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EXISTENCE OF ZERO-ORDER MEROMORPHIC SOLUTIONS IN DETECTING q-DIFFERENCE PAINLEVÉ EQUATIONS

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Abstract. The existence of zero-order meromorphic solutions is used as a sufficient condition in detecting q-difference equations of Painlevé type. It is shown that demanding the existence of at least one non-rational zero-order meromorphic solution $w(z)$ is sufficient to reduce a canonical class of q-difference equations with rational coefficients into a short list of Painlevé type q-difference equations, unless $w(z)$ is simultaneously a solution of a q-difference Riccati equation of a specific form.

1. Introduction

Over a century ago Painlevé [19,20], Fuchs [4] and Gambier [5] completed a substantial classification work, which was comprised of sieving through a large class of second-order differential equations by making use of a criterion proposed by Picard [21], now known as the Painlevé property. Painlevé and his colleagues discovered six new equations, later named Painlevé equations, which were not solvable in terms of known functions.

There are several candidates for the discrete analogue of the Painlevé property. The singularity confinement test by Grammaticos, Ramani and Papageorgiou [6] has been successful in finding new discrete equations of Painlevé type [22]. The first approach using a low growth criterion of iterates as a sign of integrability is due to Veselov [27] who showed that generic exponential growth rate of the degree of iterates corresponds to the non-existence of certain types of first integrals. Hietarinta and Viallet suggested that integrable equations have solutions with zero algebraic entropy, and for solutions of non-integrable equations the algebraic entropy should be strictly positive [13]. A method due to Roberts and Vivaldi relies on orbit dynamics over finite fields [23], while Halburd's Diophantine integrability [8] makes use of polynomial growth of heights of iterates over number fields.

Halburd and the first author [10] implemented an approach suggested by Ablowitz, Halburd and Herbst [1], who proposed that the existence of sufficiently many finite-order meromorphic solutions of a difference equation is a good indicator that the equation is of Painlevé type. In [10] it was shown that if

$$w(z+1)+w(z-1)=R(z,w(z)), \tag{1.1}$$

where $R(z,w(z))$ is rational in $w(z)$ with meromorphic coefficients, has an admissible meromorphic solution of finite order, then the class (1.1) reduces into a short

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list of equations which consist solely of difference Painlevé equations, integrable autonomous difference equations solvable in terms of elliptic functions, and linear or linearizable equations. The finite-order growth condition was weakened to encompass a class of infinite-order meromorphic functions by Halburd, the first author and Tohge in [9].

In [11] the equation

$$w(z+1)w(z-1) = \frac{c_2(z)(w(z)-c_+(z))(w(z)-c_-(z))}{(w(z)-a_+(z))(w(z)-a_-(z))} \tag{1.2}$$

was studied in the case where c_2, $c_\pm$ and $a_\pm$ are small functions of w. In particular, it was proved that if (1.2) has an admissible finite-order meromorphic solution w such that the orders of the poles of w remain uniformly bounded, then either w satisfies a difference Riccati equation

$$w(z+1) = \frac{p(z)w(z)+q(z)}{w(z)+s(z)},$$

where p, q, s are small functions of w, or equation (1.2) can be transformed by a bilinear change in w to one of the equations

$$w(z+1)w(z-1) = \frac{\gamma(z-1)w(z)^2 + \delta(z)\lambda^z w(z) + \gamma(z)\mu(z)\lambda^{2z}}{(w(z)-1)(w(z)-\gamma(z))}$$

and

$$w(z+1)w(z-1) = \frac{w(z)^2 + \delta(z)e^{i\pi z/2}\lambda^z w(z) + \mu(z)\lambda^{2z}}{w(z)^2-1},$$

where $\lambda \in \mathbb{C}$, and $\delta,\ \mu,\ \lambda,\ \gamma$ are arbitrary finite-order periodic functions such that δ and γ have period 2 and μ has period 1. Ronkainen did a classification of

$$w(z+1)w(z-1) = R(z, w(z)) \tag{1.3}$$

where $R(z, w(z))$ is rational in $w(z)$ and has meromorphic coefficients, in which he was able to remove the condition on the uniform boundedness of the orders of the poles [24]. However, the method used in the classification of (1.3) in [24] leads to situations where the coefficients of (1.3) can only be said to satisfy a difference equation, the general solution of which could not be fully determined. Various alternate forms of the difference Painlevé III equation are obtained when the coefficients are special solutions of these equations.

In this paper we consider the existence of zero-order meromorphic solutions as a detector of q-difference Painlevé equations. It turns out that the existence of just one non-rational meromorphic solution of order zero is enough to single out known q-difference Painlevé equations from within a natural subclass of

$$w(qz)w(z/q) = R(z, w(z)) =: \frac{P(z, w(z))}{Q(z, w(z))}, \tag{1.4}$$

where $P(z, w(z))$ and $Q(z, w(z))$ are polynomials in $w(z)$ without common factors having rational coefficients in z. The advantages of considering zero-order solutions of (1.4) rather than finite-order solutions of (1.2) are twofold. Handling rational coefficients does not lead into complicated and somewhat artificial technical difficulties in the proofs, as is at least to some extent the case with arbitrary meromorphic ones, and we will be able to pinpoint coefficients of (1.3) explicitly and exactly.

The method of proof of our main result is in part based on careful counting arguments using suitably selected Nevanlinna counting functions of certain special values of solutions. We will show that the pole density becomes too high for zero-order solutions of (1.4) unless the coefficients satisfy certain conditions, the implementation of which gives us the list of q-difference Painlevé equations within the class (1.4). This method enables us, on one hand, to avoid any additional assumptions on the orders of poles of the solutions, and, on the other hand, allows us to get explicit expressions for the obtained q-difference Painlevé and Riccati equations. Also, the zero-order growth assumption on solutions cannot be essentially weakened. We will elaborate on this point in section 2 below.

We will first show, in section 2 below, that if w is a non-rational zero-order meromorphic function that satisfies equation (1.4), then $\deg_w(R) \leq 2$. We state the main findings in this paper as follows.

