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Nyström-Clenshaw-Curtis Quadrature for the Solution of Volterra Integral Equations with Proportional Delays

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Abstract. The Nyström-Clenshaw-Curtis (NCC) quadrature, which was proposed in [S. Y. Kang, I. Koltracht, and G. Rawitscher, Math. Comp. 72, 729–756 (2003)], is a highly accurate numerical method for solving integral equations with semi-smooth kernel. In this paper, we introduce the basic idea of the NCC quadrature and derive an NCC quadrature for Volterra integral equations with proportional delays. Numerical results are presented to illustrate the high accuracy of the method.

INTRODUCTION

Integral equations occur in various areas of applied mathematics, physics and engineering, and there are rich results in numerical methods for the solution of Volterra integral equations. In this paper, we consider the following second-kind linear Volterra integral equation (VIE) with proportional delay

\[ u(t) = f(t) + \int_0^t k_1(t, s)u(s)ds + \int_0^q k_2(t, s)u(s)ds, \quad t \in [0, T], \]

where \(0 < q < 1\) and \(T > 0\), and \(f(t) \in C^p([0, T])\), \(k_1(t, s), k_2(t, s) \in C^p([0, T]^2)\) for certain positive even integer \(p\). It has been shown in [1] that (1) admits a unique solution \(u \in C^p([0, T])\).

Delay functional equations frequently arise in physical and biological processes, see for instance, [2]. The analysis of this type of questions dates back to the works in [3] and [4], and numerical methods for the equation have attracted many researchers and a lot of results have been achieved, see for instance [5, 6] and the references therein.

In this paper, we introduce the basic idea of the Nyström-Clenshaw-Curtis (NCC) quadrature and derive an NCC quadrature for Volterra integral equations with proportional delays. We note that the NCC quadrature, which was proposed in [7], is a highly accurate numerical method for solving integral equations with semi-smooth kernel. Numerical results are presented to illustrate the high accuracy of the method.

NCC QUADRATURE FOR VIE WITH PROPORTIONAL DELAY

We first apply the idea of the NCC quadrature to the following integral operator

\[ \int_{-1}^{w} h(t, s)u(s)ds, \quad -1 \leq w \leq t \leq 1, \]

where \(h(t, s) \in C^p([-1, 1]^2)\) and \(u(s) \in C^p([-1, 1])\) for certain positive integer \(p\). It is easily seen that for any fixed \(t \in [0, 1]\), \(h(t, s)u(s)\), as a function of \(s\), belongs to \(C^p([-1, 1])\). The basic idea of the NCC quadrature is to approximate the integrand (as functions of \(s\) for fixed \(t\)) by polynomials in the interval \([-1, 1]\).

Let \(T_j(s) = \cos(j \arccos(s)), \quad j = 0, 1, 2, \ldots\) be the sequence of Chebyshev polynomials and let \(\tau_l = \cos \frac{2l-1}{2n} \pi, \quad l = 1, \ldots, n\), be the roots of \(T_n(s)\). Let the interpolating polynomial of \(f(s)\) with the Chebyshev grid \(\{\tau_1, \tau_2, \ldots, \tau_n\}\) be denoted by \(\mathcal{T}_{n,f}\), i.e., \(\mathcal{T}_{n,f}\) is a polynomial of degree less than \(n\) and

\[ (\mathcal{T}_{n,f})(\tau_l) = f(\tau_l), \quad l = 1, 2, \ldots, n. \]
We have the following result for the error of the interpolating polynomials; see [8].

**Lemma 1** Suppose $f(x) \in C^p[-1, 1]$. If $n > p$, then

$$
[f(x) - (T_n f)(x)] \leq \delta_{n,p} \max_{x \in [-1,1]} |f^p(x)| = O(n^{-p} \log n), \quad x \in [-1, 1],
$$

where

$$
\delta_{n,p} = \left(\frac{\pi}{2}\right)^p (3 + \frac{2}{\pi}) \log(n - p + 2)^{-p}.
$$

Define

$$
F_j(w) = \int_{-1}^w h(\tau_i, s) u(s) ds, \quad -1 \leq w \leq \tau_1 \leq 1.
$$

Interpolating $f_i(s) \equiv h(\tau_i, s)u(s)$ at the Chebyshev grid $\{\tau_1, \tau_2, \ldots, \tau_n\}$, we get an approximation of $f_j(s)$:

$$
f_i(s) \approx (T_n f)(s) = \sum_{j=1}^n \alpha_{i,j} T_j(s), \quad -1 \leq s \leq 1,
$$

where

$$
[\alpha_{1,1}, \ldots, \alpha_{1,n}]^T = C^{-1} [h(\tau_1, x(\tau_1)), \ldots, h(\tau_n, x(\tau_n))]^T
$$

with $C$ being given by

$$
C = [T_{j-1}(\tau_i)]_{i,j=1}^n = \left[\cos \left(\frac{(2i-1)(j-1)\pi}{2n}\right)\right]_{i,j=1}^n.
$$

