A note on iterated spherical average on Lebesgue spaces

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\textbf{A B S T R A C T}

For the spherical average $A$ and the Laplace operator $\Delta$, we study the iterated operator $\Delta(A)^N$ and find the minimal positive integer $N$ to ensure the $L^1(\mathbb{R}^n)$ boundedness of $\Delta(A)^N$. Using a constructive proof, we completely solve a question posted by Belinsky et al. (2003).

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\section{Introduction}

Let

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

be the Laplacian on the $n$-dimensional Euclidean space $\mathbb{R}^n$. The Fourier transform of $\Delta(f)$ is

$$\hat{\Delta(f)}(\xi) = -|\xi|^2 \hat{f}(\xi).$$

The average on a sphere with radius $t$ ($>0$) is defined as

$$A_t(f)(x) = \int_{S^{n-1}} f(x - ty')d\sigma(y'), \quad n \geq 2,$$

where $S^{n-1}$ is the unit sphere with the surface Lebesgue measure normalized by $\int_{S^{n-1}} d\sigma(y') = 1$, and $y' = y/|y|$ is the unit vector for any $y \neq 0$.

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This averaging operator has its strong background in harmonic analysis, dating back as far as the early 1970s (see [9] and [10]). In [9], Stein proved that, in the case $n \geq 3$,

$$\lim_{t \to 0^+} A_t(f)(x) = f(x)$$

almost everywhere for $f \in L^p(\mathbb{R}^n)$ provided $p > \frac{n}{n-1}$, and the convergence result fails if $p \leq \frac{n}{n-1}$ (see also [10]). Stein’s proof is based on considering a more general spherical average $S^\gamma t$ with $S^\gamma_0 = A_t(f)$, so that the analytical interpolation method can be employed. In his proof, a crucial part is to obtain a precise $L^2(\mathbb{R}^n)$ estimate with the help of the Plancherel theorem. However, this argument fails if $n = 2$, since $p > \frac{n}{n-1}$ means $p > 2$ when $n = 2$. In 1986, J. Bourgain used some geometric methods to obtain this convergence result for all $f \in L^p(\mathbb{R}^2)$ with $p > 2$ (see [3], also [10]). The operator $A_t$ also plays a notable role in the approximation theory (see [1]). In order to obtain some equivalent forms of the K-functional in $L^p(\mathbb{R}^n)$ spaces, Belinsky, Dai and Ditzian in [1] considered the iterates $(A_t)^N$ for positive integers $N$ and obtained the following theorem.

**Theorem A ([1]).** Let $1 \leq p \leq \infty$. For $f \in L^p(\mathbb{R}^n)$, $n \geq 2$ and $N > \frac{2(n+2)}{n-1}$, we have the inequality

$$\left\| \Delta (A_t)^N (f) \right\|_{L^p(\mathbb{R}^n)} \leq t^{-2} \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } t > 0. \quad (1)$$

Following Theorem A, Belinsky, Dai and Ditzian then made a remark that they did not know how near the power $N$ required in Theorem A was to the optimal for $L^1$ or $L^\infty$ (see [1]). Their remark raises the following interesting problem.

**Problem BDD.** Find the smallest integer $N$ to ensure the inequality

$$\left\| \Delta (A_t)^N (f) \right\|_{L^1(\mathbb{R}^n)} \leq t^{-2} \|f\|_{L^1(\mathbb{R}^n)} \quad \text{for } t > 0. \quad (2)$$

This question was addressed by Fan and Zhao in [4]. With the aid of the known estimates of wave operators in [6] or [8], Fan and Zhao proved the following result.

**Theorem B ([4]).** Let $n \geq 2$ and $t > 0$. Then for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,

$$\left\| \Delta (A_t)^N (f) \right\|_{L^p(\mathbb{R}^n)} \leq t^{-2} \|f\|_{L^p(\mathbb{R}^n)}$$

if $|1/2 - 1/p| \leq \frac{N(n+1)-2}{n-1}$; For $f \in L^1(\mathbb{R}^n)$,

$$\left\| \Delta (A_t)^N (f) \right\|_{L^1(\mathbb{R}^n)} \leq t^{-2} \|f\|_{L^1(\mathbb{R}^n)}$$

if $N > \frac{n+3}{n-1}$.

Clearly, Theorem B improves Theorem A. But Problem BDD still remains open, since the necessity is unknown. With this observation and motivation, the aim of this paper is to completely solve this problem. The following two theorems are our main results.

**Theorem 1.1.** Let $t > 0$ and $N$ be positive integers. Suppose $n \neq 3$, $n \neq 5$. Then

$$\left\| \Delta (A_t)^N (f) \right\|_{L^1(\mathbb{R}^n)} \leq t^{-2} \|f\|_{L^1(\mathbb{R}^n)}$$

if and only if $N > \frac{n+3}{n-1}$. 
Note that in Theorem 1.1, if \( n = 3 \) then the condition \( N > \frac{n+3}{n-1} \) is equivalent to \( N > 3 \). Similarly, if \( n = 5 \) then the condition \( N > \frac{n+3}{n-1} \) is equivalent to \( N > 2 \). However, we have the following result at these two specific dimensions.

**Theorem 1.2.** Let \( t > 0 \) and \( N \) be positive integers.

(i) \[ \left\| \Delta (A_{t})^{N} (f) \right\|_{L^{1}(\mathbb{R}^{5})} \leq t^{-2} \| f \|_{L^{1}(\mathbb{R}^{5})} \]

if and only if \( N \geq 2 \).