Theorem 1.1. *Let $w(z)$ be a zero-order transcendental meromorphic solution of*

$$w(qz)w(z/q) = \frac{a_2(z)(w(z)-a_+(z))(w(z)-a_-(z))}{(w(z)-b_+(z))(w(z)-b_-(z))}, \tag{1.5}$$

where the coefficients are rational functions such that a_+, a_-, b_+, b_- are distinct and $a_2 \not\equiv 0$. If $|q| \notin \{0,1\}$, then either w satisfies one of the q-difference Riccati equations

$$w(qz) = \frac{b_\pm(qz)w(z)+u(z)}{w(z)-b_\mp(z)} \tag{1.6}$$

or

$$w(qz) = \frac{(1/a_\mp(qz) - a_\pm(z)u(z))\,w(z) - a_\pm(z)/a_\mp(qz)}{w(z)-a_\pm(z)}, \tag{1.7}$$

where u is a rational function (and an algebraic expression of a_+, a_-, b_+ and b_-), or (1.5) *can be transformed by a bilinear change in w to one of the equations*

$$w(qz)w(z/q) = \frac{w(z)^2-\mu z^{2\lambda}}{w(z)^2-1}, \tag{1.8}$$

$$w(qz)w(z/q) = \frac{\eta(w(z)^2+\delta z^\lambda w(z)+\mu z^{2\lambda})}{(w(z)-1)(w(z)-\eta)}, \tag{1.9}$$

$$w(qz)w(z/q) = \frac{\eta(w(z)^2+\mu z^\lambda)}{(w(z)-1)(w(z)-\eta)}, \tag{1.10}$$

where μ, η and δ are complex constants, and $\lambda \in \mathbb{Z}$.

The purpose of Theorem 1.1 is to demonstrate the effectiveness of the existence of zero-order meromorphic solutions as a reliable detector of q-difference Painlevé equations by going through a canonical example within the class (1.4). Incorporating the cases where $\deg_w R \leq 1$, and where $P(z,w)$ and $Q(z,w)$ have double roots or do not factorize over the field of rational functions, would have led into an extensive classification problem, which we wanted to avoid in this study.

Equations (1.6) and (1.7) are q-difference Riccati equations, with rational coefficients that are explicitly expressible in terms of the coefficients a_+, a_-, b_+ and b_- of equation (1.5) (see, e.g., equation (3.28) below in the proof of Theorem 1.1). The q-difference Riccati equation is linearizable into a second-order linear q-difference equation in a similar way as the difference Riccati. Equations (1.8), (1.9) and (1.10) are known forms of the difference Painlevé III equation (expressed as q-difference equations) discovered originally by Ramani, Grammaticos and Hietarinta in [22] using the singularity confinement method. Theorem 1.1 singles out all known q-difference Painlevé equations within the class (1.5) in their full generality, and no other equations. This suggests that the existence of zero-order meromorphic solutions is a reliable test in finding q-difference equations of Painlevé type. The question of existence of zero-order meromorphic solutions of these equations remains in general open, however.

2. A q-difference version of Malmquist's theorem

About a century ago Malmquist [16] proved that if the differential equation

$$w' = R(z, w(z)), \tag{2.1}$$

where $R(z, w(z))$ is rational in both its arguments, has a non-rational meromorphic solution, then (2.1) reduces into a Riccati equation

$$w' = a(z)w^2 + b(z)w + c(z)$$

with rational coefficients. Similarly, if the difference equation

$$w(z + 1) = R(z, w(z)), \tag{2.2}$$

where again $R(z, w(z))$ is rational in z and $w(z)$, admits at least one non-rational meromorphic solution of finite order (or of hyper-order strictly less than one), then (2.2) must in fact be a difference Riccati equation

$$w(z + 1) = \frac{a(z)w(z) + b(z)}{c(z)w(z) + d(z)}$$

with rational coefficients. This is a difference analogue of Malmquist's theorem, which follows by a more general result due to Yanagihara [28]. According to Valiron [26], the generalized Schröder q-difference equation

$$w(qz) = R(z, w(z)), \tag{2.3}$$

with $R(z, w(z))$ rational in z and $w(z)$ as above, admits a one-parameter family of meromorphic solutions when $q \in \mathbb{C}$ is chosen suitably. If $|q| > 1$, it was shown by Gundersen, Heittokangas, Laine, Rieppo and Yang [7] that the order of growth $\rho(w)$ of all non-rational meromorphic solutions of (2.3) satisfies

$$\rho(w) = \log_{|q|}(\deg_w R), \tag{2.4}$$

where $\log_{|q|}$ is the $|q|$-based logarithm. This result immediately implies a q-difference analogue of Malmquist's theorem: if the q-difference equation (2.3) has at least one meromorphic solution of zero order, then $\deg_w R = 1$, and so equation (2.3) must be a q-difference Riccati equation

$$w(qz) = \frac{a(z)w(z) + b(z)}{c(z)w(z) + d(z)},$$

where the coefficients are rational functions. The zero-order growth of solutions matches perfectly with the case of q-difference Riccati equations, and so in the first-order case it is exactly the requirement that will capture all the equations considered to be integrable, and no other equations.

We will now turn our attention to the second-order case. The method of Gundersen *et. al* can be generalized to obtain a growth estimate for all non-rational meromorphic solutions of the equation (1.4) in the following way. Suppose that

$w(z)$ is a meromorphic solution of (1.4). An application of a result, originally due to Valiron [25] and generalized by Mohon′ko [17] (see also, e.g., [14, pp. 29–34]), yields

$$T(r, w(qz)w(z/q)) = (\deg_w R)T(r, w) + O(\log r),$$

as r tends to infinity. Since

$$T(r, w(qz)w(z/q)) \leq T(r, w(qz)) + T(r, w(z/q)) = T(|q|r, w) + T(r/|q|, w) + O(1),$$

where the last equality follows by an observation due to Bergweiler, Ishizaki and Yanagihara [3], we have for all sufficiently large $r > 0$ the inequality

$$(\deg_w R)T(r, w) \leq (2 + \varepsilon)T(Kr, w) \tag{2.5}$$

with $K = \max\{|q|, 1/|q|\}$ and $\varepsilon > 0$. Let

$$G = \{r > 0 : \text{ inequality (2.5) is true}\}.$$

Then the logarithmic density of G satisfies

$$\overline{\text{logdens}}(G) = \limsup_{r\to\infty} \frac{\int_{[0,r]\cap G} \frac{dt}{t}}{\log r} = 1.$$

In order to give an upper bound for $\overline{\text{logdens}}(G)$ in terms of the order of w, we need the following lemma due to Hayman.