It follows that

$$
F_i(w) \approx \int_{-1}^w \sum_{j=1}^n \alpha_{i,j} T_j(s) ds = \sum_{j=1}^n \alpha_{i,j} \int_{-1}^w T_j(s) ds.
$$

By some computation, we get

$$
\int_{-1}^w T_{j-1}(s) ds = \begin{cases} T_0(w) + T_1(w), & j = 1, \\
-\frac{1}{4} T_0(w) + \frac{1}{4} T_2(w), & j = 2, \\
\left(\frac{-1}{2} - \frac{1}{j} \right) T_0(w) - \frac{1}{2} \left(\frac{1}{j-2} T_{j-2}(w) - \frac{1}{j} T_j(w)\right), & j \geq 3.
\end{cases}
$$

Let

$$
S_L = S_1 S,
$$

where $S_1$ and $S$ are $n \times (n+1)$ and $(n+1) \times n$ matrices given by

$$
[S_1]_{i,j} = \begin{cases} 1, & i = j = 1, 2, \ldots, n, \\
(-1)^j, & i = 1, j = 2, \ldots, n+1, \\
0, & \text{others;}
\end{cases} \quad [S]_{i,j} = \begin{cases} 1, & i = 2, j = 1, \\
1/(2i), & i = 3, \ldots, n+1, j = i-1, \\
-1/(2i-2), & i = 2, \ldots, n-1, j = i+1, \\
0, & \text{others.}
\end{cases}
$$

Then it follows from (4), (5), and (7) that

$$
F_i(w) \approx \sum_{j=1}^{n+1} \beta_{i,j} T_{j-1}(w),
$$

where

$$
[\beta_{1,1}, \ldots, \beta_{n,n}]^T = S_L [\alpha_{1,1}, \ldots, \alpha_{n,n}]^T
$$

and $\beta_{i,n+1} = \alpha_{n,n} / (2n)$. From Lemma 1, we know that the error in (8) is $O(n^{-p} \log n)$. 

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To apply formula (8) to Eq (1), we introduce simple affine transformations

\[ t = \frac{T}{2} + \frac{T'}{2}, \quad s = \frac{T}{2} + \frac{T'}{2}, \quad -1 \leq t', s' \leq 1. \]

Thus, Eq (1) is transformed to

\[ \tilde{u}(t') = \tilde{f}(t') + \frac{T}{2} \int_{-1}^{t'} \tilde{k}_1(t', s')\tilde{u}(s')ds' + \frac{T}{2} \int_{-1}^{q(t' + 1) - 1} \tilde{k}_2(t', s')\tilde{u}(s')ds', \quad t' \in [-1, 1], \tag{10} \]

where

\[ \tilde{u}(t') = u(t), \quad -1 \leq t' \leq 1; \quad \tilde{k}_i(t', s') = k_i(t, s), \quad -1 \leq t', s' \leq 1, \quad i = 1, 2. \]

Now we consider the discretization of \( F_l(\tau_i) = \int_{-1}^{\tau_i} \tilde{k}_1(\tau_i, s')\tilde{u}(s')ds' \). Using (2), (8), and (9), we get

\[ \int_{-1}^{\tau_i} \tilde{k}_1(\tau_i, s')\tilde{u}(s')ds' \approx \left[ \tilde{k}_1(\tau_i, \tau_1), \cdots, \tilde{k}_1(\tau_i, \tau_n) \right] \circ ([T_0(\tau_1), \cdots, T_{n-1}(\tau_1)]S_L C^{-1}) \tilde{u}_n, \]

where \( C, S_L \) are given by (3) and (6), respectively. It follows that the discrete form of \( \int_{-1}^{\tau_i} \tilde{k}_1(\tau_i, s')\tilde{u}(s')ds' \) is

\[ (K_1 \circ W)\tilde{u}_n, \]

where

\[ K_1 = [\tilde{k}_1(\tau_1, \tau_j)]_{i,j=1}^{p}, \quad W = CS_L C^{-1}, \quad \tilde{u}_n = [\tilde{u}(\tau_1), \cdots, \tilde{u}(\tau_n)]^T. \]

and \( A \circ B \) denotes the Hadamard product of \( A = [a_{ij}]_{i=1}^{p} \) and \( B = [b_{ij}]_{i=1}^{p} \), that is

\[ A \circ B = [a_{ij}]_{i=1}^{p}. \]

Note that \( K_1 = [k_i(\gamma_i, \gamma_j)]_{i,j=1}^{p} \) and \( \tilde{u}_n = [u(\gamma_i)]_{i=1}^{p} \), where \( \gamma_i = \frac{T}{2} + \frac{T'}{2} \tau_i, \quad i = 1, 2, \ldots, n. \)

