directions for $\mathrm{co}(W)$, where s is the number of the sides (or corners) of $\mathrm{co}(W)$.

According to the following lemma, each orthogonal direction corresponds to q critical directions. Thus, there exist $s\cdot q$ critical directions all together.

Lemma 3.2: *Let g be an exponential polynomial of the form* (1.2)*. Suppose that* $\arg z=\theta^{\perp}$ *is the orthogonal direction for* $\mathrm{co}(W)$ *related to two successive corner points $\overline{w}_j$ and $\overline{w}_k$. Then, for each $n\in\mathbb{Z}$,* $\arg(z)=(\theta^{\perp}+2n\pi)/q$ *is a critical direction for g.*

Proof. We define two vectors

$$\overrightarrow{\alpha}=(\Re(w_k-w_j),-\Im(w_k-w_j))\quad\text{and}\quad\overrightarrow{\beta}=(\cos\theta^{\perp},\sin\theta^{\perp}).$$

Since $\overrightarrow{\alpha}\perp\overrightarrow{\beta}$ by the definition of $\theta^{\perp}$, we have

$$\overrightarrow{\alpha}\cdot\overrightarrow{\beta}=\Re(w_k-w_j)\cos\theta^{\perp}-\Im(w_k-w_j)\sin\theta^{\perp}=0,$$

that is,

$$\Re(w_je^{i\theta^{\perp}})=\Re(w_ke^{i\theta^{\perp}}).\tag{3.1}$$

We observe that the line through the points $\overline{w}_j$,$\overline{w}_k$ is a supporting line for $\mathrm{co}(W)$ [8, pp. 74–75]. Let $k(\theta)$ denote the supporting function of $\mathrm{co}(W)$, which measures the (orthogonal) distance of the supporting line from the origin [8, pp. 74–75]. Thus, keeping (2.5) and (3.1) in mind, we deduce that

$$k(\theta^{\perp})=\Re(w_je^{i\theta^{\perp}})=\Re(w_ke^{i\theta^{\perp}})\geq\max_{t\neq j,k}\Re(w_te^{i\theta^{\perp}}).$$

The assertion follows from (2.1) and (2.2). ∎

Example 3.3: (1) The exponential polynomial $g(z) = ze^z - z$ has a double zero at the origin, and hence some multiple zeros are possible even though the conjugates of the leading coefficients are all corner points of $\mathrm{co}(W)$. In the first-order case with constant coefficients, the exponential polynomial

$$g(z) = (e^z - 1)\sin z$$

has a double zero at the origin.

(2) Infinitely many multiple zeros are also possible in the higher order case: All integers are double zeros of

$$g(z) = (e^{2\pi i z} - 1)(e^{2\pi i z^q} - 1), \quad q \geq 2.$$

We have $N_{\geq 2}(r, 1/g) \asymp r$, while $N(r, 1/g) = 2r^q + o(r^q)$.

(3) The function

$$g(z) = (e^z + 1)^2 = e^{2z} + 2e^z + 1$$

has double zeros only but the conjugate of its leading coefficient 1 is not a corner point or an interior point of $\mathrm{co}(W)$.

(4) The function

$$e^{2z} + e^z = e^z(e^z + 1)$$

has only simple zeros and the origin is not a corner point or an interior point of $\mathrm{co}(W)$.

(5) The function

$$g(z) = e^{2z} + e^z + 1$$

has the constant 1 as one of its leading coefficients, whose conjugate is not a corner point or an interior point of $\mathrm{co}(W)$. Since no zero of

$$g'(z) = 2e^{2z} + e^z = e^z(2e^z + 1)$$

is a zero of g, it follows that g has only simple zeros. Therefore the implication in Theorem 3.1 is not an equivalence.

(6) In the case of two leading coefficients

$$g(z) = H(z)e^{az^q} + K(z)e^{bz^q} = e^{az^q}(H(z) + K(z)e^{(b-a)z^q})$$

the convex hull $\mathrm{co}(W)$ is just a line segment with no interior points, but has $\overline{a}, \overline{b}$ as its corner points. Being of order of growth $\leq q-1$, the function $-H(z)/K(z)$ is a small target for the regularly growing $e^{(b-a)z^q}$. Thus, applying a variant of the second main theorem in [5, Theorem 2.5] to the function $e^{(b-a)z^q}$ with targets $0, \infty, -H(z)/K(z)$, we see that $T(r, g) \sim \overline{N}(r, 1/g)$.

Remark 3.1: (1) The exponential polynomial $g(z) = (e^{z^{q-p}} - 1)^m(e^{z^q} - 1)$ of order $q \geq 1$ has many zeros of order at least m. The set of the conjugates of leading coefficients of g is $W = \{0, 1\}$, which are all corner points of $\operatorname{co}(W) = [0, 1]$. The counting function of the common zeros of g and g', that is, the multiple zeros of g, grows in the order of $O(r^{q-p})$ not $o(r^{q-p})$. Hence, the concluding estimate is best possible, since Theorem 2.4 does not require any assumption on possible common zeros of the H_j's for g.