Lemma 2.1 ([12, Lemma 4]). *Let $C_1 > 1$, $C_2 > 1$ and $\rho \geq 0$. If $T : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that*

$$\overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} = \rho, \tag{2.6}$$

then the logarithmic density of the set

$$E := \{r : T(C_1 r) \geq C_2 T(r)\}$$

satisfies

$$\overline{\text{logdens}}(E) \leq \frac{\rho \log C_1}{\log C_2}.$$

By Lemma 2.1 and inequality (2.5) it follows that

$$\overline{\text{logdens}}(G) \leq \frac{\rho(w) \log K}{\log\left(\frac{\deg_w R}{2+\varepsilon}\right)},$$

and so

$$\log\left(\frac{\deg_w R}{2}\right) \leq \rho(w) \log K \tag{2.7}$$

by letting ε tend to zero. Hence, if equation (1.4) has a non-rational meromorphic solution of zero order, then it follows by (2.7) that

$$\deg_w R \leq 2 \tag{2.8}$$

as was claimed in the introduction. In section 3 below we will go through all the details of one generic subcase of the remaining class of equations, satisfying (2.8), by proving Theorem 1.1.

3. Proof of Theorem 1.1

The following result due to Barnett *et. al* is a q-difference version of a result due to A. Mohon′ko and V. Mohon′ko [18].

Lemma 3.1 ([2], Theorem 2.5). *Let $f(z)$ be a non-constant zero-order meromorphic solution of*

$$P(z,f)=0,$$

where $P(z,f)$ is a q-difference polynomial in $f(z)$. If $P(z,\alpha)\not\equiv 0$ for a slowly moving target α, then

$$m\left(r,\frac{1}{f-\alpha}\right)=o(T(r,f))$$

on a set of logarithmic density 1.

From now on we use the short notation $S(r,w)$ to describe a function of r with the asymptotic behavior $o(T(r,w))$ as r tends to infinity outside of a possible exceptional set of zero logarithmic density.

Suppose that w is a zero-order non-rational meromorphic solution of (1.5). Since $w=a_\pm$ and $w=b_\pm$ are not solutions of (1.5), it follows from Lemma 3.1 that

$$m\left(r,\frac{1}{w-a_\pm}\right)=S(r,w)\quad\text{and}\quad m\left(r,\frac{1}{w-b_\pm}\right)=S(r,w). \tag{3.1}$$

That is to say,

$$N\left(r,\frac{1}{w-a_+}\right)=T(r,w)+S(r,w),\qquad N\left(r,\frac{1}{w-a_-}\right)=T(r,w)+S(r,w)$$

and

$$N\left(r,\frac{1}{w-b_+}\right)=T(r,w)+S(r,w),\qquad N\left(r,\frac{1}{w-b_-}\right)=T(r,w)+S(r,w) \tag{3.2}$$

which imply, together with the assumption that w is non-rational, that there exist infinitely many zeros of $w-a_\pm$ and of $w-b_\pm$.

Suppose first that $z_i\in\mathbb{C}$ is a zero of $w-b_\pm=0$, i.e., that $w(z_i)=b_+(z_i)$ or $w(z_i)=b_-(z_i)$. Then, it follows by (1.5) that we have three possibilities:

(a) $w(qz_i)w(z_i/q)=\infty$,
(b) $w(z_i)=a_\pm(z_i)$,
(c) $a_2(z_i)=0$.

Suppose that there are infinitely many points z_i such that (b) is valid. Then there are infinitely many points such that either

$$a_+(z_i)=w(z_i)=b_\pm(z_i)\qquad\text{or}\qquad a_-(z_i)=w(z_i)=b_\pm(z_i).$$

Since $a_\pm$ and $b_\pm$ are rational functions, this means that either $a_-\equiv b_\pm$ or $a_+\equiv b_\pm$. But this implies that there is cancelation on the right-hand side of (1.5), which is in contradiction with the assumptions of Theorem 1.1. Similarly, if there are infinitely many points where (c) holds, then it follows that $a_2\equiv 0$, which is again a contradiction. Therefore, all except finitely many zeros z_i of $w-b_\pm$ satisfy property (a).

Let $A=\{z_i\in\mathbb{C}:i\in\mathbb{N}\}$ be the multiset of all zeros of

$$w(z)-b_+(z)=0$$

and of

$$w(z) - b_-(z) = 0,$$

where each zero of $w-b_\pm$ is repeated in A according to its multiplicity. To be precise, elements of A should be written as pairs (z_0, k_0), where k_0 is the multiplicity of z_0 in the multiset A. However, from now on we mostly suppress the multiplicity of the elements of A for the sake of brevity. Let

$$N\left(r, \frac{1}{w-b_\pm}\right) = \int_0^r \frac{n\left(t, \frac{1}{w-b_\pm}\right) - n\left(0, \frac{1}{w-b_\pm}\right)}{t} dt + n\left(0, \frac{1}{w-b_\pm}\right) \log r,$$

where $n\left(r, \frac{1}{w-b_\pm}\right)$ is the cardinality of the multiset $A \cap \{z \in \mathbb{C} : |z| \leq r\}$. Out of the elements z_i of A all except possibly finitely many satisfy either $w(qz_i) = \infty$ or $w(z_i/q) = \infty$ (or both). The multiset A may be therefore split into a union of multisets

$$A = A_\infty \cup A_F,$$

where

$$A_\infty = \{z_i \in A : w(qz_i) = \infty \text{ or } w(z_i/q) = \infty\},$$
$$A_F = \{z_i \in A : w(qz_i) \text{ and } w(z_i/q) \text{ are finite}\}$$

are again multisets taking into account multiplicities of their elements. This should be understood in the following way. If $w(z_0) = b_+(z_0)$ with multiplicity k_0 and $w(qz_0)$ is a pole of order $k_1 < k_0$ and $w(z_0/q)$ is finite, then z_0 appears k_1 times in the set A_∞ and $k_0 - k_1$ times in the set A_F. For example, if z_0 is a zero of $w(z) - b_+(z)$ of multiplicity 2, then $(z_0, 2) \in A$. If in addition $w(qz_0)$ is a pole of multiplicity 1, then $(z_0, 1) \in A_\infty$ and $(z_0, 1) \in A_F$. The same convention is adopted in all of the following sets.

