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FINITE ORDER SOLUTIONS OF DIFFERENCE EQUATIONS, AND DIFFERENCE PAINLEVÉ EQUATIONS IV

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ABSTRACT. In this paper, from the non-linear difference equation

$$(\overline{w}+w)(w+\underline{w})=\frac{P(z,w)}{Q(z,w)}$$

where $P(z,w)$ and $Q(r,w)$ are polynomials in $w(z)$ without common factors having small function coefficients related to $w(z)$, we present the form of difference Painlevé equation IV

$$(\overline{w}+w)(w+\underline{w})=\frac{(w^2-a^2)(w^2-b^2)}{(w+(\alpha z+\beta))^2+\pi},$$

where a, b, α, β and π are period small functions related to w. It shows that if the above difference equation admits at least one meromorphic solution $w(z)$ of finite order, then the difference equation can be transformed by Möbius tranformation in w to difference Painlevé IV, unless w is the solution of difference Riccati equations.

1. INTRODUCTION

Painlevé equations are defined by second order ordinary differential equations whose solutions have the Painlevé property: the only movable singularities are poles. Some results related to Painlevé equations could be found in [10]. As for an analogue of the Painlevé property of difference equations, Grammaticos et al. in [9] showed that there exists a family of non-autonomous integrable mappings which includes the discrete Painlevé equations P_I and P_{II}. Ramani et al. in [15] presented discrete forms of the Painlevé equations P_{III}, P_{IV} and P_V that complement the list of the discrete Painlevé equations P_I and P_{II}.

The analogues of the Painlevé property for difference equations in the complex plane have been discussed, for example, in [1], where Ablowitz et al. considered the solutions of difference Painlevé equations by Nevanlinna theory, which is a landmark in the application of Nevanlinna theory in the study of difference equations. They observed that all of the relevant difference equations have obvious analytic versions, and hence can be studied using the methods of complex analysis, and in particular those of Nevanlinna theory.

Halburd and Korhonen in [5] implemented the approach which was suggested by Ablowitz et al. in [1]. Let us recall some definitions and notation here which were

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used in their paper. A meromorphic function g is in $S(y)$ if

$$\lim_{r\to\infty}\frac{T(r,g)}{T(r,y)}=0,$$

where r runs to infinity anywhere outside of a set E satisfying $\int_E 1/t\,dt<\infty$. The field $S(y)$ is often referred to as the field of small functions with respect to y. A transcendental meromorphic solution y of a difference (or differential) equation is called admissible if all coefficients of the equations are in $S(y)$. In the following, we write $w=w(z)$, $\overline{w}=w(z+1)$, $\underline{w}=w(z-1)$ for brevity.

In [5] it was shown that if the difference equation

$$\overline{w}+\underline{w}=R(z,w), \tag{1.1}$$

where $R(r,w)$ is rational in w with meromorphic coefficients, has an admissible meromorphic solution of finite order, then the class (1.1) reduces into a short list of equations, where the difference Painlevé equation I and the difference Painlevé equation II are included.

In [6], Halburd and Korhonen showed that if the difference equation

$$\overline{w}\underline{w}=\frac{c_2(w-c_+)(w-c_-)}{(w-a_+)(w-a_-)} \tag{1.2}$$

has an admissible finite order meromorphic solution such that the order of its poles is bounded, then the equation (1.1) can be transformed by Möbius tranformation in w to a short list of equations, where the difference Painlevé III is included, unless w is the solution of difference Riccati equation. Ronkainen in [7] removed the assumption on the boundedness of the order of the poles; difference Painlevé III can be obtained from equation (1.2) when the coefficients are special solutions of equations. Similarly, difference Painlevé V is also considered by Ronkainen in [7].

In this paper, we consider the existence of finite order meromorphic solutions of the difference equation

$$(\overline{w}+w)(w+\underline{w})=R(r,w):=\frac{P(z,w)}{Q(z,w)} \tag{1.3}$$

where $P(z,w)$ and $Q(r,w)$ are polynomials in $w(z)$ without common factors having small function coefficients related to $w(z)$, which is as a detector of difference Painlevé equation IV. We will present the main result of this paper in Section 3.

2. Nevalinna theory and difference equations

A difference analogue lemma on the logarithmic derivative of meromorphic functions of finite order was found by Halburd and Korhonen [3,4], and by Chiang and Feng [2], independently. This result was recently extended to meromorphic functions of hyper order strictly less than one in [11], where it was also shown that this growth condition cannot be essentially relaxed further. Let us state the latest version of a difference analogue lemma on the logarithmic derivative of meromorphic functions as follows.

Lemma 2.1 ([11], Theorem 5.1). *Let w be a non-constant meromorphic function and $c\in\mathbb{C}$. If w is of finite order, then*

$$m\left(r,\frac{w(z+c)}{w(z)}\right)=O\left(\frac{\log r}{r}T(r,w)\right)$$

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for all r outside of a set E satisfying

$$\limsup_{r\to\infty} \frac{\int_{E\cap[1,r)} dt/t}{\log r} = 0,$$

i.e., outside of a set E of zero logarithmic density. If $\rho_2(w) = \rho_2 < 1$ and $\varepsilon > 0$, then

$$m\left(r, \frac{w(z+c)}{w(z)}\right) = o\left(\frac{T(r,w)}{r^{1-\rho_2-\varepsilon}}\right)$$

for all r outside of a set of finite logarithmic measure.