(2) If the conjugate of every leading coefficient of g is a corner point, and if the coefficients H_j are constants, then all the zeros of g are simple, apart from finitely many exceptions. Indeed, finitely many multiple zeros are possible: The exponential polynomial with constant coefficients

$$g(z) = e^z - \frac{e^{\pi/4}}{1+i}e^{(1+i)z} - \frac{i}{\sqrt{2}}$$

has a multiple zero at $z = \pi i/4$, because $g'(z) = e^z - e^{\pi/4}e^{(1+i)z}$. When $q > 1$, we replace z by z^q and find at least q distinct multiple zeros as the qth roots of $e^{\pi i/4}$.

4. Quotients of exponential polynomials

We generalize Ritt's theorem to exponential polynomials of arbitrary order $q \geq 1$. The example $(e^{2\pi i z^q} - 1)/(e^{2\pi i z} - 1)e^{z^q}$, $q \geq 2$, shows that the coefficients of the entire quotient may be quotients of exponential polynomials themselves, but of lower order and with removable singularities. Before stating our main result, we define two function classes for $p \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned} P_p &= \{f : f \text{ is an exponential polynomial and } \rho(f) \leq p\}, \\ R_p &= \{f/g : f, g \in P_p, g \not\equiv 0\}. \end{aligned}$$

Then P_0 and R_0 consist of polynomials and rational functions, respectively.

THEOREM 4.1: *Let $n, m, q \in \mathbb{N}$, and $p \leq q - 1$ be a non-negative integer. Set*

$$g(z) = \sum_{j=1}^{m} a_j(z)e^{\mu_j z^q} \quad \textit{and} \quad h(z) = \sum_{j=1}^{n} b_j(z)e^{\nu_j z^q},$$

where $a_j, b_j \in P_p$ are non-vanishing. If $f(z) = g(z)/h(z)$ is entire, then f too is an exponential polynomial, but with coefficients in R_p.

Clearly $\rho(f) \leq q$ in Theorem 4.1. Next we describe the case $\rho(f) < q$, which is critical in proving Theorem 4.1.

Lemma 4.2: *Under the assumptions of Theorem 4.1, if $\rho(f) < q$, then $f \in R_p$. In the particular case $a_j, b_j \in P_0$ we have $f \in P_0$. Equivalently, if $f \notin R_p$ (or if f is transcendental in the particular case), then $\rho(f) = q$.*

Proof. We may suppose that the coefficients μ_j are pairwise distinct and that the coefficients ν_j are also. Then write

$$g(z) - f(z)h(z) = \sum_{j=1}^{m} a_j(z)e^{\mu_j z^q} - \sum_{j=1}^{n} f(z)b_j(z)e^{\nu_j z^q} = 0.$$

Suppose that the combined coefficients μ_j, ν_k are pairwise distinct, that is, all exponential terms $e^{\mu_j z^q}, e^{\nu_k z^q}$ are linearly independent. Then $a_j(z) \equiv 0$ and $b_k(z) \equiv 0$ for every j and k by the Borel–Nevanlinna theorem [4, pp. 70, 108], which is a contradiction. Hence there exists μ_j and ν_k such that $\mu_j = \nu_k$. Then $a_j(z) - f(z)b_k(z) \equiv 0$ by [4, pp. 70, 108], and so $f \in R_p$.

If $a_j, b_j \in P_0$, then the reasoning above gives $f \in R_0$, and since f is entire, we must have $f \in P_0$. ∎

The property that the quotient $g(z)/h(z)$ of functions $g, h \in P_q$ is entire requires that each zero of $h(z)$ must be a zero of $g(z)$. This leads us to consider the value zero as a shared value, that is, the situation when both g and h vanish at the same points. We use the standard abbreviations CM (counting multiplicities) and IM (ignoring multiplicities) from the uniqueness theory of meromorphic functions.

Example 4.3: (1) Suppose that $g, h \in P_q$ share zeros CM. Then both $g(z)/h(z)$ and $h(z)/g(z)$ are entire so that $f(z) = g(z)/h(z)$ is a unit in P_q, that is, an exponential function of a polynomial of degree at most q. If the zeros are shared IM, then $f \in P_q$ does not necessarily hold even if f is entire. A counterexample is given by $g(z) = z(e^{z^q} - 1)^2$ and $h(z) = z^2(e^{z^q} - 1)$.

(2) Suppose that $g, h \in P_q$ share zeros IM such that the multiplicity of g always dominates that of h. Then $f(z) = g(z)/h(z)$ is clearly entire, but $f \in P_q$ does not necessarily hold. A counterexample is given by $g(z) = (\sin z)^2$ and $h(z) = z \sin z$.

(3) Suppose that entire functions g, h of finite lower order share 0 and 1 CM. Then either $g \equiv h$ or otherwise g, h reduce to very specific exponential polynomials by [18, Theorem 2.32].

5. Proof of Theorem 2.4

To prove Theorem 2.4, we need three auxiliary results of which the first one is essentially a modification of [16, Theorem V. 19].

LEMMA 5.1: *Let g be an exponential polynomial of the form* (1.2), *let $\{z_n\}$ be the (infinite) zero sequence of g, and let $\alpha > 1$. Then there exists a constant $A > 0$ such that $\log|g(z)| \geq -Ar^q \log r$ whenever $z = re^{i\theta}$ lies outside the closed discs*

$$|z - z_n| \leq 1/(|z_n|^q \log^\alpha |z_n|) \quad \textit{and} \quad |z| \leq e.$$

Proof. We may factorize the entire function g as $g = Pe^h$, where h is a polynomial of degree at most q, and P is a canonical product formed with the zeros z_n of g. As a product of two exponential polynomials, P is also an exponential polynomial. We may assume that $\rho(P) = q$ because that is the "worst case scenario". Clearly P is of finite type [14] and has genus q [16, p. 216].

