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Dual exponential polynomials and linear differential equations

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Abstract

We study linear differential equations with exponential polynomial coefficients, where exactly one coefficient is of order greater than all the others. The main result shows that a nontrivial exponential polynomial solution of such an equation has a certain dual relationship with the maximum order coefficient. Several examples illustrate our results and exhibit possibilities that can occur.

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1. Introduction

The solutions of the linear differential equation

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_1(z)f' + a_0(z)f = 0 \tag{1.1}$$

with entire coefficients $a_0(z), \ldots, a_{n-1}(z)$ are entire. To avoid ambiguity, we assume that $a_0(z) \not\equiv 0$. The following two classical theorems give general information concerning finite order solutions of (1.1).

Wittich's theorem. *[18, p. 6] The coefficients $a_0(z), \ldots, a_{n-1}(z)$ of (1.1) are polynomials if and only if all solutions of (1.1) are of finite order.*

Frei's theorem. *[1, p. 207] Suppose that at least one coefficient in (1.1) is transcendental, and that $a_j(z)$ is the last transcendental coefficient, that is, the coefficients $a_{j+1}(z), \ldots, a_{n-1}(z)$, if applicable, are polynomials. Then (1.1) possesses at most j linearly independent solutions of finite order.*

The following example illustrates Frei's theorem for $n = 3$ and $j = 2$.

Example 1. The functions $f_1(z) = e^z + z$, $f_2(z) = e^z - 1$ and $f_3(z) = z + 1$ are solutions of

$$f''' + \left(z - 1 + e^{-z}\right) f'' - (z + 1)f' + f = 0,$$

and any two of them are linearly independent.

Our focus is on finite order solutions of (1.1), in particular on exponential polynomial solutions. An *exponential polynomial* $f(z)$ is a function of the form

$$f(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)}, \tag{1.2}$$

where $P_j(z)$, $Q_j(z)$ are polynomials for $1 \le j \le k$. Observe that a polynomial is a special case of an exponential polynomial.

The literature contains numerous examples of exponential polynomial solutions of equations of the form (1.1) with exponential polynomial coefficients such that at least one coefficient is transcendental. This naturally occurring situation is related to the long-standing open problem of Gol'dberg, Ostrovskiǐ and Petrenko, which asks whether finite order transcendental solutions of (1.1) are always of completely regular growth whenever the coefficients of (1.1) are of completely regular growth, see Section 2.

It can also be noted that any exponential polynomial is a solution of an equation of the form (1.1) with polynomial coefficients, see [17].

Motivated by these considerations, we consider the situation when f is an exponential polynomial solution of (1.1) where exactly one coefficient, say $a_\mu(z)$, is a transcendental exponential polynomial, and all the other coefficients are exponential polynomials of order less than the order of a_μ. In this case, we show that f and $a_\mu(z)$ have a certain *dual* relationship. Several examples are given to illustrate the main result.

This paper is organized as follows. In Section 2, the completely regular growth problem is discussed. In order to state and prove the theorems, and discuss the examples, we give some preliminaries in Section 3. The theorem and examples for second order equations are given in Sections 4. The general theorem and more examples for arbitrary order differential equations (1.1) are given in Section 5. In Section 6, we observe some general growth properties of finite order solutions of a class of equations of the form (1.1). In Section 7, we give the proof of our theorem for second order equations, part of which follows from results in [7]. Then in Section 8, the proof of the general theorem for arbitrary order equations is given. Some parts of this general proof require new reasoning, while other parts are similar to the second order case.

2. Completely regular growth

The Phragmén–Lindelöf indicator function of an entire function $f(z)$ of finite order $\rho(f) > 0$ is given by

$$h_f(\theta) = \limsup_{r\to\infty} r^{-\rho(f)} \log|f(re^{i\theta})|$$

for $\theta \in [-\pi, \pi)$. We say that f is of completely regular growth (see [11, pp. 139-140] or [13, pp. 6-8]) if there exists a sequence of Euclidean discs $D(z_k, r_k)$ satisfying

$$\sum_{|z_k|\le r} r_k = o(r) \tag{2.1}$$

such that

$$\log|f(re^{i\theta})| = (h_f(\theta) + o(1))r^{\rho(f)}, \quad re^{i\theta} \notin \bigcup_k D(z_k, r_k), \tag{2.2}$$

as $r \to \infty$ uniformly in θ. For example, a transcendental exponential polynomial function is of completely regular growth, see [7, Lemma 1.3].

A set $E \subset \mathbb{C}$ which can be covered by a sequence of discs $D(z_k, r_k)$ satisfying (2.1) is known as a C_0-set. The projection of a C_0-set on the positive real axis has zero upper linear density.

Petrenko [12, pp. 104–112] has shown that transcendental solutions to linear differential equations (1.1) with polynomial coefficients are of completely regular growth. See also [16] for parallel discussions. Based on Petrenko's result, Gol'dberg and Ostrovskiĭ stated the following problem, see [5, p. 300].

Problem. Suppose that f is a finite order transcendental solution of (1.1) whose coefficients are entire functions of completely regular growth. Is it true that f is of completely regular growth?

Without the condition of completely regular growth for the coefficients, the answer to this problem is negative, see [5, p. 300].

It is natural to consider linear differential equations with transcendental exponential polynomial coefficients as an important special case of this problem. For the second order case, partial results are given in [7], where exponential polynomial solutions of second order equations with exponential polynomial coefficients are studied.

These considerations are additional motivations for our study of exponential polynomial solutions of equations of the form (1.1) with exponential polynomial coefficients, where exactly one coefficient is of order greater than all the others.