The following result on the Nevanlinna characteristic is essential in investigating the difference equation (1.3).

Lemma 2.2 ([10], Proposition B. 15 or [8]). *For three meromorphic functions f, g and h the Nevanlinna characteristic satisfies*

$$T(r, fg + gh + hf) \leq T(r,f) + T(r,g) + T(r,h) + O(1).$$

The following lemma is an improved version of [3, Lemma 2.2].

Lemma 2.3 ([11], Lemma 8.3). *Let $T : [0,\infty) \to [0,\infty)$ be a non-decreasing continuous function and let $s \in (0,\infty)$. If the hyper order of T is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \rho_2 < 1,$$

and $\delta \in (0, 1-\rho_2)$, then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right),$$

where r runs to infinity outside of a set of finite logarithmic measure.

The difference version of a result due to A. Mohon′ko and V. Mohon′ko [14], was found by Halburd and Korhonen [4], and Laine and Yang [12, Theorem 2.4], independently. Let us state this result as follows.

Lemma 2.4 ([4], Corollary 3.4). *Let $w(z)$ be a non-constant finite order meromorphic solution of*

$$P(z,w) = 0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a) \not\equiv 0$ for a meromorphic function $a \in S(w)$, then

$$m\left(\frac{1}{w-a}\right) = S(r,w)$$

for all r outside of a possible exceptional set with finite logarithmic measure.

The following result is the difference analogue of the Clunie lemma due to Laine and Yang, which is the generalization version of the result in [4, Theorem 3.1]. Let us state it as follows.

Lemma 2.5 ([12], Theorem 2.3). *Let w be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$H(z,w)P(z,w) = Q(z,w)$$

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where $H(z,f)$, $P(z,w)$, $Q(z,f)$ are difference polynomials such that the total degree n in $w(z)$ and its shifts, and the total degree of $Q(z,w)$ is $\le n$. Then, for each $\varepsilon > 0$,

$$m(r, P(z,w)) = O(r^{\rho-1+\varepsilon}) + S(r,w),$$

possibly outside of an exceptional set of finite logarithmic measure.

3. Main result

Equation (1.3) can be written as

$$\overline{w}\underline{w} + w\underline{w} + \overline{w}w = \frac{P(r,w) - w^2 Q(r,w)}{Q(r,w)} := K(z,w). \tag{3.1}$$

By application of a result originally due to Valiron [16] and generalized by Mohon′ko [13], Lemma 2.3 and Lemma 2.2 with $f = \overline{w}$, $g = w$ and $h = \underline{w}$, yields

$$\begin{aligned}(\deg_w K)T(r,w) &\le T(r,w) + T(r,\overline{w}) + T(r,\underline{w}) + S(r,w)\\ &\le (3+2\varepsilon)T(r,w) + S(r,w),\end{aligned}$$

for every $\varepsilon > 0$, which implies that $\deg_w(K) \le 3$. It might happen when $q \le 3$ and $p \le 5$, since P and Q have no common factors.

Next, we proceed to prove that $q = 3$ is impossible. Let

$$G(r,w) := P(r,w) - w^2 Q(z,w).$$

We see that $p = 5$ by following $q = 3$ and $\deg_w(K) \le 3$. Since P and Q have no common factors, it is clear that G and Q also have no common factors. It shows the degree of G is at most 3. Thus, the pole of w is not the pole of the right hand side of (3.1). It implies

$$N\left(\frac{G(r,w)}{Q(r,w)}\right) \le N(r,\overline{w}) + N(r,\underline{w}) + S(r,w).$$

By Lemma 2.1 and (3.1), it is shown that

$$m\left(\frac{G(r,w)}{Q(r,w)}\right) \le m(r,\overline{w}) + m(r,\underline{w}) + S(r,w).$$

Thus, we have

$$\begin{aligned}3T(r,w) = T\left(\frac{G(r,w)}{Q(r,w)}\right) + S(r,w) &\le T(r,\overline{w}) + T(r,\underline{w}) + S(r,w)\\ &\le (2+\varepsilon)T(r,w) + S(r,w),\end{aligned}$$

for every $\varepsilon > 0$, which is impossible. In short, we have the limitations

$$q \le 2 \quad \text{and} \quad p \le 4.$$

In this paper, we discuss the equations with $p = 4$ and $q = 2$. We state the result as follows.

Theorem 3.1. *If w is an admissible finite order meromorphic solution of the equation*

$$(\overline{w} + w)(w + \underline{w}) = \frac{c(w-a_1)(w-a_2)(w-a_3)(w-a_4)}{(w-b_+)(w-b_-)}, \tag{3.2}$$

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where the coefficients are distinct meromorphic functions and $c(z) \not\equiv 0$, then either w satisfies a difference Riccati equation

$$\overline{w} = \frac{\mu w + \delta}{w + \nu}, \tag{3.3}$$

where $\mu, \nu, \delta \in S(w)$, or equation (3.2) can be transformed by Möbius tranformation in w to the equation:

$$(\overline{w} + w)(w + \underline{w}) = \frac{w^4 + k_3(-1)^z w^3 + \pi_1 w^2 + k_4(-1)^z w + k_5}{w^2 + (k_1 z + k_2)w + \pi_2}, \tag{3.4}$$

where $k_i (i = 1, \ldots, 5)$ and $\pi_j (j = 1, 2)$ are period functions $\in S(w)$.