We begin estimating P similarly as Tsuji [16] by first writing

$$\Psi(z, z_n) = \log \frac{z_n}{z_n - z} - \left(\frac{z}{z_n} + \cdots + \frac{1}{q}\left(\frac{z}{z_n}\right)^q\right),$$

and then noting that

$$\log^+ \frac{1}{|P(z)|} \leq \sum_{\frac{r}{r_n} \geq \frac{1}{2}} \Re^+(\Psi(z, z_n)) + \sum_{\frac{r}{r_n} < \frac{1}{2}} |\Psi(z, z_n)| =: \Sigma_1 + \Sigma_2,$$

where $r_n = |z_n|$. In Σ_2, we have

$$|\Psi(z, z_n)| \leq \sum_{j=q+1}^{\infty} \frac{1}{j}\left(\frac{r}{r_n}\right)^j \leq \left(\frac{r}{r_n}\right)^{q+1} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j = 2\left(\frac{r}{r_n}\right)^{q+1}.$$

Since

$$n(r, 1/P) \leq \frac{1}{\log 2} \int_r^{2r} \frac{n(t, 1/P)}{t}\, dt \leq \frac{N(2r, 1/P)}{\log 2} = O(r^q), \tag{5.1}$$

an application of Riemann–Stieltjes integration and integration by parts yields

$$\begin{aligned}\Sigma_2 \leq & 2\sum_{r_n>2r}\left(\frac{r}{r_n}\right)^{q+1} = 2r^{q+1}\int_{2r}^{\infty}\frac{dn(t)}{t^{q+1}}\\ \leq & 2(q+1)r^{q+1}\int_{2r}^{\infty}\frac{n(t)}{t^{q+2}}\,dt = O(r^q).\end{aligned}$$

In Σ_1, we have

$$\begin{aligned}\Re^+(\Psi(z,z_n)) & \leq \log^+\left|\frac{z_n}{z-z_n}\right| + \sum_{j=1}^{q}\frac{1}{j}\left(\frac{r}{r_n}\right)^j\\ & \leq \log^+\left|\frac{z_n}{z-z_n}\right| + \left(\frac{r}{r_n}\right)^q\sum_{j=1}^{q}\left(\frac{r}{r_n}\right)^{j-q}\\ & \leq \log^+\left|\frac{z_n}{z-z_n}\right| + 2^q\left(\frac{r}{r_n}\right)^q.\end{aligned}$$

If z lies outside the discs $|z-z_n|\leq 1/(|z_n|^q\log^\alpha|z_n|)$ and $|z|\leq e$, it follows that

$$\log^+\left|\frac{z_n}{z-z_n}\right| \leq \log(r_n^{q+1}\log^\alpha r_n) = O(\log r),$$

where we have used the fact that $r_n \leq 2r$. Thus

$$\begin{aligned}\Sigma_1 \leq & O(n(2r)\log r) + 2^p r^q\int_e^{2r}\frac{dn(t)}{t^q}\\ \leq & O(r^q\log r) + 2^p q r^q\int_e^{2r}\frac{n(t)}{t^{q+1}}\,dt = O(r^q\log r).\end{aligned}$$

The reasoning above shows that there exists a constant $B>0$ such that $\log|P(z)| \geq -Br^q\log^\alpha r$ whenever $z=re^{i\theta}$ lies outside the aforementioned closed discs. Since h is a polynomial of degree at most q, there exists a constant $C>0$ such that $\log|e^{h(z)}| \geq \log e^{-|h(z)|} \geq -Cr^q$ for any $|z|>e$. The assertion now follows by choosing $A=B+C$. ∎

Regarding the discs in Lemma 5.1, we deduce via Riemann–Stieltjes integration, integration by parts and (5.1) that

$$\sum_{r_n\geq e}\frac{1}{r_n^q\log^\alpha r_n} = \int_e^{\infty}\frac{dn(t)}{t^q\log^\alpha t} \leq (q+\alpha)\int_e^{\infty}\frac{n(t)}{t^{q+1}\log^\alpha t} < \infty.$$

Thus the projection of the discs on the positive real axis has a finite linear measure.

LEMMA 5.2: *Let g be an exponential polynomial of the form* (1.2)*, let* $\{\zeta_n\}$ *be any sequence of (non-zero) complex points tending to infinity, and let* $p > 0$*. Then in each disc of the form* $|z - \zeta_n| \leq |\zeta_n|^{-p}$*, the function g has at most K zeros, where K is independent of n.*

Proof. We make use of [17, Theorem 3] by choosing $s = 2$, $t = 1$, $z_0 = \zeta_n$, $R = |\zeta_n|^{-p} \to 0$, and by noticing that the function $\psi(x)$ in [17, Theorem 3] is bounded as $x \to 0^+$. This yields the assertion. ∎

LEMMA 5.3: *Let g be an exponential polynomial of the form* (1.2)*, and let θ^*'s denote the cusps of the indicator $h_g(\theta)$. Let $\psi_k(\theta) = \Re(w_k e^{iq\theta})$, $p \leq q$ and*

$$2a = \min\{|\psi_k'(\theta) - \psi_j'(\theta)| : \psi_k(\theta) = \psi_j(\theta),\ k \neq j\}. \tag{5.2}$$

For a given $c > 0$ there exists an $R_0 > 0$, with the following property: If $z = re^{i\theta} \notin \Lambda_p(\theta^*, c)$ for any θ^* and any $r \geq R_0$, and if $h_g(\theta) = \psi_j(\theta)$, then

$$h_g(\theta) > \max_{k\neq j} \psi_k(\theta) + ac\frac{\log r}{r^p}.$$

Remark 5.1: (1) For Lemma 5.3 itself, any positive term tending to zero could be used to replace $ac\frac{\log r}{r^p}$. This particular term, however, is needed later.