(ii) \[ \left\| \Delta (A_{t})^{N} (f) \right\|_{L^{1}(\mathbb{R}^{3})} \leq t^{-2} \| f \|_{L^{1}(\mathbb{R}^{3})} \]

if and only if \( N \geq 3 \).

From Theorem 1.1, we see that, for the \( L^{1}(\mathbb{R}^{n}) \) boundedness of \( \Delta (A_{t})^{N} \), the sufficient condition on \( N \) obtained in Theorem B is also the necessary condition for all dimensions \( n \), except at \( n = 3 \) and \( n = 5 \). However, an interesting phenomenon observed in Theorem 1.2 is that there exists a slight difference on the \( L^{1}(\mathbb{R}^{n}) \) boundedness between the spherical averages and the wave operators. Remember that the multiplier of a wave operator is defined as

\[ W_{\gamma} (\xi) = \frac{e^{ic|\xi|}}{|\xi|^n} \psi (\xi) \]

where \( \gamma \geq 0 \), \( c \) is a nonzero constant and \( \psi \) is a \( C^{\infty} \) function that is equal to 0 in a neighborhood of the origin and is equal to 1 if \( |\xi| > C \) for a fixed positive number \( C \). The well-known estimate of wave operators (see [6,8]) says that \( W_{\gamma} (\xi) \) is an \( L^{p} (\mathbb{R}^{n}) \) multiplier if and only if \( |1/2 - 1/p| \leq \frac{2}{n-1} \) when \( 1 < p < \infty \), and that \( W_{\gamma} (\xi) \) is an \( L^{1} (\mathbb{R}^{n}) \) multiplier if and only if

\[ \gamma > \frac{n - 1}{2} \]

In the second section, we will see that, without loss of generality, the multiplier of the spherical average \( A_{1} \) is given by

\[ V^{n-2} (|\xi|) = \sqrt{\frac{2}{\pi}}\cos \left( \frac{|\xi| - \frac{(n-1)\pi}{4}}{|\xi|^{n-1}} \right) \psi (\xi) + B(\xi) \]

where \( B(\xi) \) is an \( L^{1} (\mathbb{R}^{n}) \) multiplier. Thus we cannot conclude that \( V^{n-2} (|\xi|) \) is an \( L^{1} (\mathbb{R}^{n}) \) multiplier, or \( A_{1} \) is bounded on \( L^{1} (\mathbb{R}^{n}) \) if we use the sharp \( L^{1} \) estimate for the wave operators by writing

\[ \cos \left( \frac{|\xi| - \frac{(n-1)\pi}{4}}{|\xi|^{n-1}} \right) \psi (\xi) = \frac{e^{-i\frac{(n-1)\pi}{4}}e^{i|\xi|}}{|\xi|^{rac{n-1}{2}}} \psi (\xi) + \frac{e^{i\frac{(n-1)\pi}{4}}e^{-i|\xi|}}{|\xi|^{rac{n-1}{2}}} \psi (\xi), \]

the sum of two multipliers of different wave operators. On the other hand we can easily see that \( A_{1} \) is bounded on \( L^{1} (\mathbb{R}^{n}) \) if we use the Minkowski inequality and recall the definition of \( A_{1} (f) \). This phenomenon is the essence of (ii) in Theorem 1.2.

This paper is organized as follows. In Section 2, we list some preliminary knowledge. The proof on Theorem 1.1 can be found in Section 3. The proof also illustrates the main idea of the proof for the case \( n \leq 5 \). Thus, in Section 4, to prove the case \( n = 5 \) and the case \( n = 3 \), we will only give details for those different from Section 3.

Throughout this article, the symbol \( A \preceq B \) means that there exists a constant \( C > 0 \) independent of all essential variables such that \( A \leq CB \). We use the notation \( A \approx B \) if \( A \preceq B \) and \( B \preceq A \). The symbol \( A \simeq B \) means that there exists a constant \( C \) independent of all essential variables such that \( A = CB \). Also, we use \( a_{k}, b_{k}, c_{k}, k = 1, 2, \ldots, \) to denote some constants that may be different at each of their appearances.
2. Some preliminary knowledge

Fix a measurable function $m$, the Fourier multiplier operator $T_m(f)$ is defined via the Fourier transform by

$$
\hat{T_m(f)}(\xi) = m(\xi)\hat{f}(\xi),
$$

where $m$ is called the multiplier (or symbol) of $T_m(f)$. If $T_m$ is bounded on $L^p(\mathbb{R}^n)$, then we say that $m$ is an $L^p(\mathbb{R}^n)$ multiplier. Also, up to a constant multiple, $T_m$ is a convolution operator with the kernel

$$
K(x) = \int_{\mathbb{R}^n} m(\xi)e^{i\xi \cdot x}d\xi.
$$

It is well known that a scaling argument yields that the operator norms satisfy

$$
\|T_{m(\cdot)}\|_{L^p \to L^p} = \|T_{m(t \cdot)}\|_{L^p \to L^p} \quad (\text{3})
$$

uniformly on $t > 0$. The following multiplier theorem is well known.

**Milhlin Multiplier Theorem.** Let $1 < p < \infty$. Suppose that $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ satisfies

$$
|\nabla^k m(\xi)| \leq |\xi|^k
$$

for $0 \leq k \leq n/2 + 1$. Then $m(\xi)$ is an $L^p(\mathbb{R}^n)$ multiplier.