In particular, if we take $k_3 \equiv 0$ and $k_4 \equiv 0$ in (3.4), then (3.4) can be written as

$$(\overline{w} + w)(w + \underline{w}) = \frac{(w^2 - a^2)(w^2 - b^2)}{(w + (\alpha z + \beta))^2 + \pi},$$

where a, b, α, β and π are period meromorphic functions $\in S(w)$, which is the difference Painlevé equation IV. Equation (3.4) can be seen as the general case of difference Painlevé IV.

4. Proof of Theorem 3.1

The equation we consider now is

$$(\overline{w} + w)(w + \underline{w}) = \frac{c(w - a_1)(w - a_2)(w - a_3)(w - a_4)}{(w - b_+)(w - b_-)},$$

where $a_*(z)$, $b_\pm(z)$, $c(z) \in S(w)$ such that $a_*(z)$, $b_\pm(z)$ are distinct and $c(z) \not\equiv 0$, (we denote $a_i(z)(i = 1, 2, 3, 4)$ by $a_*(z)$ here) can be rewritten as the form of (3.1). If $c \not\equiv 1$, the equation

$$\overline{w}\underline{w} + w\underline{w} + \overline{w}w = \frac{c(w - a_1)(w - a_2)(w - a_3)(w - a_4) - w^2(w - b_+)(w - b_-)}{(w - b_+)(w - b_-)}$$

implies by the result due to Mohon′ko and Lemma 2.3 that

$$4T(r, w) \leq 3T(r, w) + S(r, w),$$

which is impossible. Thus, we have $c(z) \equiv 1$. Now, the equation (3.2) reads as

$$(\overline{w} + w)(w + \underline{w}) = \frac{(w - a_1)(w - a_2)(w - a_3)(w - a_4)}{(w - b_+)(w - b_-)}. \tag{4.1}$$

Suppose that $w(z)$ is a finite order meromorphic solution of (4.1) and let

$$P_1(z, w) := (\overline{w} + w)(\underline{w} + w)(w - b_+)(w - b_-) - (w - a_1)(w - a_2)(w - a_3)(w - a_4).$$

Since $w = b_\pm$ are not the solutions of $P_1(z, w) = 0$, it follows from Lemma 2.4 that

$$N\left(r, \frac{1}{w - b_\pm}\right) = 2T(r, w) + S(r, w), \tag{4.2}$$

which implies, together with the assumption w is transcendental, that the measure of the set of zeros of $w - b_\pm = 0$ is not zero density.

Next, suppose that $z_0 \in \mathbb{C}$ is a zero of $w - b_\pm = 0$, i.e., that $w(z_0) = b_+(z_0)$ or $w(z_0) = b_-(z_0)$. Then, there are three possibilities:

(I) $w(z_0) = a_*(z_0)$;
(II) $w(z_0 + 1) = \infty$ or $w(z_0 - 1) = \infty$.

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Let $A = \{z_i \in \mathbb{C} : i \in \mathbb{N}\}$ be the multiset of all zeros of

$$w(z) - b_+(z) = 0 \quad \text{and} \quad w(z) - b_-(z) = 0$$

satisfying (II), and its cardinality of the multiset $A \cap \{z \in \mathbb{C} : |z| \leq r\}$ be denoted by $n_A\left(r, \frac{1}{w-b_\pm}\right)$ according to its multiplicity, with counting functions denoted by $N_A\left(r, \frac{1}{w-b_\pm}\right)$. We proceed to prove the zeros of $w(z) - b_\pm = 0$ are "almost all" satisfying (II) as follows. That is to say

$$N_A\left(r, \frac{1}{w-b_\pm}\right) = N\left(r, \frac{1}{w-b_\pm}\right) + S(r,w) = 2T(r,w) + S(r,w). \tag{4.3}$$

Suppose that there are infinitely many points z_0 such that (I) is valid; then it is clear that these points satisfy

$$a_*(z_0) = w(z_0) = b_\pm(z_0).$$

Since $a_*(z)$ and $b\pm(z)$ are distinct meromorphic functions such that $a_*(z), b_\pm(z) \in S(w)$, we denote by $n_I\left(r, \frac{1}{w-b_\pm}\right)$ the number of zeros of $w - b_\pm = 0$ satisfying (I) according to its multiplicity as $|z| \leq r$, and its integrated counting function by $N_I\left(r, \frac{1}{w-b_\pm}\right)$. It yields

$$N_I\left(r, \frac{1}{w-b_\pm}\right) \leq N_I\left(r, \frac{1}{a_*-b_\pm}\right) = S(r,w),$$

that is, the zeros of $w(z) - b_\pm = 0$ are "almost all" satisfying (II).