Finally, consider \( F_l(w_i) = \int_{-1}^{\tau_i} \tilde{k}_2(\tau_i, s')\tilde{u}(s')ds' \), where \( w_i = q(\tau_i + 1) - 1 \) satisfying \(-1 \leq w_i \leq \tau_i \). By using (8) we get

\[ F_l(w_i) \approx \sum_{j=1}^{n+1} \beta_{i,j} T_{j-1}(w_i), \tag{11} \]

where

\[ [\beta_{i,1}, \cdots, \beta_{i,n+1}]^T = S_L' [\alpha_{i,1}, \cdots, \alpha_{i,n}]^T. \tag{12} \]

Here \( S_L' = S_L S \) with \( S \) being given by (7) and \( S'_L \) being an \((n + 1) \times (n + 1)\) matrix given by

\[ [S'_L]_{i,j} = \begin{cases} 1, & i = j = 1, 2, \ldots, n + 1, \\ (-1)^j, & i = 1, j = 2, \ldots, n + 1, \\ 0, & \text{others.} \end{cases} \]

By using (2), (11), and (12), we get

\[ \int_{-1}^{\tau_i} \tilde{k}_2(\tau_i, s')\tilde{u}(s')ds' \approx \left[ \tilde{k}_2(\tau_i, \tau_1), \cdots, \tilde{k}_2(\tau_i, \tau_n) \right] \circ ([T_0(w_i), \cdots, T_n(w_i)]S'_L C^{-1}) \tilde{u}_n. \]

Let \( C' = [T_l(w_i)]_{i,j=1}^{p} = [\cos(j \arccos(w_i))]_{i,j=1}^{p}, \quad V = C'S'_L C^{-1}, \quad \text{and} \quad K_2 = [\tilde{k}_2(\gamma_i, \gamma_j)]_{i,j=1}^{p} \), we get the discrete form of \( \int_{-1}^{q(t' + 1) - 1} \tilde{k}_1(\tau', s')\tilde{u}(s')ds' \), which is given by

\[ (K_2 \circ V)\tilde{u}_n, \]

Based on the above discussion, we obtain the discretization of equation (10)

\[ \left[ I - \frac{T}{2}(K_1 \circ W + K_2 \circ V) \right] \tilde{u}_n = \tilde{f}_n, \]

where \( \tilde{f}_n = [f(\gamma_1), f(\gamma_2), \ldots, f(\gamma_n)]^T. \)
NUMERICAL EXAMPLES

As an illustration, we present numerical results for the following example, which is from literature [1, 6]. Let \( \hat{\mathbf{u}}_n \) and \( \mathbf{u}_n \) be the vectors consist of the function values of the exact solution at \( \gamma_1, \gamma_2, \ldots, \gamma_n \) and the numerical solution with \( n \) quadrature points, respectively. We use \( \| \mathbf{u}_n - \hat{\mathbf{u}}_n \|_\infty \) to measure the error in the numerical solutions.

**Example 1.** Consider

\[
    u(t) = f(t) - \int_0^t u(s) ds + \frac{1}{2} \int_0^t u(s) ds, \quad t \in [0, T],
\]

where \( T = 10 \) and \( f(t) = \frac{1}{2}(1 + e^{-qt}) \), and the exact solution to the above equation is \( u(t) = e^{-t} \). Three delays are chosen: \( q = 0.2, q = 0.5, \) and \( q = 0.9 \).

The absolute errors of the NCC solutions are shown in Table 1.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( q = 0.2 )</th>
<th>( q = 0.5 )</th>
<th>( q = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.20e–003</td>
<td>1.34e–003</td>
<td>7.90e–004</td>
</tr>
<tr>
<td>16</td>
<td>1.90e–009</td>
<td>1.98e–009</td>
<td>1.80e–009</td>
</tr>
<tr>
<td>32</td>
<td>2.54e–015</td>
<td>2.31e–015</td>
<td>2.90e–015</td>
</tr>
</tbody>
</table>

For comparison, we cite numerical results for the collocation solutions from [6]; see Table 2.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( q = 0.2 )</th>
<th>( q = 0.5 )</th>
<th>( q = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>5.31e–008</td>
<td>3.92e–008</td>
<td>1.87e–006</td>
</tr>
<tr>
<td>400</td>
<td>1.14e–009</td>
<td>1.02e–009</td>
<td>8.88e–009</td>
</tr>
<tr>
<td>800</td>
<td>8.78e–012</td>
<td>1.03e–011</td>
<td>2.72e–011</td>
</tr>
</tbody>
</table>

The above tables show that the proposed NCC quadrature is of very high accuracy and much better than the collocation method proposed in [6].

ACKNOWLEDGMENTS

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REFERENCES