We have already established that the set A_F contains finitely many elements at most. By (3.1), we therefore have

$$N\left(r, \frac{1}{w-b_\pm}\right) = N_{A_\infty}\left(r, \frac{1}{w-b_\pm}\right) + O(\log r) = 2T(r, w) + S(r, w),$$

where $N_{A_\infty}\left(r, \frac{1}{w-b_\pm}\right)$ is the integrated counting function corresponding to the set $A_\infty \cap \{z \in \mathbb{C} : |z| \leq r\}$. Let B_∞ be the subset of A_∞ of all elements z_i such that either $w(q^2 z_i) = b_\pm(q^2 z_i)$, or $w(z_i/q^2) = b_\pm(z_i/q^2)$ (or both identities hold). Denote by $N_{B_\infty}(r, \frac{1}{w-b_\pm})$ the corresponding integrated counting function, and by

$$N_{A_\infty \setminus B_\infty}\left(r, \frac{1}{w-b_\pm}\right) = N_{A_\infty}\left(r, \frac{1}{w-b_\pm}\right) - N_{B_\infty}\left(r, \frac{1}{w-b_\pm}\right)$$

the integrated counting function for those $b_\pm$-points of w such that $w(z_i) = b_\pm(z_i)$, and $w(qz_i) = \infty$ or $w(z_i/q) = \infty$, but $w(q^2 z_i) \neq b_\pm(q^2 z_i)$ and $w(z_i/q^2) \neq b_\pm(z_i/q^2)$. Therefore, for each two points in $B_\infty \cap \{z \in \mathbb{C} : |z| \leq r\}$ there is exactly one pole in the disc of radius $\max\{|q|r, |q|^{-1}r\} =: \tilde{q}r$ centred at the origin, which can be uniquely associated with them. We denote by $\widetilde{B_\infty}$ the set of all poles of w that are associated this way with the elements of the set B_∞, and by $N_{\widetilde{B_\infty}}(r, w)$ the corresponding integrated counting function. Then we have

$$2N_{\widetilde{B_\infty}}(r, w) \leq N_{B_\infty}\left(\tilde{q}r, \frac{1}{w-b_\pm}\right) \leq 2N_{\widetilde{B_\infty}}(\tilde{q}^2 r, w),$$

and so

$$2N_{\widetilde{B_\infty}}(r,w) = N_{B_\infty}\left(r, \frac{1}{w-b_\pm}\right) + S(r,w) \tag{3.3}$$

by Lemma 2.1. The details of how the identity

$$N_{\widetilde{B_\infty}}(\tilde{q}^2 r,w) = N_{\widetilde{B_\infty}}(r,w) + S(r,w) \tag{3.4}$$

follows from Lemma 2.1 can be found in [29], where an analogous property was shown for the usual pole counting function $N(r,w)$. Since the reasoning is identical, we omit further details. Similarly as above, for each point in $(A_\infty \setminus B_\infty) \cap \{z \in \mathbb{C} : |z| \leq r\}$ one can associate at least one pole of w in the disc $\{z \in \mathbb{C} : |z| \leq \tilde{q}r\}$. By denoting the set of these poles by $\widetilde{A_\infty \setminus B_\infty}$, and the integrated counting function for the poles of w in the set $(\widetilde{A_\infty \setminus B_\infty}) \cap \{z \in \mathbb{C} : |z| \leq r\}$ by $N_{\widetilde{A_\infty \setminus B_\infty}}(r,w)$, we have

$$N_{A_\infty \setminus B_\infty}\left(r, \frac{1}{w-b_\pm}\right) \leq N_{\widetilde{A_\infty \setminus B_\infty}}(\tilde{q}r,w). \tag{3.5}$$

Suppose now that

$$N_{B_\infty}\left(r, \frac{1}{w-b_\pm}\right) \leq 2\alpha T(r,w) + S(r,w), \tag{3.6}$$

where $0 \leq \alpha < 1$. Then by (3.3), we have

$$N_{\widetilde{B_\infty}}(r,w) \leq \alpha T(r,w) + S(r,w). \tag{3.7}$$

On the other hand, by (3.2)

$$\begin{aligned} 2T(r,w) &= N\left(r, \frac{1}{w-b_\pm}\right) + S(r,w) \\ &= N_{B_\infty}\left(r, \frac{1}{w-b_\pm}\right) + N_{A_\infty \setminus B_\infty}\left(r, \frac{1}{w-b_\pm}\right) + S(r,w) \\ &\leq 2N_{\widetilde{B_\infty}}(r,w) + N_{\widetilde{A_\infty \setminus B_\infty}}(\tilde{q}r,w) + S(r,w) \\ &\leq N(\tilde{q}r,w) + N_{\widetilde{B_\infty}}(\tilde{q}r,w) + S(r,w). \end{aligned} \tag{3.8}$$

Hence, by (3.7), we have

$$2T(r,w) \leq (1+\alpha)T(\tilde{q}r,w) + S(r,w), \tag{3.9}$$

where $\alpha < 1$. But now, since w is of zero order, it follows by Lemma 2.1 in a similar fashion as in (3.4) that $T(\tilde{q}r,w) = T(r,w) + S(r,w)$, and so (3.9) becomes

$$2T(r,w) \leq (1+\alpha)T(r,w) + S(r,w) \tag{3.10}$$

which is impossible. Hence the assumption (3.6) must be false, and so

$$N_{B_\infty}\left(r, \frac{1}{w-b_\pm}\right) = 2T(r,w) + S(r,w), \tag{3.11}$$

which means roughly speaking that almost all $b_\pm$-points of w belong to the set B_∞. This leads to the following four cases:

(a) $\quad w(z_i) = b_-(z_i)$ and $w(q^2z_i) = b_+(q^2z_i)$,

(b) $\quad w(z_i) = b_+(z_i)$ and $w(q^2z_i) = b_-(q^2z_i)$,

(c) $\quad w(z_i) = b_+(z_i)$ and $w(q^2z_i) = b_+(q^2z_i)$,

(d) $\quad w(z_i) = b_-(z_i)$ and $w(q^2z_i) = b_-(q^2z_i)$.