Let B be the subset of A of all points such that at least one of

$$w(z_0+2) = b_+(z_0+2), \quad w(z_0+2) = b_-(z_0+2),$$

holds. We denote by $N_B\left(r, \frac{1}{w-b_\pm}\right)$ the corresponding counting function, and by

$$N_{A\setminus B}\left(r, \frac{1}{w-b_\pm}\right) = N_A\left(r, \frac{1}{w-b_\pm}\right) - N_B\left(r, \frac{1}{w-b_\pm}\right)$$

the counting function for those points $z_0 \in A \setminus B$. Therefore, for each two points in $B \cap \{z \in \mathbb{C} : |z| \leq r\}$ there is exactly one pole in the disc $\{z \in \mathbb{C} : |z| \leq r+1\}$, which can be uniquely associated with them. Then we have

$$2N_B(r,w) \leq N_B\left(r+1, \frac{1}{w-b_\pm}\right) \leq 2N(r+2,w),$$

and so

$$2N_B(r,w) = N_B\left(r, \frac{1}{w-b_\pm}\right) + S(r,w) \tag{4.4}$$

by Lemma 2.3. Similarly, if there is one point in $A \setminus B \cap \{z \in \mathbb{C} : |z| \leq r\}$ there exists at least one pole in the disc $\{z \in \mathbb{C} : |z| \leq r+1\}$. Therefore, we have

$$N_{A\setminus B}\left(r, \frac{1}{w-b_\pm}\right) \leq N_{A\setminus B}(r+1,w). \tag{4.5}$$

We proceed to prove that the points in the set A are "almost all" in the set B. Now suppose that

$$N_B\left(r, \frac{1}{w-b_\pm}\right) \leq \alpha_1 N_A\left(r, \frac{1}{w-b_\pm}\right) + S(r,w) \tag{4.6}$$

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where $0 \le \alpha_1 < 1$. According to (4.3) and (4.4), it follows

$$N_B(r,w) \le \frac{\alpha_1}{2} N_A\left(r, \frac{1}{w-b_\pm}\right) + S(r,w) = \alpha_1 T(r,w) + S(r,w). \tag{4.7}$$

Therefore, from (4.3),(4.4), (4.5) and (4.7), we have

$$\begin{aligned}
2T(r,w) &= N_A\left(r, \frac{1}{w-b_\pm}\right) + S(r,w)\\
&= N_B\left(r, \frac{1}{w-b_\pm}\right) + N_{A\setminus B}\left(r, \frac{1}{w-b_\pm}\right) + S(r,w)\\
&\le 2N_B(r+1,w) + N_{A\setminus B}(r+1,w) + S(r,w)\\
&\le N_B(r+1,w) + N_A(r+1,w) + S(r,w)\\
&\le (1+\alpha_1)T(r,w) + S(r,w),
\end{aligned}$$

where $0 \le \alpha_1 < 1$. It is a contradiction to Lemma 2.3, hence the assumption (4.6) does not hold, which means

$$N_B\left(r, \frac{1}{w-b_\pm}\right) = N_A\left(r, \frac{1}{w-b_\pm}\right) + S(r,w) = 2T(r,w) + S(r,w). \tag{4.8}$$

That is to say, the zeros of $w(z) - b_\pm(z) = 0$ "almost all" belong to the B, which leads to the following case:

$$\begin{aligned}
&\text{(a)} \qquad w(z_0) = b_-(z_0) \quad \text{and} \quad w(z_0+2) = b_+(z_0+2),\\
&\text{(b)} \qquad w(z_0) = b_+(z_0) \quad \text{and} \quad w(z_0+2) = b_-(z_0+2),\\
&\text{(c)} \qquad w(z_0) = b_+(z_0) \quad \text{and} \quad w(z_0+2) = b_+(z_0+2),\\
&\text{(d)} \qquad w(z_0) = b_-(z_0) \quad \text{and} \quad w(z_0+2) = b_-(z_0+2).
\end{aligned}$$

In the following, we denote by B_a the subset of B such that (a) is valid, similarly, by B_b, B_c and B_d, respectively. It follows by (4.8) one of the following four statements holds:

$$\begin{aligned}
&\text{(i)} \quad N_{B_a}\left(r, \frac{1}{w-b_\pm}\right) \ne S(r,w), \quad N_{B_b}\left(r, \frac{1}{w-b_\pm}\right) \ne S(r,w),\\
&\text{(ii)} \quad N_{B_c}\left(r, \frac{1}{w-b_\pm}\right) \ne S(r,w), \quad N_{B_d}\left(r, \frac{1}{w-b_\pm}\right) \ne S(r,w),\\
&\text{(iii)} \quad N_{B_a}\left(r, \frac{1}{w-b_\pm}\right) \ne S(r,w), \quad N_{B\setminus B_a}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w),\\
&\text{(iv)} \quad N_{B_b}\left(r, \frac{1}{w-b_\pm}\right) \ne S(r,w), \quad N_{B\setminus B_b}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w).
\end{aligned}$$

Now let us rewrite (4.1) as follows:

$$\overline{w}w + w\underline{w} + \overline{w}\underline{w} = \frac{(A_3w^3 + A_2w^2 + A_1w + A_0) - (B_1w^3 + B_0w^2)}{w^2 + B_1w + B_0}, \tag{4.9}$$

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where

$$\begin{aligned} A_3(z) =& -(a_1(z)+a_2(z)+a_3(z)+a_4(z)),\\ A_2(z) =& a_1(z)a_2(z)+a_1(z)a_3(z)+a_1(z)a_4(z)\\ &+a_2(z)a_3(z)+a_2(z)a_4(z)+a_3(z)a_4(z),\\ A_1(z) =& -(a_1(z)a_2(z)a_3(z)+a_1(z)a_2(z)a_4(z)+a_2(z)a_3(z)a_4(z)),\\ A_0(z) =& a_1(z)a_2(z)a_3(z)a_4(z), \end{aligned}$$

and

$$B_1(z) = -(b_+(z)+b_-(z)), \qquad B_0(z) = b_+(z)b_-(z).$$

It is clear that $A_i (i = 0, \ldots, 3)$ and $B_j (j = 0, 1)$ are small meromorphic functions related to w.