Let $J_\nu(r)$ be the Bessel function of order $\nu$. We have the following asymptotic form of $J_\nu(r)$.

**Lemma 2.1 (Proposition 5.1 in [4]).** Let $r$ be a positive number and $\nu > -1/2$. For any positive integer $L$, on the interval $[1, \infty)$ we have

$$
J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos \left( r - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) + \sum_{j=0}^{L} a_j e^{ir\rho - \frac{1}{2} - j} + \sum_{j=0}^{L} b_j e^{-ir\rho - \frac{1}{2} - j} + E(r),
$$

where $a_j$ and $b_j$ are constants for all $j$, and $E(r)$ is a $C^\infty$ function satisfying

$$
\left| E^{(k)}(r) \right| \leq r^{-\frac{1}{2} - L - 1}
$$

for any $k = 0, 1, 2, \ldots$.

Write

$$
V_\nu(r) = \frac{J_\nu(r)}{r^{\nu}}.
$$

By checking the Fourier transform (see [1]), we have that

$$
\Delta (A_t)^N(f)(\xi) \simeq t^{-2} \hat{f}(\xi) \mathcal{F}\left( N, \frac{n-2}{2}, |t\xi| \right),
$$

where

$$
\mathcal{F}(N, \gamma, |\xi|) = |\xi|^2 (V_\gamma(|\xi|))^N.
$$

Let $\eta$ be a radial $C^\infty(\mathbb{R}^n)$ function satisfying $\eta(\xi) \equiv 1$ if $|\xi| \leq 10$ and $\text{supp}(\eta) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 20\}$. Let $\Psi(\xi) = 1 - \eta(\xi)$. We write

$$
|\xi|^2 \left( V_{n-2}(|\xi|) \right)^N = m_1(\xi) + m_2(\xi),
$$
where
\[ m_1 (\xi) = \eta(\xi) |\xi|^2 \left( V_{n-2} (|\xi|) \right)^N, \]
\[ m_2 (\xi) = \Psi(\xi) |\xi|^2 \left( V_{n-2} (|\xi|) \right)^N. \]

We have the following lemma.

**Lemma 2.2.** The \( m_1 \) defined above is an \( L^p \) multiplier for \( p \geq 1 \) and any positive integer \( N \).

**Proof.** \( T_{m_1} \) is a convolution operator \( T_{m_1} (f) = K_1 * f \), where
\[ K_1 (x) \simeq \int_{\mathbb{R}^n} \eta(\xi) |\xi|^2 \left( V_{n-2} (|\xi|) \right)^N e^{i\xi \cdot x} d\xi. \]

By Young’s inequality
\[ \| K_1 * f \|_{L^p} \leq \| K_1 \|_{L^1} \| f \|_{L^p}, \]
it suffices to show that \( K_1 \) is an integrable function. In fact, using the well known formula
\[ V_\delta (t) = O(1) \text{ if } |t| < 100, \]
we know that if \( |x| < 100 \) then
\[ |K_1 (x)| \leq 1. \]

On the other hand, if \( |x| \geq 100 \), without loss of generality we assume \( |x_1| \geq |x|/n \). Using the derivative formula for the Bessel function
\[ \frac{dV_\delta (t)}{dt} = -tV_{\delta+1} (t), \]
we take integration by parts on \( \xi_1 \) variable to obtain that for \( |x| \geq 100 \)
\[ |K_1 (x)| \leq \frac{1}{|x_1|^{n+1}} \leq \frac{1}{|x|^{n+1}}. \]

These estimates imply that \( K_1 \) is an integrable function.

The following lemma can be found in page 171 of [5].

**Lemma 2.3.** Let \( \gamma \neq -1 \) and \( \sigma > 0 \). Then
\[ \int_0^\infty e^{-\sigma r} r^\gamma e^{isr} dr = ie^{\frac{2\pi i \gamma}{2}} \Gamma (\gamma + 1) (s + i\sigma)^{-\gamma-1}, \]
where \( \Gamma (\gamma + 1) \) is the Gamma function.

### 3. Proof of Theorem 1.1

The proof for the case of \( n = 2 \) and \( n = 4 \) is the same as the proof for \( n \geq 6 \), with only minor modifications. We leave the proof to the reader. Thus, in this section, we mainly focus to prove the case of \( n \geq 6 \). In this case, by Theorem B, the inequality in (2) holds if \( N \geq 2 \). Therefore, it remains to show that the inequality in (2) fails if \( N = 1 \). Noticing
\[ \Delta (\hat{A}_{\ell}) (f) (\xi) = Ct^{-\frac{n}{2}} \hat{f} (\xi) |\xi|^2 V_{n-2} (|t\xi|), \]
by (3) we can assume \( t = 1 \) and we only need to show that \( \mu (\xi) = |\xi|^2 V_{n-2} (|\xi|) \) is not an \( L^1 (\mathbb{R}^n) \) multiplier. To this end, we state the following more general result.
**Theorem 3.1.** Let $n \geq 6$ and $1 \leq p \leq \infty$. Then $|\xi|^2 V_{n-2} (|\xi|) \psi (\xi)$ is an $L^p$ multiplier if and only if

$$|1/2 - 1/p| \leq 1/2 - \frac{2}{n-1}.$$ 

**Proof.** The “if” part of the theorem is the first part of Theorem B with $N = 1$. We only need to prove the “only if” part. Let functions $\eta$ and $\Psi$ be defined in Section 2. By Lemma 2.2 and a duality argument, it suffices to show that, for $1 \leq p \leq 2$,

$$m(\xi) = |\xi|^2 V_{n-2} (|\xi|) \psi (\xi)$$

is an $L^p$ multiplier only if

$$1/p \leq 1 - \frac{2}{n-1}.$$ 

(4)