In what follows, we denote by B^a_∞ the set of points $z_i \in B_\infty$ such that (a) holds, and similarly by B^b_∞, B^c_∞ and B^d_∞ the subsets of B_∞ consisting of points z_i such that (b), (c) and (d) hold, respectively. From (3.11) it follows that one of the following statements must be valid:

$$\text{(i)} \begin{cases} N_{B^c_\infty}\left(r, \frac{1}{w-b_\pm}\right) \neq S(r,w), \\ N_{B^d_\infty}\left(r, \frac{1}{w-b_\pm}\right) \neq S(r,w), \end{cases}$$

$$\text{(ii)} \begin{cases} N_{B^a_\infty}\left(r, \frac{1}{w-b_\pm}\right) \neq S(r,w), \\ N_{B^b_\infty}\left(r, \frac{1}{w-b_\pm}\right) \neq S(r,w), \\ N_{B^c_\infty}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w), \\ N_{B^d_\infty}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w), \end{cases}$$

$$\text{(iii)} \begin{cases} N_{B^a_\infty}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w), \\ N_{B^b_\infty}\left(r, \frac{1}{w-b_\pm}\right) \neq S(r,w), \\ N_{B^c_\infty}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w), \\ N_{B^d_\infty}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w), \end{cases}$$

$$\text{(iv)} \begin{cases} N_{B^a_\infty}\left(r, \frac{1}{w-b_\pm}\right) \neq S(r,w), \\ N_{B^b_\infty}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w), \\ N_{B^c_\infty}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w), \\ N_{B^d_\infty}\left(r, \frac{1}{w-b_\pm}\right) = S(r,w). \end{cases}$$

Suppose first that (i) is true. Then for all points $z_i \in B^c_\infty$ (i.e., for those points for which (c) is valid), we have the equation

$$w(z_i)w(q^2z_i) = b_+(z_i)b_+(q^2z_i) = a_2(qz_i). \tag{3.12}$$

On one hand by assumption (i) we have

$$N_{B^c_\infty}\left(r, \frac{1}{w-b_\pm}\right) \neq S(r,w),$$

and on the other hand,

$$T(r, b_+(z)b_+(q^2z) - a_2(qz)) = O(\log r)$$

since b_+ and a_2 are rational. Therefore it follows that

$$a_2(z) \equiv b_+(qz)b_+(z/q). \tag{3.13}$$

Similarly as in the case (3.13), according to (d), we obtain (3.12) but with the subscript "+" replaced by "−". Following the line of reasoning above yields

$$a_2(z) \equiv b_-(qz)b_-(z/q). \tag{3.14}$$

From equations (3.13) and (3.14) it follows that $b_+ \equiv b_-$ or $b_+ \equiv -b_-$. The former contradicts the assumption that b_+ and b_- are distinct, so we have $b_+ \equiv -b_-$.

Suppose that (ii) is true. Then, in the same way as in (3.13) and (3.14), we obtain

$$b_+(qz)b_-(z/q) \equiv b_-(qz)b_+(z/q) \equiv a_2(z), \tag{3.15}$$

which implies that $b_-/b_+ \equiv \eta_b$, where $\eta_b \in \mathbb{C}$ is a constant.

Suppose now that (iii) holds, and let $\widetilde{B^a_\infty}$, $\widetilde{B^b_\infty}$, $\widetilde{B^c_\infty}$ and $\widetilde{B^d_\infty}$ denote the set of poles that are part of the sequences of the corresponding type (a), (b), (c) or (d). Then by assumption (iii) the pole counting functions $N_{\widetilde{B^a_\infty}}(r,w)$, $N_{\widetilde{B^c_\infty}}(r,w)$ and $N_{\widetilde{B^d_\infty}}(r,w)$ are all of the growth $S(r,w)$, but

$$N_{\widetilde{B^b_\infty}}(r,w) \neq S(r,w).$$

In addition, there may be poles of w that are not in any of the sets $\widetilde{B^a_\infty}$, $\widetilde{B^b_\infty}$, $\widetilde{B^c_\infty}$ or $\widetilde{B^d_\infty}$. For instance, there may be poles of $w(z)$ such that $w(q^2z) = 0$ and $w(qz) \neq b_\pm(qz)$. Let $\widetilde{C}$ denote the set of all poles of w in the complement of $\widetilde{B^a_\infty} \cup \widetilde{B^b_\infty} \cup \widetilde{B^c_\infty} \cup \widetilde{B^d_\infty}$, and suppose that the corresponding integrated counting function $N_{\widetilde{C}}(r,w)$ satisfies

$$N_{\widetilde{C}}(r,w) \neq S(r,w). \tag{3.16}$$

This means that there is a constant $\beta > 0$ and a set $F \subset (0,\infty)$ of strictly positive logarithmic density such that

$$N_{\widetilde{C}}(r,w) \geq \beta T(r,w)$$

for all $r \in F$. Since $N(r,w) = N_{\widetilde{C}}(r,w) + N_{\widetilde{B_\infty}}(r,w)$, it follows that

$$N_{\widetilde{B_\infty}}(r,w) \leq (1-\beta)T(r,w) \tag{3.17}$$

for all $r \in F$. Now, by (3.8) and Lemma 2.1, we have

$$2T(r,w) \leq N(r,w) + N_{\widetilde{B_\infty}}(r,w) + S(r,w),$$

and so (3.17) yields

$$2T(r,w) \leq (2-\beta)T(r,w) + S(r,w)$$

for all $r \in F$. This is impossible since the exceptional set associated with the error term $S(r,w)$ is of zero logarithmic density, and F has strictly positive logarithmic density. We conclude that (3.16) cannot be valid, and so

$$N_{\widetilde{B^a_\infty} \cup \widetilde{B^c_\infty} \cup \widetilde{B^d_\infty} \cup \widetilde{C}}(r,w) = S(r,w), \tag{3.18}$$

meaning that "almost all" poles of w are in the set $\widetilde{B^b_\infty}$.