Now suppose that (i) is valid. For all the points $z_0 \in B_a$, we take $z = z_0 + 1$ in (4.9), leave only $w(z_0 + 2)w(z_0)$ on the left hand side, and move the other terms to the right hand side. Keeping in mind that $w(z_0 + 1) = \infty$, and comparing both sides of (4.9), we have

$$b_+(z_0+2)b_-(z_0) = A_2(z_0+1) - B_0(z_0+1) - B_1(z_0+1)(b_+(z_0+2)+b_-(z_0))$$

and

$$b_+(z_0+2)+b_-(z_0) = A_3(z_0+1) - B_1(z_0+1).$$

Assume that either of the following equations is not valid

$$\overline{\overline{b}}_+ + b_- \equiv \overline{A}_3 - \overline{B}_1, \tag{4.10}$$

$$\overline{\overline{b}}_+ b_- \equiv \overline{A}_2 - \overline{B}_0 - \overline{B}_1(\overline{\overline{b}}_+ + b_-). \tag{4.11}$$

From the two equations above, it is clear that the points $z_0 \in B_a$ should exactly be the zeros of the equations (4.10) or (4.11); then

$$\begin{aligned} N_{B_a}\left(r, \frac{1}{w-b_\pm}\right) \leq& N\left(r+1, \frac{1}{(\overline{\overline{b}}_+ + b_-) - (\overline{A}_3 - \overline{B}_1)}\right)\\ &+ N\left(r+1, \frac{1}{\overline{\overline{b}}_+ b_- - (\overline{A}_2 - \overline{B}_0 - \overline{B}_1(\overline{\overline{b}}_+ + b_-))}\right)\\ =& S(r, w), \end{aligned}$$

which is a contradiction to (4.8). Hence, it shows that both (4.10) and (4.11) hold. Similarly, for all the points $z_0 \in B_b$, we obtain

$$\overline{\overline{b}}_- + b_+ \equiv \overline{A}_3 - \overline{B}_1, \tag{4.12}$$

$$\overline{\overline{b}}_- b_+ \equiv \overline{A}_2 - \overline{B}_0 - \overline{B}_1(\overline{\overline{b}}_- + b_+). \tag{4.13}$$

From (4.10)-(4.13), it yields that $\overline{\overline{b}}_+ b_- = \overline{\overline{b}}_- b_+$ and $\overline{\overline{b}}_+ + b_- = \overline{\overline{b}}_- + b_+$. According to the assumption that b_+ and b_- are distinct, it follows $\overline{\overline{b}}_+ = b_+$ and $\overline{\overline{b}}_- = b_-$, and the following equation can be deduced immediately:

$$B_0(z+2) = B_0(z). \tag{4.14}$$

By combining $\overline{\overline{b}}_+ = b_+$ and $\overline{\overline{b}}_- = b_-$ with (4.10) and (4.12), it yields

$$B_1(z+2) - 2B_1(z) + B_1(z-1) = 0.$$

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By solving this equation, we have

$$B_1(z) = k_1(z)z + k_2(z), \tag{4.15}$$

where k_1 and k_2 are meromorphic functions with period 1.

Suppose that (ii) is valid. In the same way as the case of (4.11) and (4.10), we obtain that

$$\overline{\overline{b}}_+ + b_+ \equiv \overline{A}_3 - \overline{B}_1, \tag{4.16}$$

$$\overline{\overline{b}}_+ b_+ \equiv \overline{A}_2 - \overline{B}_0 - \overline{B}_1(\overline{\overline{b}}_+ + b_+), \tag{4.17}$$

$$\overline{\overline{b}}_- + b_- \equiv \overline{A}_3 - \overline{B}_1, \tag{4.18}$$

$$\overline{\overline{b}}_- b_- \equiv \overline{A}_2 - \overline{B}_0 - \overline{B}_1(\overline{\overline{b}}_- + b_-). \tag{4.19}$$

From (4.16)-(4.19), it yields that $\overline{\overline{b}}_+ b_+ = \overline{\overline{b}}_- b_-$ and $\overline{\overline{b}}_+ + b_+ = \overline{\overline{b}}_- + b_-$, which implies that $\overline{\overline{b}}_+ = b_-$ and $\overline{\overline{b}}_- = b_+$. In the same way as the case of (4.14) and (4.15), we can also obtain these two equations.