3. Preliminaries

Suppose that f is a transcendental exponential polynomial. Then from (1.2), f can be written in the *normalized* form

$$f(z) = F_0(z) + F_1(z)e^{w_1 z^q} + \cdots + F_m(z)e^{w_m z^q}, \tag{3.1}$$

where $q = \max\{\deg(Q_j)\} \geq 1$, the w_j are pairwise distinct non-zero constants, the coefficients $F_j(z)$ are exponential polynomials of order $\leq q-1$ such that $F_j(z) \not\equiv 0$ for $1 \leq j \leq m$, and $m \leq k$. From now on, we always assume that if a transcendental exponential polynomial is given in a normalized form, then its coefficient functions are exponential polynomials of strictly lower order.

Denote the set of conjugate leading coefficients of the function f in (3.1) by $W_f = \{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_m\}$, where $\overline{w}_0 = 0$ is related to the coefficient $F_0(z)$ when $F_0(z) \not\equiv 0$, and $W_f = \{\overline{w}_1, \ldots, \overline{w}_m\}$ when $F_0(z) \equiv 0$. The convex hull $\mathrm{co}(W_f)$ of the set W_f is the intersection of all convex sets containing W_f. Typically, $\mathrm{co}(W_f)$ is a polygon, unless the conjugate leading coefficients $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_m$ are collinear. In the latter case, $\mathrm{co}(W_f)$ reduces to a line segment. The vertices of the polygon $\mathrm{co}(W_f)$ are formed by some (possibly all) of the conjugate leading coefficients $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_m$. The circumference $C(\mathrm{co}(W_f))$ of $\mathrm{co}(W_f)$ plays an important role in describing the value distribution of f, see [14].

Now let h be a quotient of two transcendental exponential polynomials, say

$$h(z) = f(z)/g(z),$$

where f is of the form (3.1) and g is an exponential polynomial of the normalized form

$$g(z) = G_0(z) + G_1(z)e^{w_1 z^q} + \cdots + G_m(z)e^{w_m z^q}.$$

In these representations of f and g for the quotient h, we allow that some of the coefficients $F_j(z)$ or $G_t(z)$ may vanish identically, but we suppose that the matching coefficients $F_j(z)$ and $G_j(z)$ do not both vanish identically for any j. For the quotient h, define the set $W_h = \{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_m\}$. The proximity function of h is

$$m(r,h) = \big(C(\mathrm{co}(W_h)) - C(\mathrm{co}(W_g))\big)\frac{r^q}{2\pi} + o(r^q), \tag{3.2}$$

see [15, Satz 1]. In particular, if $g \equiv 1$, then $W_g = \{0\}$ and $C(\mathrm{co}(W_g)) = 0$. This yields [14, Satz 1] as a special case, namely

$$m(r,f) = C(\mathrm{co}(W_f^0))\frac{r^q}{2\pi} + o(r^q),$$

where $W_f^0 = W_f \cup \{0\}$.

Suppose that f is given in the normalized form (3.1). If the nonzero conjugate leading coefficients $\overline{w}_1, \ldots, \overline{w}_m$ of f all lie on one particular ray $\arg(z) = \theta$, then f is called *a simple exponential polynomial*. If g is another simple exponential polynomial such that $\rho(g) = \rho(f)$, where the non-zero conjugate leading coefficients of g all lie on the opposite ray $\arg(z) = -\theta$, then f and g are called *dual exponential polynomials*. For instance, $f(z) = e^z + e^{2z}$ and $g(z) = e^{-4z}$ are dual exponential polynomials.

4. Second order equations

To have a nontrivial finite order solution of (1.1) in the second order case

$$f'' + A(z) f' + B(z) f = 0, \tag{4.1}$$

it follows from the theorems of Wittich and Frei that either both entire coefficients $A(z), B(z)$ are polynomials or the coefficient $A(z)$ has to be transcendental. In the latter case essentially at most one finite order solution could exist because the others would be constant multiples of this solution.

The following is our main result on the second order equation (4.1), extending [7, Theorem 3.4].

Theorem 1. *Suppose that f is a nontrivial exponential polynomial solution of (4.1), where $A(z)$ and $B(z)$ are exponential polynomials satisfying $\rho(B) < \rho(A)$. Then f and $A(z)$ are dual exponential polynomials of order $q \in \mathbb{N}$, and f has the normalized representation*

$$f(z) = c + F_1(z)e^{w_1 z^q} + \cdots + F_m(z)e^{w_m z^q}, \tag{4.2}$$

where the non-zero constants w_j have the same argument, $m \in \mathbb{N}$, and $c \in \mathbb{C} \setminus \{0\}$.

In particular, if $\rho(Af') < q$, then $q = 1$ and

$$f(z) = c + \beta e^{\alpha z}, \quad A(z) = \gamma e^{-\alpha z} \quad \textit{and} \quad B(z) = \mu, \tag{4.3}$$

where $\alpha, \beta, \gamma, \mu \in \mathbb{C} \setminus \{0\}$.

For the reasons mentioned in Section 1, the proof of Theorem 1 will be given separately in Section 7. The next two examples illustrate Theorem 1.

Example 2. Following [3, Example 1], let $B(z) \not\equiv 0$ be a polynomial in z, and let $A(z) = -1 - B(z)(1 + e^{-z})$. Then $f(z) = e^z + 1$ is a solution of (4.1), where $f(z)$ and $A(z)$ are dual exponential polynomials.

Example 3. [2] The equation

$$f'' + e^{-z} f' + \alpha f = 0 \tag{4.4}$$

where $\alpha \neq 0$ is a constant, has a subnormal solution if and only if $\alpha = -m^2$ for a positive integer m. The subnormal solution is a polynomial of degree m in e^z, that is

$$f(z) = C_0 + C_1 e^z + \cdots + C_m e^{mz}, \tag{4.5}$$

where $C_0, \dots, C_m \in \mathbb{C}$ with $C_m \neq 0$. Thus, f in (4.5) and the coefficient e^{-z} in (4.4) are dual exponential polynomials. For example, $f(z) = 1 + 4e^z + 6e^{2z}$ solves (4.4) with $\alpha = -4$, while $f(z) = 1 + e^z$ solves (4.4) with $\alpha = -1$. The latter situation illustrates the particular case in Theorem 1.