We will now show that (3.18) leads to w satisfying a q-difference Riccati equation. To this end, we define an auxiliary function

$$u(z) = (w(z) - b_+(z))(w(qz) - b_-(qz)). \tag{3.19}$$

Those poles of w which are in the set $\widetilde{B^b_\infty}$ are not poles of u, since they cancel out with the zeros of $w - b_\pm$ both when $w(z_i) = \infty$ and when $w(qz_i) = \infty$. Therefore the only possible poles of u are the poles of the coefficients b_- and b_+, and the poles of w in the set $\widetilde{B^a_\infty} \cup \widetilde{B^c_\infty} \cup \widetilde{B^d_\infty} \cup \widetilde{C}$. By the assumption that $b_\pm$ are rational, and by (3.18), we have

$$N(r,u) = S(r,w). \tag{3.20}$$

We will now show that also the proximity function of u is of the growth $S(r,w)$. In order to do this we need a q-difference analogue of the Clunie lemma due to Laine and Yang.

Lemma 3.2 ([15], Theorem 2.5)**.** *Let $f(z)$ be a transcendental meromorphic solution of order zero of*

$$U(z,f)P(z,f) = Q(z,f),$$

where $U(z,f)$, $P(z,f)$ and $Q(z,f)$ are q-difference polynomials with rational coefficients such that $\deg U(z,f) = n$ and $\deg Q(z,f) \le n$. If $U(z,f)$ contains just one term of maximal degree in $f(z)$ and its q-shifts, then

$$m(r, P(z,f)) = S(r,f).$$

By applying Lemma 3.2 to the upshifted equation (1.5) with

$$P(z,w) = w(z), \qquad U(z,q) = w(q^2z)(w(qz) - b_+(qz))(w(qz) - b_-(qz))$$

and

$$Q(z,w) = a_2(qz)(w(qz) - a_+(qz))(w(qz) - a_-(qz)),$$

it follows that

$$m(r,w) = S(r,w). \tag{3.21}$$

By upshifting (1.5) twice and using Lemma 3.2 as above, we have

$$m(r, w(qz)) = S(r,w), \tag{3.22}$$

and so

$$m(r,u) = S(r,w) \tag{3.23}$$

by combining equations (3.19), (3.21) and (3.22). Hence it follows by (3.20) and (3.23) that $T(r,u) = S(r,w)$, which means that u is a small function with respect to w, and so equation (3.19) becomes a q-difference Riccati equation with small meromorphic coefficients.

We will now show that u is, in fact, a rational function. From (3.19) it follows that w satisfies

$$w(qz) = \frac{b_-(qz)w(z) + u(z) - b_-(qz)b_+(z)}{w(z) - b_+(z)}$$

and

$$w(z/q) = \frac{b_+(z/q)w(z) + u(z/q) - b_-(z)b_+(z/q)}{w(z) - b_-(z)}.$$

Since, on the other hand, w is a solution of (1.5), it follows that

$$A(z)w(z)^2 + B(z)w(z) + C(z) = 0, \tag{3.24}$$

where

$$\begin{cases} A(z) = a_2(z) - b_-(qz)b_+(z/q), \\ B(z) = -a_2(z)(a_+(z) + a_-(z)) - b_-(qz)u(z/q) - b_+(z/q)u(z) \\ \qquad\qquad + b_+(z/q)b_-(qz)(b_+(z) + b_-(z)), \\ C(z) = -(u(z) - b_-(qz)b_+(z))(u(z/q) - b_-(z)b_+(z/q)) \\ \qquad\qquad + a_2(z)a_+(z)a_-(z). \end{cases} \tag{3.25}$$

Now since $T(r,u) = S(r,w)$ and $b_\pm$ are rational, it follows that $T(r,A) = S(r,w)$, $T(r,B) = S(r,w)$ and $T(r,C) = S(r,w)$. Therefore (3.24) becomes an algebraic equation for w with meromorphic coefficients that are small with respect to w, and so it follows that $A \equiv B \equiv C \equiv 0$. Using this fact with (3.25), we have

$$\begin{cases} a_2(z) = b_-(qz)b_+(z/q), \\ u(z/q) = -\frac{b_+(z/q)}{b_-(qz)}u(z) + a_2(z)(b_+(z) - a_+(z) + b_-(z) - a_-(z)), \\ a_2(z)a_+(z)a_-(z) = (u(z) - b_-(qz)b_+(z))(u(z/q) - b_-(z)b_+(z/q)). \end{cases} \tag{3.26}$$