Suppose that (iii) is valid. In this case, we have already shown that (4.10) and (4.11) hold. Let us define a meromorphic function

$$U(z) := (w(z) - b_+(z))(w(z+1) - b_-(z+1)). \tag{4.20}$$

The poles of w which are in the set of B_a are not the poles of U, since they cancel out with the zeros of $\overline{w} - \overline{b}_- = 0$. Similarly, the poles of $\overline{w}$ are not. Therefore, the only possibility is that the poles of U in the set of B are from the poles of $b_+\overline{b}_-$ or are a complement of set B_a. By the assumption of (iii), we have

$$N_B(r, U) = S(r, w).$$

From (4.20), it follows that

$$w(z+1) = \frac{U(z)}{w(z) - b_-(z)} + b_+(z+1), \quad w(z-1) = \frac{U(z-1)}{w(z) - b_+(z)} + b_-(z-1).$$

By substituting these two equations into (4.9), it reads

$$A(z)w^2(z) + B(z)w(z) + C(z) = 0, \tag{4.21}$$

where

$$\begin{cases} A(z) = & U(z) + U(z-1), \\ B(z) = & B_1(z)(A_2(z) - 2B_0(z) - B_1(z)A_3(z) + B_1^2(z)) + B_0(z)A_3(z) \\ & +U(z)(b_-(z-1) - b_+(z)) + U(z-1)(b_+(z+1) - b_-(z)) - A_1(z), \\ C(z) = & U(z)U(z-1) + B_0(z)(A_2(z) - B_0(z) - B_1(z)A_3(z) + B_1^2(z)) \\ & -U(z)b_-(z)b_-(z-1) - U(z-1)b_+(z)b_+(z+1) - A_0(z). \end{cases}$$

Equation (4.21) can be written as

$$A(z)w^2(z) = -B(z)w(z) - C(z).$$

Since $N_B(r, Aw^2) = 2N_B(r, w) + S(r, w)$ and $N_B(r, Bw + C) = N_B(r, w) + S(r, w)$, it follows that

$$N_B(r, w) = S(r, w),$$

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which is a contradiction to (4.4) and (4.8). So it implies that $A(z) \equiv B(z) \equiv C(z) \equiv 0$. By solving this functional system, we obtain $U(z)$ is a small function related to w, which is given by

$$U(z) = \frac{B_1(z)(A_2(z) - 2B_0(z) - B_1(z)A_3(z) + B_1^2(z)) + B_0(z)A_3(z) - A_1(z)}{(b_+(z+1) - b_-(z)) - (b_-(z-1) - b_+(z))}.$$

It leads us to the fact that w satisfies the difference Riccati equation, which is equation (3.3) of Theorem (3.1).

If (iv) holds, then it can be seen exactly in the same way as in (iii) that w satisfies the difference Riccati equation

$$\overline{w} = \frac{\overline{b}_- w + U - \overline{b}_- b_+}{w - b_+},$$

where U is a small function related to w given in a similar way to (iii).

In the following, we transform the equation (4.1) by $w \to \frac{1}{w}$, and obtain

$$\frac{(\overline{w} + w)(w + \underline{w})}{\overline{w}\underline{w}} = \frac{\frac{a_1 a_2 a_3 a_4}{b_- b_+}(w - \frac{1}{a_1})(w - \frac{1}{a_2})(w - \frac{1}{a_3})(w - \frac{1}{a_4})}{(w - \frac{1}{b_+})(w - \frac{1}{b_-})} \tag{4.22}$$

which has a similar form to (4.1). By applying Lemma 2.5 to equation (4.22) with $Q(z, w) = (w - 1/b_-)(w - 1/b_+)(\overline{w} + w)(w + \underline{w})$ and

$$H(z, w) = a_2 a_3 a_4 \overline{w}\underline{w}(w - 1/a_2)(w - 1/a_3)(w - 1/a_4)/(b_- b_+),$$

it follows that

$$m\left(r, \frac{1}{w - 1/a_1}\right) = O(r^{\rho-1+\varepsilon}) + S(r, w),$$

for every $\varepsilon > 0$, possibly outside of an exceptional set of finite logarithmic measure, where ρ is the growth order of w. Similarly, we have

$$m\left(r, \frac{1}{w - 1/a_*}\right) = O(r^{\rho-1+\varepsilon}) + S(r, w),$$

for every $\varepsilon > 0$, possibly outside of an exceptional set of finite logarithmic measure, that is,

$$N\left(r, \frac{1}{w - 1/a_*}\right) = 4T(r, w) + S(r, w).$$

Let $z_1 \in \mathbb{C}$ be the zeros of $w - 1/a_* = 0$. We denote by A_* the set of all the zeros of $w - 1/a_* = 0$ such that

$$w(z_1) + w(z_1 + 1) = 0 \quad \text{or} \quad w(z_1) + w(z_1 - 1) = 0.$$

Similarly as the case in (4.3), we have

$$N_{A_*}\left(r, \frac{1}{w - 1/a_*}\right) = N\left(r, \frac{1}{w - 1/a_*}\right) + S(r, w) = 4T(r, w) + S(r, w). \tag{4.23}$$

Let B_* be the subset of A_* such that $P_*(z, \overline{w}) = 0$, where

$$P_*(z, w) = (w - 1/a_1)(w - 1/a_2)(w - 1/a_3)(w - 1/a_4).$$

We proceed to prove that the points of A_* should "almost all" belong to the set B_*. Assume that

$$N_{B_*}\left(r, \frac{1}{w - 1/a_*}\right) \le 4\alpha_2 T(r, w) + S(r, w), \tag{4.24}$$