Assume that $T_m$ is bounded on $L^p$ for some $p \geq 1$. Namely we have a constant $C_1 > 0$ such that

$$\|T_m (f)\|_{L^p} \leq C_1 \|f\|_{L^p}.$$ 

Define the Abel mean $G_\sigma (f)$ of $f$, $\sigma > 0$, by

$$\hat{G_\sigma (f)} (\xi) = e^{-\sigma |\xi|} \hat{f} (\xi).$$ 

It is well known that for all $1 \leq p \leq \infty$, there is a constant $C_2$ independent of $\sigma > 0$ such that

$$\|G_\sigma (f)\|_{L^p (\mathbb{R}^n)} \leq C_2 \|f\|_{L^1 (\mathbb{R}^n)},$$

for all $f \in L^p (\mathbb{R}^n)$. Pick a function $f \in S (\mathbb{R}^n)$ such that the support of $f$ lies in the unit ball and $f(x) = 1$ for $|x| < 1/2$. We may assume $0 \leq f(x) \leq 1$. Set

$$f_\sigma (x) = f \left( \frac{x}{\sigma} \right).$$

Then

$$\int_{\mathbb{R}^n} |f_\sigma (x)|^p \, dx \simeq \sigma^n.$$ 

On the other hand, we have

$$\frac{\|G_\sigma (T_m (f_\sigma))\|_{L^p}}{\|f_\sigma\|_{L^p}} \leq C_1 C_2.$$ 

Now we choose a sequence $\{\sigma_k\} \downarrow 0$ for which

$$\lim_{k \to \infty} \frac{\|G_{\sigma_k} (T_m (f_{\sigma_k}))\|_{L^p}}{\|f_{\sigma_k}\|_{L^p}} \leq C_1 C_2.$$ 

(5)

Next, we will use an argument of contradiction to show that it is impossible to have (5) if $1/p > 1 - \frac{2}{n-1}$, which leads to (4) as we desire.

Noting that $G_\sigma T_m (f_\sigma)$ is a convolution

$$G_\sigma T_m (f_\sigma) \simeq K_{\sigma,m} * f_\sigma,$$

we now estimate the behavior of the kernel

$$K_{\sigma,m} (x) = \int_{\mathbb{R}^n} |\xi|^2 V_{n-2} (|\xi|) e^{-\sigma |\xi|} \psi (\xi) e^{i\xi \cdot x} \, d\xi.$$
in the set
\[ \Omega = \{ x \in \mathbb{R}^n : |1 - x| < 0.01 \} . \]
At the polar coordinate system (see Chapter 4 in [11]) we have
\[ K_{\sigma, m}(x) \simeq \int_0^\infty r^{n+1} V_{n-2} \left( \frac{r}{2} \right) e^{-\sigma r} \Psi(r) V_{n-2} \left( r |x| \right) dr. \]
By the asymptotic form of \( J_\nu(r) \), we have that for \( x \in \Omega \) and \( r \geq 1 \)
\[ r^{n+1} V_{n-2} \left( \frac{r}{2} \right) = \sum_{j=0}^{3} a_j e^{x|x|} \left( r |x| \right)^{n+3}_{x} - j + \sum_{j=1}^{3} b_j e^{-i r} \left( r^{n+3}_{x} - j + O \left( r^{n+5}_{x} \right) \right) \]
and
\[ V_{n-2} \left( r |x| \right) = \sum_{j=0}^{3} a_j e^{x|x|} \left( r |x| \right)^{n+3}_{x} - j + \sum_{j=0}^{3} b_j e^{-i r} \left( r |x| \right)^{n+3}_{x} - j + O \left( r^{n+5}_{x} \right). \]
Therefore, for \( x \in \Omega \) and \( r \geq 1 \), we have
\[
K_{\sigma, m}(x) \simeq \int_0^\infty \frac{\left( a_0 b_0 r^2 e^{r(1-|x|)} + a_0 b_0 r^2 e^{-r(1-|x|)} \right)}{|x|^{n+1}_{x}} e^{-\sigma r} \Psi(r) dr \\
+ \int_0^\infty \frac{\left( a_0 b_0 r^2 e^{r(1+|x|)} + b_0 r^2 e^{-r(1+|x|)} \right)}{|x|^{n+1}_{x}} e^{-\sigma r} \Psi(r) dr \\
+ \sum_{j=1}^{3} \int_0^\infty \frac{\left( c_j e^{r(1-|x|)} + d_j e^{-r(1-|x|)} \right)}{|x|^{n+1+j}_{x}} r^{2-j} e^{-\sigma r} \Psi(r) dr \\
+ \sum_{j=1}^{3} \int_0^\infty \frac{\left( \alpha_j e^{r(1+|x|)} + \beta_j e^{-r(1+|x|)} \right)}{|x|^{n+1+j}_{x}} r^{2-j} e^{-\sigma r} \Psi(r) dr \\
+ \int_0^\infty A(r,x) e^{-\sigma r} \Psi(r) dr,
\]
where all \( a_0, b_0, c_j, d_j, \alpha_j, \beta_j \) are constants, and \( A(r,x) = O(r^{-2}) \) uniformly on \( x \in \Omega \).
Recalling that \( \Psi(r) = 0 \) if \( r < 10 \), we have
\[
\int_0^\infty A(r,x) e^{-\sigma r} \Psi(r) dr = O(1), \quad x \in \Omega. \tag{6}
\]
For \( x \in \Omega \), we write
\[
\int_0^\infty \frac{\left( a_0 b_0 r^2 e^{r(1-|x|)} + a_0 b_0 r^2 e^{-r(1-|x|)} \right)}{|x|^{n+1}_{x}} e^{-\sigma r} \Psi(r) dr \\
= \int_0^\infty \frac{\left( a_0 b_0 r^2 e^{r(1-|x|)} + a_0 b_0 r^2 e^{-r(1-|x|)} \right)}{|x|^{n+1}_{x}} e^{-\sigma r} dr \\
- \int_0^\infty \frac{\left( a_0 b_0 r^2 e^{r(1-|x|)} + a_0 b_0 r^2 e^{-r(1-|x|)} \right)}{|x|^{n+1}_{x}} e^{-\sigma r} \eta(r) dr.
\]
Here, by the choice of \( \eta \), we know that
\[
\int_0^\infty \frac{\left( a_0 b_0 r^2 e^{r(1-|x|)} + a_0 b_0 r^2 e^{-r(1-|x|)} \right)}{|x|^{n+1}_{x}} e^{-\sigma r} \eta(r) dr = O(1) \tag{7}
\]
uniformly on \( x \in \Omega \) and \( \sigma > 0 \). By Lemma 2.3, we have that, for \( x \in \Omega \),