Last, we observe that the coefficient e^{-z} in (4.4) has only one transcendental term while the solution f in (4.5) has m transcendental terms (where m can be arbitrarily large) by the following lemma.

Lemma 1. *In (4.5), we have $C_j \neq 0$ for $0 \leq j \leq m$.*

Proof. First we notice that $C_0 \neq 0$ follows from [3, Theorem 3]. Now suppose that $C_j = 0$ for some index j satisfying $0 < j < m$. Then there must exist an integer p such that $C_p = 0$ and $C_{p+1} \neq 0$. From substitution of (4.5) into (4.4), it can be seen that there is one term of the form

$$(p+1)C_{p+1}e^{pz} \not\equiv 0,$$

and no other term with e^{pz}. Since this is impossible from Borel's theorem [19, p. 77], this completes the proof. $\square$

5. The general result

In Section 4, we stated Theorem 1 for the exponential polynomial solutions of second order linear differential equations with exponential polynomial coefficients $A(z), B(z)$, where $\rho(A) > \rho(B)$. Theorem 1 is included in our general result for arbitrary order equations, Theorem 2 below, except that in the particular case $\rho(Af') < q$ in Theorem 1, less hypothesis is assumed and the stronger conclusion (4.3) is obtained.

The following two preliminary examples illustrate the duality phenomenon for third order linear differential equations. Differing from the second order case, the dominating coefficient can be in two different positions.

Example 4. The function $f(z) = e^{-z} + z - 1$ satisfies

$$f''' + (e^z - z)f'' - zf' + f = 0.$$

The coefficient $e^z - z$ and the solution $e^{-z} + z - 1$ are dual exponential polynomials.

Example 5. The function $f(z) = e^z - 1$ solves the equation

$$f''' - 2f'' + e^{-z}f' + f = 0.$$

The coefficient e^{-z} and the solution $e^z - 1$ are dual exponential polynomials.

Theorem 2. *Suppose that f is an exponential polynomial solution of (1.1), where for precisely one index $\mu \in \{1, \cdots, n-1\}$, $a_\mu(z)$ is a transcendental exponential polynomial such that for all $j \neq \mu$, $a_j(z)$ is an exponential polynomial satisfying $\rho(a_j) < \rho(a_\mu)$. Then either f is a* *polynomial of degree* $\le \mu - 1$ *or* f *and* $a_\mu(z)$ *are dual exponential polynomials of order* $q \in \mathbb{N}$. *In the latter case,* f *has the normalized representation*

$$f(z) = S(z) + F_1(z)e^{w_1 z^q} + \cdots + F_m(z)e^{w_m z^q}, \tag{5.1}$$

where the non-zero constants w_j *have the same argument,* $m \in \mathbb{N}$, *and* $S(z)$ *is a polynomial satisfying* $S(z) \not\equiv 0$ *and* $\deg(S) \le \mu - 1$.

In particular, if $\rho(a_\mu f^{(\mu)}) < q$ *and* $a_j(z)$ *are polynomials for* $j \neq \mu$, *then*

$$f(z) = S(z) + Q(z)e^{P(z)} \quad \text{and} \quad a_\mu(z) = R(z)e^{-P(z)}, \tag{5.2}$$

where $P(z)$, $Q(z)$, $R(z)$ *are polynomials and* $\deg(P) = q$.

The proof of Theorem 2 is postponed to Section 8. We now give several further examples which illustrate Theorem 2 and which also show possibilities that can occur.

Example 6. Differentiating Frei's equation (4.4) with $\alpha = -m^2$ gives

$$f''' + e^{-z} f'' - (e^{-z} + m^2) f' = 0. \tag{5.3}$$

Adding (4.4) and (5.3) yields

$$f''' + (e^{-z} + 1) f'' - m^2 f' - m^2 f = 0, \tag{5.4}$$

for which the function f in (4.5) is a solution. For instance, when $m = 2$, we see that $f(z) = 1 + 4e^z + 6e^{2z}$ satisfies

$$f''' + (e^{-z} + 1) f'' - 4f' - 4f = 0.$$

Example 7. The function $f(z) = 16 - 27e^{-2z} + 27e^{-3z}$ satisfies

$$f''' + (1/9)(9 + 9e^z + 4e^{2z}) f'' - 5f' + 3f = 0.$$

Here the coefficient $(1/9)(9+9e^z+4e^{2z})$ and the solution f both have two transcendental terms.

Example 8. Here are examples with $S(z) = cz + d$ in (5.1), where $c, d \in \mathbb{C}$. For any entire function $A(z)$ and any constants $c \neq 0$ and d, the linear polynomial $f(z) = cz + d$ satisfies

$$f''' + A(z) f'' - (z + d/c) f' + f = 0. \tag{5.5}$$

For the case when $c \neq 0$ and the solution is transcendental, $f(z) = e^z + z$ solves

$$f''' + ze^{-z} f'' - f = 0.$$

For the case when $c = 0$, $f(z) = 1 + e^{-z}$ satisfies

$$f''' - f'' + 2e^z f' + 2f = 0.$$

Observe that Example 1 also illustrates these situations.

The next example gives solutions of order two of third order differential equations.