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where $0 \leq \alpha_2 < 1$. For each point of zeros of $w + \overline{w} = 0$ in $B_* \cap \{z \in \mathbb{C} : |z| \leq r\}$, there are exactly two points of zeros of $w - 1/a_* = 0$ in $B_* \cap \{z \in \mathbb{C} : |z| \leq r\}$. Then we have

$$2N_{B_*}\left(r, \frac{1}{w+\overline{w}}\right) \leq N_{B_*}\left(r, \frac{1}{w - 1/a_*}\right) \leq 2N_{B_*}\left(r+1, \frac{1}{w+\overline{w}}\right),$$

which, by Lemma 2.3, implies that

$$2N_{B_*}\left(r, \frac{1}{w+\overline{w}}\right) = N_{B_*}\left(r, \frac{1}{w - 1/a_*}\right) + S(r, w). \tag{4.25}$$

For each point of zeros of $w - 1/a_* = 0$ in $A_* \setminus B_* \cap \{z \in \mathbb{C} : |z| \leq r\}$, there is at least one point of zero of $w + \overline{w} = 0$ in $A_* \setminus B_* \cap \{z \in \mathbb{C} : |z| \leq r\}$. Then we have

$$N_{A_*\setminus B_*}\left(r, \frac{1}{w - 1/a_*}\right) \leq N_{A_*\setminus B_*}\left(r+1, \frac{1}{w+\overline{w}}\right). \tag{4.26}$$

Therefore, from (4.23)-(4.26), we have

$$\begin{aligned} 4T(r,w) &= N_{A_*}\left(r, \frac{1}{w - 1/a_*}\right) + S(r,w) \\ &= N_{B_*}\left(r, \frac{1}{w - 1/a_*}\right) + N_{A_*\setminus B_*}\left(r, \frac{1}{w - 1/a_*}\right) + S(r,w) \\ &\leq 2N_{B_*}\left(r, \frac{1}{w+\overline{w}}\right) + N_{A_*\setminus B_*}\left(r+1, \frac{1}{w+\overline{w}}\right) + S(r,w) \\ &\leq N_{B_*}\left(r+1, \frac{1}{w+\overline{w}}\right) + N_{A_*}\left(r+1, \frac{1}{w+\overline{w}}\right) + S(r,w) \\ &\leq (2+2\alpha_2)T(r+1,w) + S(r,w), \end{aligned}$$

where $0 \leq \alpha_2 < 1$. It is a contradiction to Lemma 2.3. Then, the assumption (4.24) is not valid. That is,

$$N_{B_*}\left(r, \frac{1}{w - 1/a_*}\right) = 4T(r,w) + S(r,w). \tag{4.27}$$

In the following, we denote by H_j^i the set of points $z_1 \in B_*$ such that $w(z_1) = 1/a_i(z_1)$ and $w(z_1+1) = 1/a_j(z_1+1)$ hold. Since there are sixteen possibilities such that $w(z_1) = 1/a_i(z_1)$ and $w(z_1+1) = 1/a_j(z_1+1)$, where $i, j \in \{1,2,3,4\}$, together with (4.27) and a_* are distinct with each other, one of the following holds:

$$\begin{aligned} &\text{(v)}\ N_{H_j^i}\left(\frac{1}{w - 1/a_*}\right) \neq S(r,w),\ N_{B_*\setminus H_j^i}\left(\frac{1}{w - 1/a_*}\right) = S(r,w),\ N(H_j^i) = 4, \\ &\text{(vi)}\ N_{H_j^i}\left(\frac{1}{w - 1/a_*}\right) \neq S(r,w),\ N_{B_*\setminus H_j^i}\left(\frac{1}{w - 1/a_*}\right) = S(r,w),\ N(H_j^i) = 3, \\ &\text{(vii)}\ N_{H_j^i}\left(\frac{1}{w - 1/a_*}\right) \neq S(r,w),\ N_{B_*\setminus H_j^i}\left(\frac{1}{w - 1/a_*}\right) = S(r,w),\ N(H_j^i) = 2, \end{aligned}$$

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where $N(H_j^i)$ is the number of the set H_j^i. For example, if

$$N_{H_1^2}\left(\frac{1}{w-1/a_*}\right) \neq S(r,w), \quad N_{H_2^3}\left(\frac{1}{w-1/a_*}\right) \neq S(r,w),$$
$$N_{H_3^4}\left(\frac{1}{w-1/a_*}\right) \neq S(r,w), \quad N_{B_*\setminus(H_1^2\cap H_2^3\cap H_3^4)}\left(\frac{1}{w-1/a_*}\right) = S(r,w),$$

then $N(H_j^i)=3$. We will discuss every case in the following.