(2) We show that the constant a in (5.2) is strictly positive. The identity $\psi_k(\theta) = \psi_j(\theta)$ in (5.2) can hold if θ is one of the finitely many cusps θ^*. Suppose that in addition $\psi_k'(\theta) = \psi_j'(\theta)$ holds. Denote $w_n = a_n + ib_n$. Then $\psi_n(\theta) = a_n \cos(q\theta) - b_n \sin(q\theta)$. From the two assumed identities, we get the equations

$$\begin{aligned}(a_k - a_j)\cos(q\theta) =& (b_k - b_j)\sin(q\theta),\\ (a_j - a_k)\sin(q\theta) =& (b_k - b_j)\cos(q\theta).\end{aligned}$$

If $a_k = a_j$, then $b_k = b_j$, and so $w_k = w_j$, which is a contradiction. If $a_k \neq a_j$, then $b_k \neq b_j$, and further $a_k - a_j = \pm i(b_k - b_j)$, which is again a contradiction. Thus $\psi_k'(\theta) \neq \psi_j'(\theta)$ for any $\theta = \theta^*$, and, a fortiori, $a > 0$.

Proof of Lemma 5.3. If the claim is not true, there exists a sequence $z_n = r_n e^{i\theta_n}$ with $z_n \notin \Lambda_p(\theta^*, c)$ for any θ^*, $\lim_{n\to\infty} r_n = \infty$, $h_g(\theta_n) = \psi_j(\theta_n)$, and indices $k = k_n \neq j$ such that

$$h_g(\theta_n) \leq \psi_k(\theta_n) + ac\frac{\log r_n}{r_n^p}.$$

The sequence $\{\theta_n\}$ of angles may have several accumulation points. After passing to a subsequence, we may suppose that k is independent of n and the angles θ_n converge to some number θ_0. This is possible even if $z_n \notin \Lambda_p(\theta^*, c)$ since the domains $\Lambda_p(\theta^*, c)$ are essentially contained in arbitrarily small ε-sectors; see the discussion in Remark 2.1.

By letting $n \to \infty$ in the inequalities

$$0 \leq \max_l \{\psi_l(\theta_n)\} - \psi_k(\theta_n) = \psi_j(\theta_n) - \psi_k(\theta_n) \leq ac \log r_n / r_n^p$$

and using continuity, we obtain $h_g(\theta_0) = \psi_j(\theta_0) = \psi_k(\theta_0)$. By definition, θ_0 must be a cusp of the indicator $h_g(\theta)$ (i.e., one of the θ^*'s). In the neighborhood of θ_0, we have

$$\begin{aligned}|\psi_k(\theta) - \psi_j(\theta)| =& |\psi_k(\theta) - \psi_k(\theta_0) - (\psi_j(\theta) - \psi_j(\theta_0))| \\ =& \left| \frac{\psi_k(\theta) - \psi_k(\theta_0)}{\theta - \theta_0} - \frac{\psi_j(\theta) - \psi_j(\theta_0)}{\theta - \theta_0} \right| |\theta - \theta_0| \\ \geq& (2a - o(1))|\theta - \theta_0|, \quad \theta \to \theta_0.\end{aligned}$$

But for $z_n \notin \Lambda_p(\theta_0, c)$ and $n \to \infty$, we obtain

$$|\theta_n - \theta_0| \geq c \frac{\log r_n}{r_n^p} \geq \frac{1}{a} |\psi_k(\theta_n) - \psi_j(\theta_n)| \geq (2 - o(1))|\theta_n - \theta_0|,$$

which is a contradiction. ∎

Proof of Theorem 2.4. Suppose that $q \geq 2$ and that $1 \leq p \leq q - 1$. We first consider the case where all coefficients $H_j(z)$ are non-polynomials. There exist constants $A > 0$ and $r_0 > 0$ such that

$$\log M(r, H_j) \leq A r^{q-p}, \quad j \in \{0, \ldots, m\}, \tag{5.3}$$

whenever $r \geq r_0$. Let $\{z_{jn}\}$ denote the zeros of $H_j(z)$. We conclude by Lemma 5.1 that there exists a constant $A > 0$ such that

$$\log |H_j(z)| \geq -A r^{q-p} \log r, \quad j \in \{0, \ldots, m\}, \tag{5.4}$$

whenever $|z - z_{jn}| > 1/|z_{jn}|^{q-p} \log^\alpha |z_{jn}|$ and $|z| > e$. We may suppose that the constant A in (5.4) is the same as that in (5.3) by choosing the larger of the two. Noting that

$$r^q \psi_k(\theta) = \Re(w_k z^q),$$

we make use of Lemma 5.3 by choosing $c > \frac{3A}{a}$: If $z = re^{i\theta} \notin \Lambda_p(\theta^*, c)$ and if r is large enough, then