Example 9. The function $f(z) = \exp\left(z^2\right) - 1$ solves the following two equations:

$$f''' + \left(\exp\left(-z^2\right) - 2z - 1\right) f'' - 4f' + \left(4z^2 + 2\right) f = 0,$$
$$f''' - 2zf'' - (2 + 2e^{-z^2})f' - 4zf = 0.$$

The function $f(z) = \exp\left(z^2/2 + z\right) + z + 1$ is a solution of

$$f''' + \left(\exp\left(-z^2/2 - z\right) - z - 1\right) f'' - f' - (z+1)f = 0.$$

The following example illustrates the particular case in Theorem 2 when $\rho(a_\mu f^{(\mu)}) < q$ and $a_j(z)$ are polynomials for $j \neq \mu$.

Example 10. The function $f(z) = \exp\left(z^2\right) + \sqrt{2}z + 2$ solves

$$f''' + \left(\left(z + \frac{5}{2}\sqrt{2}\right)\exp\left(-z^2\right)\right) f'' - (4z^2 - \sqrt{2}z + 5)f' - \left(2\sqrt{2}z^2 + 2z\right) f = 0,$$

and (5.2) is satisfied.

Examples 1–10 are illustrations of Theorem 2 for second and third order equations. Below is an example of Theorem 2 for higher order equations.

Example 11. Let n be an even number and μ be an integer such that $0 < \mu < n$. If $a_\mu(z) = e^{-z}$, $a_j = (-1)^j$ for $\mu < j < n$ and $a_j = (-1)^{j+1}$ for $0 \le j < \mu$, then $f(z) = e^z + 1$ solves the equation

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_\mu(z)f^{(\mu)} + \cdots + a_1f' + a_0f = 0.$$

6. Finite order solutions

It is natural to ask what properties exponential polynomial solutions of equations of the form in Theorem 2 have in common with the larger class of finite order solutions. Theorem 3 below gives such a comparison for the class of equations in Theorem 2, where the coefficients $a_j(z)$ are polynomials for $j \neq \mu$.

An R-set in the complex plane is a countable union of discs whose radii have a finite sum. Thus any R-set is clearly a C_0-set. The union of two R-sets is again an R-set, and the set of angles θ for which the ray $\arg(z) = \theta$ meets infinitely many discs of a given R-set has linear measure zero, see [9, pp. 84-85].

It is known that the majority of the zeros of a given transcendental exponential polynomial $a(z)$ of the normalized form (3.1) lie within finitely many ε-sectors with arbitrarily small openings (ε-logarithmic strips to be more exact, see [8,14] for more details). In any sector between these ε-sectors, $|a(z)|$ has three possible growth types:

(1) polynomial growth without an exceptional set;
(2) blows up exponentially outside of a possible exceptional R-set;
(3) decays to zero exponentially without an exceptional set.

This is clear if $m = 1$. In general, in between any two consecutive ε-sectors there exists a unique index $j \in \{0, \dots, m\}$ such that

$$a(z) = F_j(z)e^{w_j z^q}(1 + o(1))$$

as $z \to \infty$ outside of a possible R-set, see [8,14]. The index j changes when moving from one sector to another. The R-set contains the zeros of the coefficients $F_j(z)$, and hence disappears for polynomial coefficients.

The growth type (2) must always exist because $a(z)$ is a transcendental exponential polynomial, while neither of the growth types (1) or (3) is necessary. In particular, the growth types for $a(z)$ can occur in three possible combinations: only (2), (2) + (3) and (1) + (2).

Theorem 3. *Suppose that f is a finite order solution of (1.1), where $a_\mu(z)$ is a transcendental exponential polynomial for precisely one index $\mu \in \{1, \dots, n-1\}$, while $a_j(z)$ $(j \neq \mu)$ are polynomials. Then either f is a polynomial of degree $\leq \mu - 1$ or $\rho(f) \geq \rho(a_\mu)$. Moreover, the following conditions hold.*

(a) *If $a_\mu(z)$ has only the growth type (2), then f is a polynomial of degree $\leq \mu - 1$.*
(b) *If $a_\mu(z)$ has only the growth types (2) and (3), then*

$$\rho(f) \leq 1 + \max_{j \neq \mu} \left\{ \frac{\deg(a_j)}{n - j} \right\}. \tag{6.1}$$

(c) *If $a_\mu(z)$ has only the growth types (1) and (2), and if $|a_\mu(z)| \leq \alpha |z|^\beta$ holds in any sector of type (1), then*

$$\rho(f) \leq 1 + \max \left\{ \frac{\beta}{n - \mu}, \max_{j \neq \mu} \frac{\deg(a_j)}{n - j} \right\}. \tag{6.2}$$

Choosing $A(z) = e^z + e^{-z}$ in (5.5) gives an example of Part (a). Example 10 has a solution for which (6.1) is an equality. Moreover, in Part (b), we see that

$$\rho(a_\mu) \leq 1 + \max_{j \neq \mu} \left\{ \frac{\deg(a_j)}{n - j} \right\} \tag{6.3}$$

must hold in order for transcendental finite order solutions to exist. Consequently, if equality in (6.3) holds, then all transcendental finite order solutions f satisfy $\rho(f) = \rho(a_\mu)$. Example 10 shows that equality in (6.3) is possible. In Part (c), each of the three equations in Example 9 has a solution for which (6.2) is an equality.

Proof of Theorem 3. We see immediately from (1.1) that $\rho(f) \geq \rho(a_\mu)$ for any solution f, unless $f^{(\mu)} \equiv 0$. If $f^{(\mu)} \equiv 0$, then f is a polynomial with degree at most $\mu - 1$.