Suppose that (v) and (i) are true. There are eighteen possibilities in this case. First we discuss

$$N_{H_1^1}\left(\frac{1}{w-1/a_*}\right) \neq S(r,w), \quad N_{H_2^2}\left(\frac{1}{w-1/a_*}\right) \neq S(r,w),$$
$$N_{H_3^3}\left(\frac{1}{w-1/a_*}\right) \neq S(r,w), \quad N_{H_4^4}\left(\frac{1}{w-1/a_*}\right) \neq S(r,w).$$

In the following, we proceed to prove each i and j satisfying

$$\overline{a}_i + a_j \equiv 0, \tag{4.28}$$

where $1 \leq i, j \leq 4$. We note that $i=j$ may happen here. For all the points $z_1 \in H_1^1$, it yields

$$w(z_1)+w(z_1+1) = \frac{1}{a_1(z_1)} + \frac{1}{a_1(z_1+1)} = 0.$$

Therefore, we have $\overline{a}_1 = -a_1$. Otherwise, let us assume $a_1+\overline{a}_1 \neq 0$; then

$$N_{H_1^1}\left(\frac{1}{w-1/a_*}\right) \leq N\left(r, \frac{\overline{a}_1 a_1}{\overline{a}_1 + a_1}\right) = S(r,w),$$

which is a contradiction. For the points $z_1 \in H_2^2$, H_3^3, H_4^4, we have $\overline{a}_2 = -a_2$, $\overline{a}_3 = -a_3$, $\overline{a}_4 = -a_4$, respectively. Hence, the equation (4.28) has been proved. From (4.28), we can deduce that

$$\overline{A}_3 = -A_3, \qquad \overline{A}_2 = A_2, \qquad \overline{A}_1 = -A_1, \qquad \overline{A}_0 = A_0. \tag{4.29}$$

By solving the equations $\overline{A}_1 = -A_1$ and $\overline{A}_3 = -A_3$, we obtain that

$$A_3(z) = k_3(z)(-1)^z \qquad \text{and} \qquad A_1(z) = k_4(z)(-1)^z, \tag{4.30}$$

where k_3 and k_4 are meromorphic functions with period 1. In the case (i), we have already shown that $\overline{\overline{b}}_- = b_-$, $\overline{\overline{b}}_+ = b_+$, and (4.10) hold. By substituting (4.14), (4.15) and (4.30) into (4.9), it follows that

$$\overline{w}w + w\underline{w} + \overline{w}\underline{w} = \frac{(k_3(-1)^z - k_1 z - k_2)w^3 + \pi_1 w^2 + k_4(-1)^z w + k_5}{w^2 + (k_1 z + k_2)w + \pi_2}$$

where π_1, π_2 are meromorphic functions with period 2, and k_5 is a meromorphic function with period 1, which is the equation (3.4) in Theorem 3.1. In a similar way, for the other seventeen cases, we can obtain this same result.

Suppose that (v) and (ii) are true, and similarly we have the equation (3.4) in Theorem 3.1.

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Suppose that (vi) and (i) are true. There are seventy-two possibilities in this case. Let us take one of them as an example to discuss as follows:

$$N_{H_1^2}\left(\frac{1}{w-1/a_*}\right)\neq S(r,w),\quad N_{H_2^3}\left(\frac{1}{w-1/a_*}\right)\neq S(r,w),$$
$$N_{H_3^4}\left(\frac{1}{w-1/a_*}\right)\neq S(r,w),\quad N_{B_*\setminus(H_1^2\cap H_2^3\cap H_3^4)}\left(\frac{1}{w-1/a_*}\right)= S(r,w).$$

In the same way as the cases (i) and (v), we have $\overline{a}_2=-a_1$, $\overline{a}_3=-a_2$, $\overline{a}_4=-a_3$. In the case (i), we have shown that (4.10) and (4.12) hold, that is, $\overline{A}_3=-A_3$. From the discussion above, we obtain

$$\overline{A}_3=-(\overline{a}_1+\overline{a}_2+\overline{a}_3+\overline{a}_4)=-(-a_1-a_2-a_3+\overline{a}_1)=-A_3,$$

which implies that $\overline{a}_1=-a_4$. It shows that equations (4.29) hold, and at the same time, we have the equation (4.30). For the other seventy-one possibilities, we can also deduce that (4.29) and (4.30) hold. Therefore, in this situation, we have the equation (3.4) of Theorem 3.1.

Suppose that (vi) and (ii) are true. Similarly, we have the equation (3.4) in Theorem 3.1.

Suppose that (vii) and (i) are true. There are three subcases in this case. We only choose one of them to discuss because it is similar to the others. For example,

$$N_{H_1^2}\left(\frac{1}{w-1/a_*}\right)\neq S(r,w),\quad N_{H_3^4}\left(\frac{1}{w-1/a_*}\right)\neq S(r,w).$$

We conclude that $\overline{a}_2=-a_1$ and $\overline{a}_3=-a_4$. In the case (i), we have already shown that (4.10) and (4.11) hold. That is to say, $\overline{A}_3=-A_3$ and $\overline{A}_2=A_2$. From the discussion above, it is easy to check that

$$a_2+a_4+\overline{a}_1+\overline{a}_3=0\qquad\text{and}\qquad a_2a_4=\overline{a}_1\overline{a}_3.$$

By solving the equations, it follows that one of the following cases hold:

$$\overline{a}_1=-a_2\quad\text{and}\quad\overline{a}_3=-a_4\qquad\text{or}\qquad\overline{a}_1=-a_4\quad\text{and}\quad\overline{a}_3=-a_1,$$

which implies that equations (4.29) and (4.30) hold. For the other two subcases, we can also deduce that the equations (4.29) and (4.30) hold again. Therefore, in this case, we have the equation (3.4) of Theorem 3.1.

If one of the cases (iii) and (iv) holds together with any of the cases (v)-(vii), we obtain the difference Riccati equation (3.3) in Theorem 3.1.

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