$$\Re(w_j z^q) \geq \Re(w_k z^q) + 3Ar^{q-p} \log r \tag{5.5}$$

for all j and for all $k \neq j$. If in addition $|z - z_{jn}| > 1/|z_{jn}|^{q-p} \log^\alpha |z_{jn}|$ for some j and for all n, then the estimates (5.3)–(5.5) applied to (1.2) yield

$$\begin{aligned} |g(z)e^{-w_j z^q}| \geq& |H_j(z)| - \sum_{k\neq j} |H_k(z)||e^{(w_k - w_j)z^q}| \\ \geq& |H_j(z)| - \sum_{k\neq j} M(r, H_k)e^{-3Ar^{q-p}\log r} \\ \geq& e^{-Ar^{q-p}\log r} - me^{-2Ar^{q-p}\log r} > 0. \end{aligned} \tag{5.6}$$

Since $|g(z)e^{-w_j z^q}|$ is indeed bounded away from zero, it follows that the zeros of g which are not in the domains $\Lambda_p(\theta^*, c)$ must be in the discs

$$U_{jn} : |z - z_{jn}| \leq 1/|z_{jn}|^{q-p} \log^\alpha |z_{jn}|$$

or in some neighborhood $|z| \leq r_0$ of the origin. The discs U_{jn} and $|z| \leq r_0$ contain all zeros of the coefficients $H_1(z), \ldots, H_m(z)$, which are of order $\leq q-p$ and of mean type. The estimate $N_{\Lambda_p}(r) = O(r^{q-p})$ then follows from the fact that the number of zeros of g in each of the discs U_{jn} is bounded from above by some constant $K > 0$ by Lemma 5.2.

If some (but not all) coefficients are polynomials, the previous reasoning simplifies. Indeed, if a particular coefficient $H_j(z)$ is a polynomial, then the growth of $|H_j(z)|$ is comparable to $|z|^{\deg(H_j)}$ for r large enough. This is easy to take into account in the estimates (5.6). Consideration of the discs U_{jn} is not necessary for this particular coefficient.

Suppose then that $q \geq 1$ is any integer and $p = q$, that is, all of the coefficients $H_j(z)$ are polynomials. This covers all remaining cases. There exist constants $A_1 > 0$ and $A_2 > 0$ such that

$$|H_j(z)| \leq A_1|z|^d \quad \text{and} \quad |H_j(z)| \geq A_2|z|^{d_j},$$

where $d_j = \deg(H_j)$ and $d = \max\{d_j\}$. Suppose that $z = re^{i\theta} \notin \Lambda_p(\theta^*, c)$ and that r is large enough. In Lemma 5.3 choose $c > \frac{3d}{a}$ if $d > 0$, and then

$$\begin{aligned} |g(z)e^{-w_j z}| \geq& |H_j(z)| - A_1 m r^d e^{-3d\log r} \\ \geq& A_2 r^{d_j} - A_1 m r^{-2d} > 0. \end{aligned} \tag{5.7}$$

If $d = 0$, the coefficients $H_j(z)$ are nonzero constants. In this case we may choose $c > \frac{1}{a}$ and proceed similarly as above. Thus g can have at most finitely many zeros outside the domains $\Lambda_p(\theta^*, c)$, and so $N_{\Lambda_p}(r) = O(\log r)$. ∎

Remark 5.2: (1) The majority of zeros of g are in finitely many domains Λ by (2.4). Those zeros of g that are not in the domains Λ are near the zeros of the coefficients $H_j(z)$ of g. However, the coefficients are exponential polynomials also, so the majority of their zeros are in finitely many domains Λ, and so on. In this way we deduce that all of the zeros of g are in finitely many domains Λ, or in the disc $|z| \leq r_0$. In particular, there are finitely many cusps θ^* in all of these finitely many collections. As noted in [14], if θ is different from any of the θ^*'s, then the ray $\arg(z) = \theta$ is, at least eventually, outside the domains Λ, and so

$$h_g(\theta) = \lim_{r\to\infty} r^{-q} \log |g(re^{i\theta})|.$$

In other words, "lim sup" in the definition of the indicator can be replaced with "lim", apart from finitely many exceptions on the interval $[-\pi, \pi)$.

(2) Suppose that $z = re^{i\theta}$ is such that $\theta_l^* + \varepsilon \leq \theta \leq \theta_{l+1}^* - \varepsilon$, where $\arg(z) = \theta_l$ and $\arg(z) = \theta_{l+1}$ are two consecutive critical rays for g. Then the reasoning related to (5.6) and (5.7) shows that there is a unique index j such that

$$g(z) = H_j(z)e^{w_j z^q}(1 + o(1)), \quad r \to \infty,$$

outside a small collection of discs enclosing the zeros of $H_j(z)$. That is, g is dominated by precisely one exponential term.

6. Proof of Theorem 3.1

Without loss of generality, we can assume that the corner points of co(W) are organized in the counterclockwise direction from 1 to $s \leq m$, and denote them again by $\overline{w}_1, \ldots, \overline{w}_k, \ldots, \overline{w}_s$. If $s = m$, then all points $\overline{w}_k$ were corner points of co(W) to begin with. (That is, none of them are interior points.)