Suppose that $|a_\mu(z)|$ blows up exponentially in a sector $S_1 = \{z : \theta_1 \le \operatorname{Arg} z \le \theta_2\}$, where $\theta_1 < \theta_2$, outside of a possible exceptional R-set. Using the reasoning in the proof of [4, Theorem 1], we deduce that $|f^{(\mu)}(z)|$ decays to zero exponentially in the sector $S(\varepsilon) = \{z : \theta_1 + \varepsilon \le \operatorname{Arg} z \le \theta_2 - \varepsilon\}$, outside of a possible exceptional R-set. Let $\operatorname{Arg} z = \varphi$ be any ray in the sector $S(\varepsilon)$ that meets at most finitely many discs of the R-set. Since

$$f^{(\mu-1)}(re^{i\varphi}) = f^{(\mu-1)}(0) + \int_0^{re^{i\varphi}} f^{(\mu)}(\zeta)\, d\zeta,$$

we see that $f^{(\mu-1)}(z)$ is bounded on the ray $\operatorname{Arg} z = \varphi$. Since this reasoning works for almost every ray in the sector $S(\varepsilon)$, and since f is of finite order, we conclude by the Phragmén–Lindelöf principle that $f^{(\mu-1)}(z)$ is bounded in $S(\varepsilon)$. Thus, if $a_\mu(z)$ has only the growth type (2), we deduce by the Phragmén–Lindelöf principle that $f^{(\mu-1)}(z)$ is bounded everywhere, and hence reduces to a constant by Liouville's theorem. This proves Part (a).

Suppose then that there exists a sector S_2 in which $|a_\mu(z)|$ decays to zero. We make use of the general growth estimate [6, Theorem 5.1], according to which

$$|f(re^{i\theta})| \le C \exp\left(n_c \int_0^r \max_{0\le j\le n-1} |a_j(te^{i\theta})|^{\frac{1}{n-j}}\, dt \right), \tag{6.4}$$

where $n_c \in \{1, \dots, n\}$ is the number of nonzero coefficients, $\theta \in [\theta_1, \theta_2]$ is arbitrary, and the constant $C > 0$ depends on the initial values of f but not on θ. Since $|a_\mu(z)|$ decays to zero, we may ignore the index $j = \mu$ in (6.4). By the Phragmén–Lindelöf principle, we have

$$\log^+ |f(z)| = O\left(|z|^{1+\max_{j\ne\mu}\left\{\frac{\deg(a_j)}{n-j}\right\}} \right)$$

uniformly in S_2. By the previous paragraph, if $|a_\mu(z)|$ blows up exponentially in some sector, then f has at most polynomial growth in a slightly smaller sector. Using the Phragmén–Lindelöf principle again to avoid the ε-sectors, we conclude the proof of Part (b).

Finally we suppose that there exists at least one sector S_3 in which $|a_\mu(z)| \le \alpha |z|^\beta$. We use the general growth estimate (6.4) to obtain

$$\log^+ |f(z)| = O\left(|z|^{1+\max\left\{\frac{\beta}{n-\mu}, \max_{j\ne\mu} \frac{\deg(a_j)}{n-j}\right\}} \right)$$

uniformly in S_3. The discussion above shows that f has polynomial growth whenever $a_\mu(z)$ has the growth type (2). The ε-sectors in between the two growth types can once again be dealt with by using the Phragmén–Lindelöf principle. This proves Part (c). $\square$

7. Proof of Theorem 1

It can be seen from [7, Theorem 3.6] that $\rho(f)=\rho(A)\geq 1$, and that f has the normalized form

$$f(z)=c+F_1(z)e^{w_1z^q}+\cdots+F_m(z)e^{w_mz^q}, \tag{7.1}$$

where $c\neq 0$ is a constant and where the coefficients $F_j(z)$ are exponential polynomials of order $\leq q-1$. Differentiating f yields

$$f'(z)=G_1(z)e^{w_1z^q}+\cdots+G_m(z)e^{w_mz^q},$$

where $G_j(z)=F_j'(z)+qw_jz^{q-1}F_j(z)\not\equiv 0$ for $j=1,\ldots,m$. Denote the set of conjugate leading coefficients of f by $W_f=\{\overline{w}_0=0,\overline{w}_1,\ldots,\overline{w}_m\}$.

We now write $A(z)$ in the normalized form

$$A(z)=A_0(z)+A_1(z)e^{\lambda_1z^q}+\cdots+A_p(z)e^{\lambda_pz^q}. \tag{7.2}$$

Since $\rho(f)<\infty$ and $B(z)$ is an exponential polynomial of order $<q$, it follows from (4.1) that

$$m\left(r,\frac{Af'}{f}\right)=O(r^{q-1}). \tag{7.3}$$

Suppose that $\rho(Af')=q$. According to Examples 2 and 3, either of the cases $A_0(z)\not\equiv 0$ or $A_0(z)\equiv 0$ is possible. If $A_0(z)\not\equiv 0$, then the set of conjugate leading coefficients of the exponential polynomial $A(z)f'$ is

$$W_{Af'}=\{\overline{w}_1,\ldots,\overline{w}_m,\overline{w}_i+\overline{\lambda}_j\}, \tag{7.4}$$

where $i=1,\ldots,m$ and $j=1,\ldots,p$. If $A_0(z)\equiv 0$, then

$$W_{Af'}=\{\overline{w}_i+\overline{\lambda}_j\}, \tag{7.5}$$

where again $i=1,\ldots,m$ and $j=1,\ldots,p$. Following [15], for the quotient $g=A(z)f'/f$, we set $W_g=W_{Af'}\cup W_f$. In both cases (7.4) and (7.5), we have

$$W_g=\{\overline{w}_0,\overline{w}_1,\ldots,\overline{w}_m,\overline{w}_i+\overline{\lambda}_j\}. \tag{7.6}$$

By [15, Satz 1], we deduce that

$$m\left(r,\frac{Af'}{f}\right)=\big(C(\operatorname{co}(W_g))-C(\operatorname{co}(W_f))\big)\frac{r^q}{2\pi}+o\left(r^q\right).$$

But (7.3) then gives us

$$C(\operatorname{co}(W_g))=C(\operatorname{co}(W_f)). \tag{7.7}$$

Since $W_f \subset W_g$, we have $\mathrm{co}(W_f) \subset \mathrm{co}(W_g)$. If $\mathrm{co}(W_f) \neq \mathrm{co}(W_g)$, then at least one of the points $\overline{w}_i + \overline{\lambda}_j$ must be a vertex of $\mathrm{co}(W_g)$ but not a vertex of $\mathrm{co}(W_f)$. However, introducing a new vertex ζ to any convex polygon $\mathcal{C}$ increases its circumference because ζ has to be an exterior point of $\mathcal{C}$ in order to maintain convexity. But this contradicts with (7.7), and so $\mathrm{co}(W_f) = \mathrm{co}(W_g)$ must hold.