\[
\int_0^\infty \left( a_0 b_0 r^2 e^{i r (1-|x|)} + a_0 b_0 r^2 e^{-i r (1-|x|)} \right) e^{-\sigma r} dr \\
\approx \int_0^\infty \left( r^2 e^{i r (1-|x| + i \sigma)} + r^2 e^{-i r (1-|x| + i \sigma)} \right) dr \\
\approx \frac{1}{((1-|x|) + i \sigma)^3} - \frac{1}{((1-|x|) - i \sigma)^3} \\
= -i2\sigma \left\{ 3(1-|x|)^2 + \sigma^2 \right\} \\
\frac{1}{(1-|x|)^2 + \sigma^2}.
\]

Thus, we obtain that

\[
\left| \int_0^\infty \left( a_0 b_0 r^2 e^{i r (1-|x|)} + a_0 b_0 r^2 e^{-i r (1-|x|)} \right) e^{-\sigma r} dr \right| \\
\leq \frac{\sigma (1-|x|)^2}{(1-|x|)^2 + \sigma^2} \geq \frac{\sigma}{(1-|x|)^4}
\]

if \( |1-|x|| > 100\sigma \) and \( x \in \Omega \).

A similar argument involving Lemma 2.3 shows that, for \( x \in \Omega \), we have

\[
\sum_{j=1}^{2} \int_0^\infty \left\{ c_j e^{i r (1-|x|)} + d_j e^{-i r (1-|x|)} \right\} r^{2-j} \frac{e^{-\sigma r} \Psi (r)}{|x|^{n-1} + j} dr \\
\approx \sum_{j=1}^{2} c_j (1-|x| + i \sigma)^{-j} + \sum_{j=1}^{2} d_j (1-|x| - i \sigma)^{-j} + O(1),
\]

where all \( c_j \) and \( d_j \) are constants, which may not be the same as the previously appeared \( c_j \) and \( d_j \). For the term \( j = 3 \),

\[
\int_0^\infty \left\{ c_3 e^{i r (1-|x|)} + d_3 e^{-i r (1-|x|)} \right\} r^{1} \frac{e^{-\sigma r} \Psi (r)}{|x|^{n-1} + 3} dr,
\]

we cannot directly invoke Lemma 2.3. Let us estimate only the term

\[
c_3 \int_0^\infty \frac{e^{i r (1-|x|)}}{r |x|^{n-1+3}} e^{-\sigma r} \Psi (r) dr \approx \int_0^\infty \frac{e^{ir(1-|x|)+i\sigma}}{r} \Psi (r) dr,
\]

since the estimate of another term is the same, where \( x \in \Omega \). Using integration by parts we now have

\[
\int_0^\infty \frac{e^{ir(1-|x|)+i\sigma}}{r} \Psi (r) dr = \int_0^\infty \frac{i}{(1-|x|)+i\sigma} \int_0^\infty e^{ir(1-|x|)+i\sigma} \frac{d}{dr} \left( \frac{\Psi (r)}{r} \right) dr.
\]

By the definition of \( \Psi (r) \), it is easy to see

\[
\left| \int_0^\infty \frac{e^{ir(1-|x|)+i\sigma}}{r} \Psi (r) dr \right| \leq \frac{1}{|(1-|x|)+i\sigma|},
\]
A similar argument (may be easier) shows that

\[
\begin{align*}
&\left| \sum_{j=1}^{3} \int_{0}^{\infty} \frac{\alpha_j e^{i\tau(1+|x|)} + \beta_j e^{-i\tau(1+|x|)}}{|x|^{\frac{n-1}{2} + j}} e^{-\sigma r \Psi(r)} \, dr \right| \\
+ &\left| \int_{0}^{\infty} a_0^2 e^{i\tau(1+|x|)} + b_0^2 e^{-i\tau(1+|x|)} \frac{2 \sigma}{|x|^{\frac{n-1}{2}}} e^{-\sigma r \Psi(r)} \, dr \right| = O(1)
\end{align*}
\]

(11)

uniformly for \( x \in \Omega \) and \( \sigma > 0 \).