A simple differentiation gives us

$$g'(z) = G_1(z)e^{w_1 z^q} + \cdots + G_m(z)e^{w_m z^q},$$

where

$$G_j(z) = H_j'(z) + qw_j z^{q-1} H_j(z).$$

If $G_j(z) \equiv 0$ for some j, then either $H_j(z) \equiv 0$ or $\rho(H_j) = q$, which are not possible. Thus $G_j(z) \not\equiv 0$ for $j = 1, \ldots, m$. In particular, g and g' have the same leading coefficients, and the coefficients of g' are also of submaximal growth due to $\rho(G_j) \leq \rho(H_j) \leq q - p$.

It suffices to prove that

$$N(r, 1/g', 1/g) = O(r^{q-p} + \log r), \tag{6.1}$$

where $N(r, 1/g', 1/g)$ is the integrated counting function of the common zeros of g and g' in $|z| < r$, each zero being counted according to the multiplicity of the zero of g. Set $h = g/g'$ and

$$h_1(z) = \frac{g(z)}{g'(z)} - \frac{H_1(z)}{G_1(z)} = \frac{\sum_{k=2}^{m}(H_k(z) - G_k(z)\frac{H_1(z)}{G_1(z)})e^{w_k z^q}}{\sum_{k=1}^{m} G_k(z)e^{w_k z^q}} =: \frac{g_1(z)}{g'(z)}.$$

We see that w_1 is not a leading coefficient of $g_1(z)$. Suppose that w_k is not a leading coefficient of $g_1(z)$ for some $k \neq 1$. Then we must have

$$H_k(z) - G_k(z)\frac{H_1(z)}{G_1(z)} \equiv 0,$$

which can be re-written as

$$\frac{H_k'}{H_k} = \frac{H_1'}{H_1} + q(w_1 - w_k)z^{q-1}.$$

Integrating both sides in any simply connected domain D that does not include the zeros of H_1, H_k, we get

$$H_k(z) = CH_1(z)e^{(w_1-w_k)z^q}, \quad C \in \mathbb{C} \setminus \{0\},\ z \in D.$$

By the standard uniqueness theorem, this representation for $H_k(z)$ is valid everywhere in $\mathbb{C}$. Since $w_1 \neq w_k$, this contradicts the fact that $\rho(H_k) \leq q - p < q$. Thus $w_2, \ldots, w_m$ are precisely the leading coefficients of g_1.

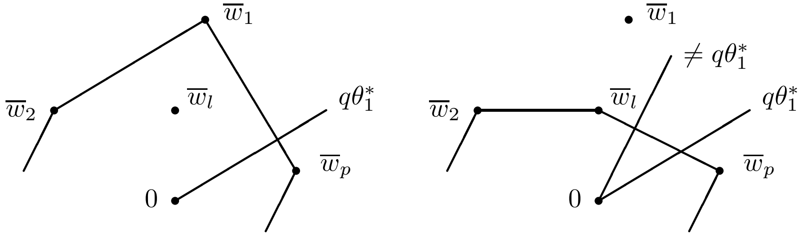

Figure 1. Removing one corner point changes the critical directions.

Let $\theta_1^* \in \mathbb{R}$ be an angle such that the ray $\arg(z) = q\theta_1^*$ (or $\theta_1^\perp$) intersects orthogonally the line through the points $\overline{w}_p, \overline{w}_1$. Then $\arg(z) = \theta_1^* + \frac{2\pi j}{q}$ (or $(\theta_1^\perp + 2\pi j)/q$), $j = 0, \dots, q-1$, are critical directions for g' by Lemma 3.2, and none of them are critical directions for g_1.

For a meromorphic F and $\theta \in \mathbb{R}$, $\varepsilon > 0$, let $N(r, F, \theta)$ be the integrated counting function of the poles of F in the intersection of the disc $|z| < r$ and the sector $\theta - \varepsilon/2 \le \arg(z) \le \theta + \varepsilon/2$. The counting function $N(r, 1/F, \theta)$ for the zeros of F is defined analogously.

From the reasoning above, the poles of h_1 near the directions

$$\arg(z) = \theta_1^* + \frac{2\pi j}{q}, \quad j = 0, \dots, q-1,$$

coincide substantially with the zeros of g'. Since these directions are not critical for g_1, we have by Theorem 2.4 that

$$N\Big(r, \frac{1}{g_1}, \theta_1^* + \frac{2\pi j}{q}\Big) = O(r^{q-p} + \log r), \quad j = 0, \dots, q-1.$$

Thus for $j = 0, \dots, q-1$,

$$N\Big(r, h_1, \theta_1^* + \frac{2\pi j}{q}\Big) = N\Big(r, \frac{1}{g'}, \theta_1^* + \frac{2\pi j}{q}\Big) + O(r^{q-p} + \log r).$$

Since $h_1(z) = h(z) - \frac{H_1(z)}{G_1(z)}$, the functions h_1 and h differ at most by a function of order $q - p$ and of mean type. Therefore

$$N(r, h) = N(r, h_1) + O(r^{q-p} + \log r),$$

so that

$$N\Big(r, h, \theta_1^* + \frac{2\pi j}{q}\Big) = N\Big(r, \frac{1}{g'}, \theta_1^* + \frac{2\pi j}{q}\Big) + O(r^{q-p} + \log r)$$

for $j = 0, \dots, q-1$.