By combining the three equations in (3.26) we obtain an algebraic equation of degree two for u with rational coefficients. By solving this equation we obtain an explicit algebraic expression for u in terms of the rational coefficients of (1.5). From (3.19) it follows that u is meromorphic, and hence, in fact, rational. We conclude that w satisfies a q-difference Riccati equation

$$w(qz) = \frac{b_-(qz)w(z) + u(z) - b_-(qz)b_+(z)}{w(z) - b_+(z)} \tag{3.27}$$

with rational coefficients, where u depends only on a_+, a_-, b_+ and b_-. The explicit expression of $u(z)$ can be easily determined and is given by

$$\begin{aligned} u(z) =\Big(&- b_-(qz)a_-(z) + b_-(qz)b_+(z) - b_-(qz)a_+(z) \\ &- b_-(z) + b_+(z) + b_-(z)b_-(qz) + \Big(2b_-(qz)^2a_+(z)a_-(z) \\ &- 2b_-(qz)b_+(z)^2 + b_-(qz)^2b_+(z)^2 - 4a_+(z)a_-(z) \\ &+ 2b_+(z)b_-(z) + b_+(z)^2 + 2b_-(qz)^2b_+(z)b_-(z) \\ &+ b_-(qz)^2a_-(z)^2 + b_-(qz)^2a_+(z)^2 + b_-(z)^2 - 2b_-(z)^2b_-(qz) \\ &+ b_-(z)^2b_-(qz)^2 + 2b_-(qz)a_-(z)b_-(z) - 2b_-(qz)^2a_-(z)b_-(z) \\ &+ 2b_-(qz)a_+(z)b_-(z) - 2b_-(qz)^2a_+(z)b_-(z) \\ &+ 2b_-(qz)b_+(z)a_+(z) + 2b_-(qz)b_+(z)a_-(z) - 2b_-(qz)^2b_+(z)a_-(z) \\ &- 2b_-(qz)^2b_+(z)a_+(z) - 4b_-(qz)b_+(z)b_-(z)\Big)^{1/2}\Big)\frac{b_-(qz)}{2}. \end{aligned} \tag{3.28}$$

In the several remaining similar cases leading to a q-difference Riccati equation we will not state the rather complicated expressions for $u(z)$ explicitly. Obtaining these expressions by the same way as above is straightforward.

If (iv) holds, then it can be seen exactly in the same way as in (iii) that w satisfies the q-difference Riccati equation

$$w(qz) = \frac{b_+(qz)w(z) + u(z) - b_+(qz)b_-(z)}{w(z) - b_-(z)}, \tag{3.29}$$

where u is a rational function obtained by a similar method as in (3.28).

In the next phase of the proof we transform equation (1.5) using $w \to \frac{1}{\omega}$, to obtain

$$\omega(qz)\omega(z/q) = \frac{\frac{b_+(z)b_-(z)}{a_2(z)a_+(z)a_-(z)}(\omega(z) - 1/b_+(z))(\omega(z) - 1/b_-(z))}{(\omega(z) - 1/a_+(z))(\omega(z) - 1/a_-(z))}, \tag{3.30}$$

which has the same exact form as (1.5) but different coefficients. Define, similarly as in the first part of the proof,

$$A_0 = \{z_i \in \mathbb{C} : \omega(z_i) = 1/a_\pm(z_i), \text{ and } \omega(qz_i) = \infty \text{ or } \omega(z_i/q) = \infty\},$$

and let $B_0 \subset A_0$ consist of all elements z_i such that either $\omega(q^2 z_i) = b_\pm(q^2 z_i)$, or $\omega(z_i/q^2) = b_\pm(z_i/q^2)$ (or both identities hold). Again, this leads to four cases:

$$\begin{array}{llcl}
(e) & \omega(z_i) = 1/a_-(z_i) & \text{and} & \omega(q^2 z_i) = 1/a_+(q^2 z_i),\\
(f) & \omega(z_i) = 1/a_+(z_i) & \text{and} & \omega(q^2 z_i) = 1/a_-(q^2 z_i),\\
(g) & \omega(z_i) = 1/a_+(z_i) & \text{and} & \omega(q^2 z_i) = 1/a_+(q^2 z_i),\\
(h) & \omega(z_i) = 1/a_-(z_i) & \text{and} & \omega(q^2 z_i) = 1/a_-(q^2 z_i).
\end{array}$$

We denote by B_0^e the subset of points of B_0 for which (e) holds, and by B_0^f, B_0^g and B_0^h the subsets of B_0 such that (f), (g) and (h) are valid, respectively. We also denote by $\widetilde{B_0^e}$, $\widetilde{B_0^f}$, $\widetilde{B_0^g}$ and $\widetilde{B_0^h}$ the sets of poles that are part of the sequences of the corresponding types (e), (f), (g) or (h). Similarly as above, we have the following four possibilities:

$$\text{(v)} \begin{cases} N_{B_0^g}\left(r, \frac{1}{w-1/a_\pm}\right) \neq S(r,w),\\ N_{B_0^h}\left(r, \frac{1}{w-1/a_\pm}\right) \neq S(r,w), \end{cases}$$

$$\text{(vi)} \begin{cases} N_{B_0^e}\left(r, \frac{1}{w-1/a_\pm}\right) \neq S(r,w),\\ N_{B_0^f}\left(r, \frac{1}{w-1/a_\pm}\right) \neq S(r,w),\\ N_{B_0^g}\left(r, \frac{1}{w-1/a_\pm}\right) = S(r,w),\\ N_{B_0^h}\left(r, \frac{1}{w-1/a_\pm}\right) = S(r,w), \end{cases}$$

$$\text{(vii)} \begin{cases} N_{B_0^e}\left(r, \frac{1}{w-1/a_\pm}\right) = S(r,w),\\ N_{B_0^f}\left(r, \frac{1}{w-1/a_\pm}\right) \neq S(r,w),\\ N_{B_0^g}\left(r, \frac{1}{w-1/a_\pm}\right) = S(r,w),\\ N_{B_0^h}\left(r, \frac{1}{w-1/a_\pm}\right) = S(r,w), \end{cases}$$

$$\text{(viii)} \begin{cases} N_{B_0^e}\left(r, \frac{1}{w-1/a_\pm}\right) \neq S(r,w),\\ N_{B_0^f}\left(r, \frac{1}{w-1/a_\pm}\right) = S(r,w),\\ N_{B_0^g}\left(r, \frac{1}{w-1/a_\pm}\right) = S(r,w),\\ N_{B_0^h}\left(r, \frac{1}{w-1/a_\pm}\right) = S(r,w). \end{cases}$$