Combining estimates (6)–(11), we can find a small \( \rho > 0 \) and a positive number \( D \) such that for \( \sigma \leq 10^{-3} \rho \) and \( \rho > |1 - |x|| > 50\sigma \) we have

\[ |K_{\sigma,m}(x)| \geq \frac{\sigma}{(1 - |x|)^4} - \frac{D}{(1 - |x|)^2}. \]

Note that

\[
\int_{A} \left| \int_{\mathbb{R}^n} G_{\sigma}(K)(x-y)f_{\sigma}(x) \, dx \right|^{p} dy \neq 0
\]

only if \( |x| < \sigma \). By the above estimate on \( K_{\sigma,m}(x) \), we have that a constant \( c \) depending on \( p \) such that

\[
\| G_{\sigma}(T_m(f_{\sigma})) \|_{p} \geq \int_{A} \left| \int_{\mathbb{R}^n} K_{\sigma,m}(x-y)f_{\sigma}(y) \, dy \right|^{p} dx
\]

\[
\simeq \left\{ \int_{200\sigma < 1 - |x| < \rho/2} c \left( \frac{\sigma}{(1 - |x|)^4} \right)^{p} - \left( \frac{D}{(1 - |x|)^2} \right)^{p} dx \right\} \|f_{\sigma}\|_{L^1}^{p}.
\]

Now, for fixed \( \rho > 0 \),

\[
\left\{ c \int_{200\sigma < 1 - |x| < \rho/2} \left( \frac{\sigma}{(1 - |x|)^4} \right)^{p} dy \right\} \|f_{\sigma}\|_{L^1}^{p}
\]

\[
\simeq \sigma^{pn+p} \int_{200\sigma < 1 - |x| < \rho/2} \frac{1}{(1 - |x|)^{4p}} dy
\]

\[
\simeq \sigma^{pn+p} \sigma^{-4p+1} = \sigma^{p(n-3)+1},
\]

and

\[
\|f_{\sigma}\|_{L^1}^{p} \left\{ \int_{200\sigma < 1 - |x| < \rho/2} \frac{1}{(1 - |x|)^{2p}} dy \right\}
\]

\[
\simeq \sigma^{pn}\sigma^{-2p+1}.
\]

This shows that for small \( \sigma \)

\[ \| G_{\sigma}(T_m(f_{\sigma})) \|_{p} \simeq \sigma^{p(n-3)+1}, \]

and

\[ \frac{\| G_{\sigma}(T_m(f_{\sigma})) \|_{L^p}}{\|f_{\sigma}\|_{L^p}} \simeq \sigma^{p(n-3)+1-n}. \]

On the other hand by the assumption we know that

\[ \lim_{k \to \infty} \frac{\| G_{\sigma_k}(T_m(f_{\sigma_k})) \|_{L^p}}{\|f_{\sigma_k}\|_{L^p}} \leq 1, \]
for a sequence \( \{\sigma_k\} \downarrow 0 \). Hence we must require \( p(n - 3) + 1 - n \geq 0 \), which is equivalent to

\[
\frac{1}{p} \leq 1 - \frac{2}{n - 1}.
\]

Theorem 1.1 is proved.

Remark. By checking the “only if” part of the theorem, it is not difficult to see that the proof works also for the case of \( n = 5 \). Hence, combining to Theorem B, we have the following corollary from this fact.

Corollary 3.2. At \( n = 5 \), \( |\xi|^2 V_{\frac{3}{2}} (|\xi|) \) is bounded on \( L^p (\mathbb{R}^5) \) if and only if \( p = 2 \).

4. Proof of Theorem 1.2

The proofs for \( n = 3 \) and \( n = 5 \) share the same idea. We prove only the case \( n = 5 \) and leave the proof for another case to the reader. The following is the main result in this section.

Proposition 4.1. At \( n = 5 \), \( |\xi|^2 \left( V_{\frac{3}{2}} (|\xi|) \right)^N \) is an \( L^1(\mathbb{R}^5) \) multiplier if and only if \( N = 2 \).

Proof. By Corollary 3.2, it suffices to show that \( |\xi|^2 \left( V_{\frac{3}{2}} (|\xi|) \right)^2 \) is an \( L^1(\mathbb{R}^5) \) multiple. Let \( \psi (\xi) \) be defined in Section 2. By Lemma 2.2, \( |\xi|^2 \left( V_{\frac{3}{2}} (|\xi|) \right)^2 \) is an \( L^1(\mathbb{R}^5) \) multiplier if and only if \( m (\xi) = \psi (\xi) |\xi|^2 \left( V_{\frac{3}{2}} (|\xi|) \right)^2 \) is an \( L^1(\mathbb{R}^5) \) multiplier. By the definition of \( V_{\frac{3}{2}} (|\xi|) \) and the known formula

\[
J_{\frac{3}{2}} (r) = \sqrt{\frac{2}{\pi r}} \left( \frac{\sin r}{r} - \cos r \right),
\]
we have that

\[
\psi (\xi) |\xi|^2 V_{\frac{3}{2}} (|\xi|)^2
\]

\[
\simeq \psi (\xi) \frac{\cos^2 |\xi|}{|\xi|^2} + \psi (\xi) \frac{\sin^2 |\xi|}{|\xi|^4} - \psi (\xi) \frac{\sin 2 |\xi|}{|\xi|^3}
\]

\[
= m (\xi) + \mu (\xi) + \omega (\xi).
\]

We need to show that all \( m, \mu \) and \( \omega \) are \( L^1(\mathbb{R}^5) \) multipliers. To this end, we will only prove the fact for \( \psi (\xi) \frac{\cos^2 |\xi|}{|\xi|^2} \), since the proofs to other two multipliers are similar and even more simpler.