What we have done with the corner $\overline{w}_1$ we can do successively with all s corners $\overline{w}_1, \dots, \overline{w}_s$ of the convex hull $\mathrm{co}(W)$. We form the auxiliary functions

$$h_k(z) := h(z) - \frac{H_k(z)}{G_k(z)} =: \frac{g_k(z)}{g'(z)}, \quad k = 2, \dots, s.$$

Then g_k has precisely the leading coefficients $\{w_1, \dots, w_m\} \backslash \{w_k\}$. Let $\theta_k^* \in \mathbb{R}$ be an angle such that the ray $\arg(z) = q\theta_k^*$ intersects orthogonally the line through the points $\overline{w}_{k-1}, \overline{w}_k$, $k \geq 2$. Similarly as before, the directions $\arg(z) = \theta_k^* + \frac{2\pi j}{q}$, $j = 0, \ldots, q-1$, are critical for g', but none of them are critical for g_k. We have

$$N\Big(r, h, \theta_k^* + \frac{2\pi j}{q}\Big) = N\Big(r, \frac{1}{g'}, \theta_k^* + \frac{2\pi j}{q}\Big) + O(r^{q-p} + \log r)$$

for $k = 2, \ldots, s$ and $j = 0, \ldots, q-1$. The poles of h are among the zeros of g' most of which are in the union V of the aforementioned ε-sectors around the $s \cdot q$ critical directions by Theorem 2.4. We conclude that

$$\begin{aligned} N(r, h) =& N\Big(r, \frac{g}{g'}, V\Big) + N\Big(r, \frac{g}{g'}, V^c\Big) \\ =& N\Big(r, \frac{1}{g'}\Big) + O(r^{q-p} + \log r). \end{aligned}$$

Thus g and g' have a few common zeros in the sense of (6.1). ∎

7. Proof of Theorem 4.1

We will modify the reasoning used in [7, 11]. We may suppose that the leading coefficients μ_j, ν_j are arranged such that

$$\Re\mu_1 \geq \Re\mu_2 \geq \cdots \geq \Re\mu_m \quad \text{and} \quad \Re\nu_1 \geq \Re\mu_2 \geq \cdots \geq \Re\nu_n. \tag{7.1}$$

By Lemma 4.2, we may suppose that $\rho(f) = q$. Then f is of mean type by [15, Satz 1]. The indicator $h_f(\theta)$ is defined for almost all $\theta \in [-\pi, \pi)$, and by analyticity for all θ. More precisely,

$$h_f(\theta) = \Re((\mu_k - \nu_l)e^{iq\theta})),$$

where

$$\Re(\mu_k e^{iq\theta}) = \max_j \Re(\mu_j e^{iq\theta}) \quad \text{and} \quad \Re(\nu_l e^{iq\theta}) = \max_j \Re(\nu_j e^{iq\theta}).$$

We choose a direction θ for which the maximum related to the leading coefficients ν_j occurs for one value of j only, and at the same time $h_h(\theta)$ reaches its maximum value. This can be done for the following reasons:

(1) Exponential polynomials are of mean type, hence the maximum value of their Phragmén–Lindelöf indicator is positive.
(2) By Remark 5.2, along any direction $\arg(z) = \theta$ outside the sectors $\{\varphi : |\varphi - \theta^*| < \varepsilon\}$ precisely one exponential term of $h(z)$ is dominant. Here the cusps θ^* are related to the indicator function $h_h(\theta)$.

(3) As a continuous function, $h_h(\theta)$ reaches its maximum value somewhere on $[-\pi,\pi]$. We just have to be sure that this doesn't happen at one of its cusps θ^*. But this possibility is excluded by the discussions in [8, pp. 54–55]: The one-sided derivatives exist everywhere, and the right-hand derivative is greater than or equal to the left-hand derivative.

Since $f(z)$ is an exponential polynomial if and only if $f(e^{-i\theta}z)$ is, we may suppose that $\theta = 0$. By (7.1) this yields $k = l = 1$ and, for some constant $d > 0$,

$$\Re(\nu_j - \nu_1) < -d < 0, \quad j = 2,\dots,n. \tag{7.2}$$

We construct an entire function

$$f_1(z) = f(z)b_1(z) - a_1(z)e^{(\mu_1-\nu_1)z^q}.$$

If $f_1 \in P_p$, then the proof is complete. Hence we suppose that $f_1 \notin P_p$. We observe that $f_1(z) = g_1(z)/h(z)$, where

$$g_1(z) = g(z)b_1(z) - h(z)a_1(z)e^{(\mu_1-\nu_1)z^q},$$

that is, $f_1(z)$ is of the same form as f. Using Lemma 4.2, we conclude that $\rho(f_1) = q$. Further, $f_1(z)$ is of mean type by [15, Satz 1]. For brevity, let $h_1(\theta)$ denote the indicator of $f_1(z)$, which is a continuous function. Both of the functions $f(z)b_1(z)$ and $a_1(z)e^{(\mu_1-\nu_1)z^q}$ have the same indicator at $\theta = 0$, namely $h_f(0)$. Thus

$$h_1(0) \le h_f(0). \tag{7.3}$$

For $\theta = \pi/q$ the indicator of f is $h_f(\pi/q)$, while the indicator of $a_1(z)e^{(\mu_1-\nu_1)z^q}$ is $\Re(\nu_1 - \mu_1) = -h_f(0)$. Hence $h_1(\pi/q) \le \max\{h_f(\pi/q), -h_f(0)\}$. Using

$$h_f(\theta) + h_f(\theta + \pi/q) \ge 0, \quad \theta \in [-\pi,\pi),$$

see [9, p. 56], we find that

$$h_1(\pi/q) \le h_f(\pi/q). \tag{7.4}$$

Inequalities (7.3) and (7.4) correspond to inequalities (a) and (b) in either of the references [7, 11].