Suppose first that (i) and (v) both hold. We have already shown that (i) implies $b_+(z) \equiv -b_-(z)$ and $a_2(z) \equiv b_+(qz)b_+(z/q)$. Similarly, in the case (v), we have $a_+(z) \equiv -a_-(z)$ and

$$\frac{b_+(z)b_-(z)}{a_2(z)a_+(z)a_-(z)} \equiv \frac{1}{a_+(z/q)a_+(qz)} \equiv \frac{1}{a_-(z/q)a_-(qz)}. \tag{3.31}$$

This implies that

$$\frac{a_+(qz)}{b_+(qz)} \cdot \frac{a_+(z/q)}{b_+(z/q)} \equiv \frac{a_+(z)^2}{b_+(z)^2}, \tag{3.32}$$

which can be solved as a q-difference equation with respect to a_+/b_+ to obtain $a_+/b_+ \equiv \mu z^\lambda$, where $\mu \in \mathbb{C}$ and $\lambda \in \mathbb{Z}$ (since a_+ and b_+ are rational) are arbitrary constants. Now by using equation (3.31) and the transformation $\omega(z) = y(z)/a_+(z)$, equation (3.30) takes the form

$$y(qz)y(z/q) = \frac{y(z)^2 - a_+(z)^2/b_+(z)^2}{y(z)^2-1} = \frac{y(z)^2 - \mu^2 z^{2\lambda}}{y(z)^2-1}, \tag{3.33}$$

which is equation (1.8) of Theorem 1.1.

Assume now that (ii) and (vi) are valid. We have already observed that (ii) yields equation (3.15), which implies that $b_+/b_- \equiv \eta_b$, where η_b is a complex constant. Similarly, by using the transformed equation (3.30) in the case (vi), we have

$$\frac{a_2(z)a_+(z)a_-(z)}{b_+(z)b_-(z)} \equiv a_+(z/q)a_-(qz) \equiv a_+(qz)a_-(z/q) \tag{3.34}$$

which implies that $a_+/a_- \equiv \eta_a$, where $\eta_a \in \mathbb{C}$ is a constant. By combining (3.34) and (3.15), we have

$$\left(\frac{a_+(z)a_-(z)}{b_+(z)b_-(z)}\right)^2 \equiv \frac{a_+(qz)a_-(qz)}{b_+(qz)b_-(qz)}\frac{a_+(z/q)a_-(z/q)}{b_+(z/q)b_-(z/q)}. \tag{3.35}$$

By solving this we obtain

$$\frac{a_+(z)a_-(z)}{b_+(z)b_-(z)} \equiv \tilde{\mu} z^{\tilde{\lambda}}, \tag{3.36}$$

where $\tilde{\mu} \in \mathbb{C}$ and $\tilde{\lambda} \in \mathbb{C}$ are arbitrary constants. Now, by substituting $b_+ \equiv \eta_b b_-$ and $a_+ \equiv \eta_a a_-$ into (3.36), it follows that

$$\frac{a_-(z)}{b_-(z)} \equiv \mu z^{\lambda}, \tag{3.37}$$

where $\mu = \sqrt{\eta_b \tilde{\mu}/\eta_a}$ and $\lambda = \tilde{\lambda}/2$. Note that $\eta_a \neq 0$ since otherwise we would have $a_+ \equiv a_1 \equiv 0$ which is prohibited by the assumptions of Theorem 1.1. In addition, we have $\lambda \in \mathbb{Z}$ by the fact that a_-/b_- is a rational function. Now by using equations (3.37) and (3.15), the transformation $w(z) = b_-(z)y(z)$ takes equation (1.5) to the form

$$\begin{aligned} y(qz)y(z/q) &= \frac{\eta_b(y(z) - \eta_a \frac{a_-(z)}{b_-(z)})(y(z) - \frac{a_-(z)}{b_-(z)})}{(y(z)-\eta_b)(y(z)-1)} \\ &= \frac{\eta_b(y(z)^2 + \delta z^{\lambda} y(z) + \eta_a \mu^2 z^{2\lambda})}{(y(z)-\eta_b)(y(z)-1)}, \end{aligned} \tag{3.38}$$

where $\delta = (\eta_a + 1)\mu$. This equation is the q-difference Painlevé III equation (1.9).

Suppose now that (i) and (vi) are valid. We have already shown that (i) yields the equations

$$a_2(z) \equiv b_+(qz)b_+(z/q) \equiv b_-(qz)b_-(z/q), \tag{3.39}$$

and that $b_+ \equiv -b_-$. By combining (3.34) and (3.39) it follows that equation (3.35) is valid in this case also. Hence, as above, we can solve it to obtain (3.37). Finally, by combining $a_+ \equiv \eta_a a_-$, $b_+ \equiv -b_-$ and (3.37), we obtain equation (3.38) in the special case $\eta_b = -1$. Thus we have obtained a special case of (1.9).

Consider next the case where (ii) and (v) are both valid simultaneously. In this case, equation (3.35) holds, and moreover we have $a_+ \equiv -a_-$, $b_+ \equiv \eta_b b_-$

and $a_2(z) \equiv b_+(qz)b_-(z/q)$. By combining these equations, and by performing the transformation $w \to b_- w$, equation (1.5) takes the form

$$w(qz)w(z/q) = \frac{\eta_b(w(z)^2 + \mu z^\lambda)}{(w(z)-1)(w(z)-\eta_b)}$$

which is equation (1.10) of Theorem 1.1.

If one of the cases (iii) and (iv) holds together with any of the cases (v)–(viii), we obtain one of the respective Riccati equations (3.27) or (3.29). Similarly, if we have (vii) and one of the cases (i)–(iv), it follows that w satisfies the difference q-Riccati equation

$$w(qz) = \frac{(1/a_-(qz) - a_+(z)u(z))\, w(z) - a_+(z)/a_-(qz)}{w(z) - a_+(z)},$$

where u is a rational function the form of which can be explicitly obtained in a similar way as in (3.28). Finally, if (viii) is valid together with one of the cases (i)–(iv), we have

$$w(qz) = \frac{(1/a_+(qz) - a_-(z)u(z))\, w(z) - a_-(z)/a_+(qz)}{w(z) - a_-(z)},$$

where again u is rational.

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