Let \( G_\sigma (f), \sigma > 0 \), be the Abel mean. It suffices to show that the kernel \( (G_\sigma K) (x) \) is an integrable function uniformly on \( \sigma > 0 \), where

\[
(G_\sigma K) (x) \simeq \int_{\mathbb{R}^5} e^{-\sigma |\xi|} \psi (\xi) \frac{\cos^2 |\xi|}{|\xi|^2} e^{i\xi \cdot x} d\xi.
\]

Our strategy is to decompose

\[
(G_\sigma K) (x) = K_{\sigma, 1} (x) + K_{\sigma, 2} (x)
\]
and show that both kernels \( K_{\sigma, 1} (x) \) and \( K_{\sigma, 2} (x) \) are integrable.

Notice

\[
\cos^2 |\xi| = \frac{1}{4} \left( e^{2i|\xi|} - 2 + e^{-2i|\xi|} \right).
\]
From Chapter 4 in [11], we know that

\[(G_\sigma K)(x) \simeq \int_0^\infty e^{-r\sigma} r^2 \Psi(r) \cos^2 \frac{r}{3^2} (r \mid x \mid) \, dr = K_{\sigma,1}(x) + K_{\sigma,2}(x),\]

where

\[K_{\sigma,1}(x) \simeq \frac{1}{|x|^2} \int_0^\infty \Psi(r) e^{-r\sigma} \left( e^{2ir} + e^{-2ir} \right) \left( \frac{\sin r |x|}{r} - \cos r |x| \right) \, dr,\]

\[K_{\sigma,2}(x) \simeq \frac{1}{|x|^2} \int_0^\infty \Psi(r) e^{-r\sigma} \left( \frac{\sin r |x|}{r} - \cos r |x| \right) \, dr.\]

Write

\[K_{\sigma,2}(x) \simeq \frac{1}{|x|^2} \int_0^\infty e^{-r\sigma} \Psi(r) \frac{\sin r |x|}{r} \, dr - \frac{1}{|x|^2} \int_0^\infty e^{-r\sigma} \Psi(r) \cos r |x| \, dr.\]

We will only show that the second function is integrable, since the proof to the first function is the same. By integration by parts, we obtain that

\[
- \frac{1}{|x|^3} \int_0^\infty e^{-r\sigma} \Psi(r) \cos r |x| \, dr = \frac{1}{|x|^3} \left( \int_0^\infty \sigma e^{-r\sigma} \Psi(r) \sin r |x| \, dr - \int_0^\infty e^{-r\sigma} \Psi'(r) \sin r |x| \, dr \right).
\]

Since \( \Psi'(r) = 0 \) when \( r > 20 \), we obtain that

\[
\frac{1}{|x|^3} \int_0^\infty e^{-r\sigma} \Psi(r) \cos r |x| \, dr \leq \frac{1}{|x|^3}
\]

uniformly on \( \sigma > 0 \) if \( |x| < 10 \). When \( |x| \geq 10 \), we keep using integration by parts to obtain

\[
\frac{1}{|x|^3} \int_0^\infty \frac{d}{dr} \left( e^{-r\sigma} \Psi(r) \right) \sin r |x| \, dr \leq \frac{1}{|x|^6}.
\]

Thus

\[
\|K_{\sigma,2}\|_{L^1(\mathbb{R}^5)} \leq \int_{|x|<10} \frac{dx}{|x|^2} + \int_{|x|\geq10} \frac{dx}{|x|^5} \leq 1
\]

uniformly on \( \sigma > 0 \).

Next, we write

\[
K_{\sigma,1}(x) \simeq \frac{1}{|x|^2} \int_0^\infty \Psi(r) e^{-r\sigma} \left( e^{2ir} + e^{-2ir} \right) \frac{\sin r |x|}{r} \, dr
- \frac{1}{|x|^2} \int_0^\infty \Psi(r) e^{-r\sigma} \left( e^{2ir} + e^{-2ir} \right) \cos r |x| \, dr,
\]

and we will only show that the second function is integrable, since the proof to the first function is the same.

In fact, the second function

\[
- \frac{1}{|x|^3} \int_0^\infty \Psi(r) e^{-r\sigma} \left( e^{2ir} + e^{-2ir} \right) \cos r |x| \, dr
\]

\[
\simeq \frac{1}{|x|^2} \int_0^\infty \Psi(r) e^{-r\sigma} \left( e^{2ir} + e^{-2ir} \right) \left( e^{ir|x|} + e^{-ir|x|} \right) \, dr
\]
\[
\frac{1}{|x|^2} \int_0^\infty e^{-\sigma r} \left( e^{i r (2-|x|)} + e^{-i r (2-|x|)} \right) \psi(r) \, dr \\
+ \frac{1}{|x|^2} \int_0^\infty e^{-\sigma r} \left( e^{i r (2+|x|)} + e^{-i r (2+|x|)} \right) \psi(r) \, dr \\
= G_\sigma (K_3)(x) - G_\sigma (K_4)(x).
\]

We will only show that \( G_\sigma (K_3)(x) \) is integrable. The proof for the integrability of \( G_\sigma (K_4)(x) \) is the same and is omitted.