Suppose for the sake of argument that the points $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_m$ are not collinear. Then $\mathrm{co}(W_f)$ has at least three vertices and a nonempty interior. Let

$$\overline{z}_1, \ldots, \overline{z}_k \in \{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_m\}, \quad 3 \leq k \leq m+1,$$

be the vertex points of $\mathrm{co}(W_f)$. Since $\lambda_1 \neq 0$, a simple vector calculus applied to the convex polygon $\mathrm{co}(W_f)$ shows that we must have $\overline{z}_j + \overline{\lambda}_1 \notin \mathrm{co}(W_f) = \mathrm{co}(W_g)$ for some $j = 1, \ldots, k$. But this is a contradiction, and so $\mathrm{co}(W_f) = \mathrm{co}(W_g)$ is a line segment.

We have proved that all the conjugate leading coefficients of f and of $A(z)$ are collinear. By appealing to a suitable rotation, we may suppose that the leading coefficients of f and of $A(z)$ are on the real axis. Regarding the non-zero leading coefficients of f and of $A(z)$, we may assume, without loss of generality, that $w_1 < \ldots < w_m$ and $\lambda_1 < \ldots < \lambda_p$. We now prove the following two inequalities

$$w_m\lambda_p < 0 \quad \text{and} \quad w_1\lambda_1 < 0. \tag{7.8}$$

(1) If $w_m > 0$ and $\lambda_p > 0$, then $\overline{w}_m + \overline{\lambda}_p \notin \mathrm{co}(W_f)$, which is a contradiction. If $w_m < 0$ and $\lambda_p < 0$, then $\overline{w}_1 + \overline{\lambda}_j \notin \mathrm{co}(W_f)$ for every $j = 1, \ldots, p$, which is impossible. Then $w_m\lambda_p < 0$ follows by $w_m\lambda_p \neq 0$.
(2) If $w_1 < 0$ and $\lambda_1 < 0$, then $\overline{w}_1 + \overline{\lambda}_1 \notin \mathrm{co}(W_f)$, which is a contradiction. If $w_1 > 0$ and $\lambda_1 > 0$, then $\overline{w}_m + \overline{\lambda}_j \notin \mathrm{co}(W_f)$ for every $j = 1, \ldots, p$, which is impossible. Then $w_1\lambda_1 < 0$ follows by $w_1\lambda_1 \neq 0$.

By (7.8), the non-zero leading coefficients of f and of $A(z)$ are located on the opposite sides of the origin. This final observation shows that f and $A(z)$ are dual exponential polynomials.

Now suppose that $\rho(Af') < q$. Then $\max\{\lambda(A), \lambda(f')\} < q$, where $\lambda(v)$ denotes the exponent of convergence of the sequence of zeros of an entire function $v(z)$ ($v(z) \not\equiv 0$). From the proof of [7, Theorem 3.4(3)], we deduce that $A(z)f'$ is a constant function, $q = 1$, and $m = p = 1$. In particular, $w_1 = -\lambda_1$, from which the assertions for $f(z)$ and $A(z)$ in (4.3) follow. But then $B(z)$ is a non-zero constant from [10, Theorem 1]. This proves (4.3).

8. Proof of Theorem 2

In order to prove Theorem 2, we need two lemmas.

Lemma 2. *Suppose that f is an exponential polynomial solution of (1.1), where $a_\mu(z)$ is a transcendental exponential polynomial and $a_j(z)$ ($j \neq \mu$) are exponential polynomials of order $< \rho(a_\mu)$. Then either f is a polynomial with degree at most $\mu - 1$ or $\rho(f) = \rho(a_\mu)$.*

Proof. We see immediately from (1.1) that $\rho(f) \geq \rho(a_\mu)$ for any solution f, unless $f^{(\mu)} \equiv 0$. If $f^{(\mu)} \equiv 0$, then f is a polynomial with degree at most $\mu - 1$.

To finish the proof, we suppose that $q=\rho(f)>\rho(a_\mu)\geq 1$, and aim for a contradiction. Then the exponential polynomial solution f can be written as

$$f(z)=P_{0,0}(z)+P_{1,0}(z)e^{R_1(z)}+\cdots+P_{l,0}(z)e^{R_l(z)}, \tag{8.1}$$

where the coefficients $P_{j,0}(z)$ and $R_j(z)$ are ordinary polynomials for $1\leq j\leq l$, $\deg(R_j)=q$, and $R_t-R_j\not\equiv 0$ for $t\neq j$, and $P_{0,0}(z)$ is an exponential polynomial of order $<q$. Differentiating f for i times, we get

$$f^{(i)}(z)=P_{0,i}(z)+P_{1,i}(z)e^{R_1(z)}+\cdots+P_{l,i}(z)e^{R_l(z)},$$

where $P_{j,i}=P'_{j,i-1}+R'_jP_{j,i-1}\not\equiv 0$ are polynomials for $j=1,\dots,l$ and $P_{0,i}=P_{0,0}^{(i)}$ for $1\leq i\leq n$. By substituting (8.1) into (1.1), we obtain

$$P_{0,n}+\sum_{j=0}^{n-1}a_j(z)P_{0,j}+\sum_{t=1}^{l}\left(\sum_{j=0}^{n-1}a_j(z)P_{t,j}+P_{t,n}\right)e^{R_t(z)}=0.$$