Next we construct another entire function

$$\begin{aligned} f_2(z) =& b_1(z)(f_1(z) - a_2(z)e^{(\mu_2-\nu_1)z^q}) \\ =& b_1(z)(f(z)b_1(z) - a_1(z)e^{(\mu_1-\nu_1)z^q} - a_2(z)e^{(\mu_2-\nu_1)z^q}). \end{aligned}$$

If $f_2 \in P_p$, we are done, so we suppose that $f_2 \notin P_p$. Then $f_2(z) = g_2(z)/h(z)$ is of the same form as f, where

$$g_2(z) = b_1(z)(g(z)b_1(z) - h(z)a_1(z)e^{(\mu_1-\nu_1)z^q} - h(z)a_2(z)e^{(\mu_2-\nu_1)z^q}).$$

Similarly as above, we deduce that $\rho(f_2) = q$, and that $f_2(z)$ is of mean type. Let $h_2(\theta)$ denote the indicator of $f_2(z)$. Noting by (7.1) that

$$\Re(\mu_2 - \nu_1) \le \Re(\mu_1 - \nu_1) = h_f(0),$$

we obtain, corresponding to (7.3) and (7.4), the inequalities

$$h_2(0) \le h_f(0) \quad \text{and} \quad h_2(\pi/q) \le h_f(\pi/q).$$

We continue this process by defining a sequence of entire functions

$$\begin{aligned} f_s(z) =& b_1(z)^{s-1}\left(f(z)b_1(z) - \sum_{j=1}^{s} a_j(z)e^{(\mu_j-\nu_1)z^q}\right) \\ =& b_1(z)^{s-1}\left(g(z)b_1(z) - h(z)\sum_{j=1}^{s} a_j(z)e^{(\mu_j-\nu_1)z^q}\right)\Big/h(z). \end{aligned}$$

If $f_s \in P_p$, we are done. Otherwise $\rho(f_s) = q$, and $f_s(z)$ is of mean type. Let $h_s(\theta)$ denote the indicator of $f_s(z)$. Keeping (7.1) in mind, choose $s \in \{1, \dots, m\}$ such that

$$\begin{aligned} \Re\mu_j \ge& \Re\mu_1 - d, \quad j \le s, \\ \Re\mu_j <& \Re\mu_1 - d, \quad j \ge s+1. \end{aligned} \tag{7.5}$$

The first inequality in (7.5) is certainly true for $j = 1$. Write

$$g(z) = \sum_{j=1}^{m} a_j(z)e^{\mu_j z^q} = \sum_{j=1}^{s} a_j(z)e^{\mu_j z^q} + \sum_{j=s+1}^{m} a_j(z)e^{\mu_j z^q},$$

where the second sum is empty if the second case in (7.5) fails. Now

$$f_s(z) = g_s(z)/h(z),$$

where

$$g_s(z) = b_1(z)^{s-1}\left(b_1(z)\sum_{j=s+1}^{m} a_j(z)e^{\mu_j z^q} - \sum_{j=1}^{s}\sum_{k=2}^{n} a_j(z)b_k(z)e^{(\mu_j-\nu_1+\nu_k)z^q}\right).$$

By (7.1) and (7.2) it follows that

$$\Re(\mu_j - \nu_1 + \nu_k) < \Re\mu_j - d \le \Re\mu_1 - d, \quad j \ge 1.$$

This and the second case in (7.5) then show that the exponents in $g_s(z)$ have real parts $< \Re\mu_1 - d$ for $\theta = 0$. Consequently, $h_s(0) \leq h_f(0) - d$.

Set $s = s_1$. We have proved that the entire quotient f_{s_1} of exponential polynomials has the indicator function $h_{s_1}(\theta)$ satisfying $h_{s_1}(0) \leq h_f(0) - d$. Since f_{s_1} is of the same form as f, we may apply the reasoning above to f_{s_1} in place of f. If we get a contradiction along the way, we are done. Otherwise we may proceed with the process and find an entire quotient f_{s_2} of exponential polynomials such that its indicator function $h_{s_2}(\theta)$ satisfies $h_{s_2}(0) \leq h_f(0) - 2d$. And so on. If the process continues indefinitely, we will find an entire quotient f_{s_M} of exponential polynomials such that its indicator function $h_{s_M}(\theta)$ satisfies

$$h_{s_M}(0) \leq h_f(0) - Md. \tag{7.6}$$

Here the integer M can be chosen to be large enough, so that

$$Md > h_f(0) + h_f(\pi/q) \geq 0. \tag{7.7}$$

Analogously as in (7.4), we have $h_{S_M}(\pi/q) \leq h_f(\pi/q)$. Combining this with (7.6) and (7.7), we obtain

$$h_{S_M}(0) + h_{S_M}(\pi/q) \leq h_f(0) + h_f(\pi/q) - Md < 0,$$

which contradicts with [9, p. 56]. Hence the assertion that the reduction process can be continued indefinitely is false, and the proof is complete. ∎

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