Firstly, an easy integration by parts yields that
\[
|G_\sigma (K_3)(x)| \leq |x|^{-6} \quad \text{if} \quad |x| > 10. \tag{13}
\]

Secondly, we write
\[
G_\sigma (K_3)(x) \simeq \frac{1}{|x|^2} \int_0^\infty e^{-\sigma r} \left( e^{i r (2-|x|)} + e^{-i r (2-|x|)} \right) \, dr \\
- \frac{1}{|x|^2} \int_0^\infty e^{-\sigma r} \left( e^{i 2r e^{-i r |x|}} + e^{-i 2r e^{i r |x|}} \right) \eta(r) \, dr \\
= G_\sigma (K_5)(x) - G_\sigma (K_6)(x).
\]

It is easy to check, by the definition of \( \eta \), that
\[
|G_\sigma (K_6)(x)| \leq \frac{1}{|x|^2} \quad \text{if} \quad |x| \leq 10. \tag{14}
\]

By Lemma 2.3, we have that for \(|x| \leq 10\),
\[
G_\sigma (K_5)(x) \simeq \frac{\sigma}{|x|^2 \{(1 - |x|) + i \sigma\} \{(1 - |x|) - i \sigma\}} \\
= \frac{\sigma}{|x|^2 \{(1 - |x|)^2 + \sigma^2\}}.
\tag{15}
\]

Thus let
\[
E_\sigma = \{x \in \mathbb{R}^n : 1/2 < |x| < 10\} \cap \{x \in \mathbb{R}^n : |1 - |x|| > \sigma\}.
\]

By (13)–(15), we obtain
\[
\|G_\sigma (K_3)\|_{L^1(\mathbb{R}^5)} \leq \|G_\sigma (K_5)\|_{L^1(\mathbb{R}^5)} + \|G_\sigma (K_6)\|_{L^1(\mathbb{R}^5)} \tag{16}
\]
\[
\leq \int_{|x|<10} \frac{1}{|x|^2} \, dx + \int_{|x|\geq 10} \frac{1}{|x|^2} \, dx + \int_{|1-|x||<\sigma} \frac{\sigma}{|1-|x||^2} \, dx + \int_{E_\sigma} \frac{\sigma}{|x| (1-|x|)^2} \, dx \\
\leq 1.
\]

The proposition is proved.

5. Further discussions

By a scaling argument, the inequalities in Theorem B are equivalent to
\[
\left\| \Delta (A_1)^N (f) \right\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}. \tag{17}
\]
for $p \geq 1$. Let $L^p_\alpha(\mathbb{R}^n)$ be the homogeneous Sobolev space consisting of all functions satisfying
\[
\|f\|_{L^p_\alpha(\mathbb{R}^n)} = \left\| |\Delta|^{\alpha/2} f \right\|_{L^p(\mathbb{R}^n)} < \infty,
\]
where $|\Delta|^{\alpha/2}$ is the fractional Laplacian of order $\alpha$ defined by
\[
|\Delta|^{\alpha/2} f (\xi) = |\xi|^\alpha \hat{f}(\xi).
\]
The inequality in (17) is thus equivalent to
\[
\left\| (A_1)^N (f) \right\|_{L^p_2(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)},
\]
(18)
which says that the $N$-fold convolution $(A_1)^N$ gains 2 units of regularity in the homogeneous Sobolev space:
\[
(A_1)^N : L^p(\mathbb{R}^n) \to L^p_2(\mathbb{R}^n).
\]
When $n = 1$, the average of sphere is reduced to
\[
a_t (f) (x) = \frac{1}{2} (f (x + t) + f (x - t)).
\]
It is well-known that $a_t (f)$ and its iterates $a_t^N (f)$ are not in general smoother than $f (x)$. With the increase of the dimension of space we can see a significant change in this phenomenon. An interpretation of our result is that it is sufficient to take 2-fold convolution of spherical average $A_1$ to gain 2 units of regularity in the homogeneous $L^1 (\mathbb{R}^n)$ spaces when $n \geq 5$, and it suffices to take 3-fold convolution of spherical average $A_1$ to gain 2 units of regularity in the homogeneous $L^1 (\mathbb{R}^n)$ spaces when $n = 3, 4$. Also, we need to take at least 6-fold convolution of spherical average $A_1$ to gain 2 units of regularity in the homogeneous $L^1 (\mathbb{R}^2)$ space. It is worth to notice that we can use our method to study a similar problem to $L^p_\alpha (\mathbb{R}^n)$ for any $p \geq 1$ and $\alpha > 0$.

Our method also might provide potential applications to study short random walks in high dimensions originated by Pearson [7] in 100 years ago. An $N$-step uniform walk in $\mathbb{R}^n$ starts at the origin and consists of $N$ independent steps of length 1, each of which is taken into a uniformly random direction. It is known in Theorem 2.1 of [2] that the probability density function $p_N (\frac{n-2}{2}, x)$ of such a random walk is the Fourier inverse of $(A_1)^N$. Precisely, $p_N (\frac{n-2}{2}, x)$ is the kernel of the convolution $(A_1)^N f$:
\[
(A_1)^N f = p_N (\frac{n-2}{2}, \cdot) * f.
\]
With Proposition 4.1 as a model, our method might allow us to study the performance of $p_N (\frac{n-2}{2}, \cdot)$ after $N$ steps for any integer $N$, based on the asymptotic formula of Bessel functions and Lemma 2.3.

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