By the assumption $q>\rho(a_\mu)$ and by Borel's theorem [19, p. 77], it follows that

$$P_{0,n}+a_{n-1}(z)P_{0,n-1}+\cdots+a_1(z)P_{0,1}+a_0(z)P_{0,0}\equiv 0 \tag{8.2}$$

and

$$P_{j,n}+a_{n-1}(z)P_{j,n-1}+\cdots+a_1(z)P_{j,1}+a_0(z)P_{j,0}\equiv 0,\ \ j=1,\dots,l. \tag{8.3}$$

Since $\rho(a_\mu)\geq 1$ and since all other terms in (8.3) are exponential polynomials of order $<\rho(a_\mu)$, we must have $P_{j,\mu}(z)\equiv 0$ for all $j=1,\dots,l$, which is a contradiction from above. This proves Lemma 2. $\square$

Lemma 3. *Suppose that f is a transcendental exponential polynomial in normalized form*

$$f(z)=H_0(z)+H_1(z)e^{w_1z^p}+\cdots+H_m(z)e^{w_mz^p}, \tag{8.4}$$

where $p\geq 1$. If f is a solution of a differential equation

$$f'+G(z)f=Q(z)e^{P(z)}, \tag{8.5}$$

where $P(z), Q(z), G(z)$ are polynomials with $PQG\not\equiv 0$ and $\deg(G)\geq p$, then $f(z)=Q_1(z)e^{P_1(z)}$, where $P_1(z)$, $Q_1(z)$ are polynomials such that $\deg(P_1)=\deg(P)=p$.

Proof. The derivative of f is

$$f'(z)=H_0'(z)+G_1(z)e^{w_1z^p}+\cdots+G_m(z)e^{w_mz^p},$$

where $G_i=H_i'+pw_iz^{p-1}H_i\not\equiv 0$ for $1\leq i\leq m$. Substituting f and f' into (8.5), we have

$$\sum_{i=0}^{m}(G_i + GH_i)e^{w_i z^p} = Q(z)e^{P(z)},$$

where $w_0 = 0$.

Suppose that $m \geq 2$. By Borel's theorem, it follows that there exists an $i \in \{0, 1, \ldots, m\}$ such that

$$G_i + GH_i = Q(z)e^{P(z)-w_i z^p}, \tag{8.6}$$

$$G_j + GH_j = 0, \quad j \neq i. \tag{8.7}$$

We may re-write (8.7) as

$$H_j' + \left(pw_j z^{p-1} + G(z)\right) H_j = 0, \quad j \neq i. \tag{8.8}$$

We assume that $P(z)$ is not a constant. In particular, $H_0' + G(z)H_0 = 0$ because $w_0 = 0$. This gives either $H_0(z) \equiv 0$ or $\rho(H_0) = \deg(G) + 1$, of which the latter is impossible. Since $\deg(G) \geq p$, we obviously have $pw_j z^{p-1} + G(z) \not\equiv 0$ for all $j \neq i$. Then (8.8) gives either $H_j(z) \equiv 0$ or $\rho(H_j) \geq p + 1$, which are both contradictions. Now suppose that $P(z)$ is a constant. Then from (8.6) and (8.7), we have $H_j \equiv 0$ for $j \neq 0$, which contradicts $\rho(f) = p$.

Therefore, $m = 1$. This gives (8.6) for $i = 1$, while (8.7) is $H_0' + GH_0 = 0$. Hence, $H_0(z) \equiv 0$. Thus, the solution f can be written as $f(z) = H_1(z)e^{w_1 z^q}$. Meanwhile, equation (8.6) with $i = 1$ can be re-written as

$$\left(H_1' + \left(pw_1 z^{p-1} + G(z)\right) H_1\right) e^{w_1 z^p} = Q(z)e^{P(z)},$$

which implies $\deg(P) = p$. Denoting $P_1(z) := P(z) - w_1 z^p$, we see that $H_1 \not\equiv 0$ with $\rho(H_1) \leq p - 1$ is an exponential polynomial solution of

$$H_1' + \left(pw_1 z^{p-1} + G(z)\right) H_1 = Q(z)e^{P_1(z)}.$$

If $H_1(z)$ is a polynomial, we are done. Otherwise, $H_1(z)$ is a transcendental exponential polynomial solution of

$$H_1' + G_1(z)H_1 = Q(z)e^{P_1(z)}, \tag{8.9}$$

where $G_1(z) = pw_1 z^{p-1} + G(z)$ and $\rho(H_1) = \rho(P_1) \leq p - 1$. Equation (8.9) now takes the role of equation (8.5), for which f in (8.4) is a solution. We conclude that $H_1(z)$ has only one exponential term. By continuing in this way, we finally have $f(z) = Q_1(z)e^{P_2(z)}$, where $Q_1(z)$ and $P_2(z)$ are polynomials. This proves Lemma 3. □

Proof of Theorem 2. By Lemma 2 we may suppose that $\rho(f) = \rho(a_\mu) = q \geq 1$. Write f in normalized form as

$$f(z) = F_0(z) + F_{1,0}(z)e^{w_1 z^q} + \cdots + F_{m,0}(z)e^{w_m z^q},$$

and $a_\mu(z)$ in normalized form as

$$a_\mu(z) = A_0(z) + A_1(z)e^{\lambda_1 z^q} + \cdots + A_s(z)e^{\lambda_s z^q}.$$

Then

$$f^{(\mu)}(z) = F_0^{(\mu)}(z) + F_{1,\mu}(z)e^{w_1 z^q} + \cdots + F_{m,\mu}(z)e^{w_m z^q},$$

where $F_{j,\mu}(z) = F'_{j,\mu-1}(z) + qw_j z^{q-1}F_{j,\mu-1}(z) \not\equiv 0$ for $j = 1, \dots, m$. In particular, the exponential polynomials f and $f^{(\mu)}$ have the same non-zero leading coefficients.

Writing

$$|a_\mu(z)| \le \left|\frac{f^{(n)}(z)}{f^{(\mu)}(z)}\right| + |a_{n-1}(z)|\left|\frac{f^{(n-1)}(z)}{f^{(\mu)}(z)}\right| + \cdots + |a_0(z)|\left|\frac{f(z)}{f^{(\mu)}(z)}\right|,$$

we obtain

$$m(r, a_\mu) \le m\left(r, \frac{f^{(\mu-1)}(z)}{f^{(\mu)}(z)}\right) + \cdots + m\left(r, \frac{f(z)}{f^{(\mu)}(z)}\right) + O(r^{q-1}).$$

By (3.2) we deduce that zero is a leading coefficient of at least one of $f, f', \cdots, f^{(\mu-1)}$ but not of $f^{(\mu)}$. Then $F_0(z)$ is a polynomial satisfying $F_0(z) \not\equiv 0$ and $\deg(F_0) \le \mu - 1$. Thus the set $W_f = \{\overline{w}_0 = 0, \overline{w}_1, \dots, \overline{w}_m\}$ represents the conjugate leading coefficients of f.

Suppose that $\rho(a_\mu f^{(\mu)}) = q$. Similarly as in (7.6), for the exponential quotient $h = a_\mu(z)f^{(\mu)}/f$, we set $W_h = W_{a_\mu f^{(\mu)}} \cup W_f$, which is defined by

$$W_h = \{0, \overline{w}_1, \dots, \overline{w}_m, \overline{w}_i + \overline{\lambda}_j\},$$

where $i = 1, \dots, m$ and $j = 1, \dots, p$. Since a_j $(j \ne \mu)$ satisfy $\rho(a_j) \le q - 1$, (1.1) yields

$$m\left(r, \frac{a_\mu f^{(\mu)}}{f}\right) = O(r^{q-1}).$$

Then the reasoning related to (7.7) gives us $\mathrm{co}(W_f) = \mathrm{co}(W_h)$, and finally that f and $a_\mu(z)$ are dual exponential polynomials.

Now suppose that $\rho(a_\mu f^{(\mu)}) < q$ and $a_j(z)$ are polynomials for $j \ne \mu$. Then the exponents of convergence of the zeros of $a_\mu(z)$ and $f^{(\mu)}$ satisfy $\max\{\lambda(a_\mu), \lambda(f^{(\mu)})\} < q$. The inequality $\lambda(a_\mu) < q$ is possible only if $A_0(z) \equiv 0$ and $s = 1$. Keeping in mind that $F_0(z)$ is a non-zero polynomial with degree at most $\mu - 1$, the inequality $\lambda(f^{(\mu)}) < q$ is possible only if $m = 1$. In other words,

$$a_\mu(z) = A_1(z)e^{\lambda_1 z^q} \quad \text{and} \quad f^{(\mu)}(z) = F_{1,\mu}(z)e^{w_1 z^q}.$$

Thus $w_1 = -\lambda_1$ must hold, for otherwise $\rho(Af^{(\mu)}) = q$. Substituting f and $a_\mu(z)$ into (1.1), and noting that $F_0^{(\mu)} \equiv 0$, we obtain

$$A_1(z)F_{1,\mu}(z)+\sum_{j=0}^{\mu-1}a_jF_0^{(j)}+\left(\sum_{\substack{j=0\\ j\neq\mu}}^{n-1}a_jF_{1,j}+F_{1,n}\right)e^{w_1z^q}=0.$$

Borel's theorem then yields

$$A_1(z)F_{1,\mu}(z)+\sum_{j=0}^{\mu-1}a_jF_0^{(j)}=0. \tag{8.10}$$

From (8.10) we see that $A_1(z)F_{1,\mu}(z)$ is a polynomial. Thus there exist polynomials $P(z)$, $Q_1(z)$, $R(z)$ such that $\deg(P)\leq q-1$, $Q_1(z)R(z)\not\equiv 0$ and

$$F_{1,\mu}(z)=Q_1(z)e^{P(z)} \quad \text{and} \quad A_1(z)=R(z)e^{-P(z)}.$$

The former identity can be re-written in the form

$$F'_{1,\mu-1}+qw_1z^{q-1}F_{1,\mu-1}=Q_1(z)e^{P(z)}.$$

Using Lemma 3, we have that $F_{1,\mu-1}(z)=Q_2(z)e^{P(z)}$, where $Q_2(z)$ is a polynomial. By considering the equation

$$F'_{1,\mu-2}+qw_1z^{q-1}F_{1,\mu-2}=Q_2(z)e^{P(z)}$$

and repeating the process, we get $F_{1,\mu-2}=Q_3e^{P(z)}$, where Q_3 is a polynomial. Then by repeating the process again, we can deduce that there exists a polynomial $Q_4(z)$ such that

$$F'_{1,0}+qw_1z^{q-1}F_{1,0}=Q_4(z)e^{P(z)},$$

from which $F_{1,0}(z)=Q(z)e^{P(z)}$, where $Q(z)$ is a polynomial. Hence, we obtain that

$$f(z)=F_0(z)+Q(z)e^{P(z)+w_1z^q} \quad \text{and} \quad a_\mu(z)=R(z)e^{-P(z)-w_1z^q},$$

which yields the conclusion (5.2). □

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