On generic continuity of maps to posets with metrics

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Dedicated to Sergey Antonyan on the occasion of his 65th birthday.

Abstract

A map $f : X \to Y$ from a Baire space $X$ to a topological space $(Y, T_2)$ is said to be generically continuous on $X$ if the set $G(f, T_2) := \{x \in X : f : X \to (Y, T_2) \text{ is continuous at } x\}$ includes a dense $G_\delta$-set in $X$. In the paper, we investigate the following question: What conditions on spaces $X$, $(Y, T_1)$ and $(Y, T_2)$ do imply that $f : X \to (Y, T_2)$ is generically continuous on $X$ for every continuous map $f : X \to (Y, T_1)$? We answer this question for $Y$ being a poset with a topology $T_2 := T_d$ induced by a metric $d$ on $Y$. It is proved that if $Y$ is a poset with a metric $d$ satisfying some natural conditions, then $f : X \to (Y, T_d)$ is generically continuous for every Baire metric space (or Baire space) $X$ and every continuous map $f : X \to (Y, \uparrow T_d)$, where $\uparrow T_d := \{\uparrow U : U \in T_d\}$. Based on this result, we extend some known results for $Y$ being hyperspaces and give an application to dynamical systems. Some related counterexamples are presented.

Keywords: Generic continuity; Poset; Hyperspace; Domain; Metric space; Dynamical systems

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1. Introduction

A map $f : X \to Y$ from a Baire space $X$ to a topological space $(Y, T_2)$ is said to be\textbf{ generically continuous} on $X$ if the set of continuous points

$$G(f, T_2) := \{x \in X : f : X \to (Y, T_2) \text{ is continuous at } x\}$$

of $f$ includes a dense $G_\delta$-set in $X$. In particular, for two topologies $T_1$ and $T_2$ on a set $Y$ with $T_1 \subset T_2$, the topology $T_2$ is called\textbf{ categorically related to} the topology $T_1$ on $Y$ if $G(f, T_2)$ includes a countable intersection of open dense sets in $X$ for every continuous map $f : X \to (Y, T_1)$. Thus, $f : X \to (Y, T_2)$ is generically continuous on $X$ if, furthermore, $X$ is a Baire space. The two concepts were investigated as early as in 1927 by Hill in [12] and in 1932 by Kuratowski in [18], cf [19]. More general, in [4, 6, 7, 8], the following problem was discussed: What conditions on the spaces $X$, $(Y, T_1)$ and $(Y, T_2)$ imply that $G(f, T_2)$ includes...
a countable intersection of open dense sets in $X$ for every continuous map $f : X \to (Y, \mathcal{T}_1)$. Moreover, Fort gave his results many applications to mathematical analysis in [8].

Some of the important cases are the following. Let $Y = \text{Cld}(Z)$ ($Y = \text{Com}(Z)$, respectively) be the family of all non-empty closed sets (non-empty compact sets, respectively) in a space $Z$. As well-known, there exist many natural topologies on $Y$, such as, upper topology $\mathcal{T}_u$, lower topology $\mathcal{T}_l$, Vietoris topology $\mathcal{T}_V$, Fell topology $\mathcal{T}_F$. And, if $(Z, d)$ is a metric space, the metric upper topology $\mathcal{T}_{mu}$, the metric lower topology $\mathcal{T}_{ml}$ and the topology $\mathcal{T}_{dU}$ induced by the Hausdorff metric $d_H$ also are considered.

In [4, 6, 7, 12, 18] (cf. [19, P71, Corollary 1]), for $Y = \text{Com}(Z)$, they gave some conditions such that $\mathcal{T}_V$ is categorically related to $\mathcal{T}_u$ or $\mathcal{T}_l$. In particular, Fort in [7] proved the following result:

**Theorem 1.1.** ([7], cf.[11, Theorem 4 in P31]) Let $X$ be a topological space and $(Z, d)$ a metric space. Let $f : X \to \text{Com}(Z)$ be upper semicontinuous or lower semicontinuous (that is, $f : X \to (\text{Com}(Z), \mathcal{T}_u)$ or $f : X \to (\text{Com}(Z), \mathcal{T}_l)$ is continuous). Then, the set $G(f, \mathcal{T}_V)$ of continuous points of the map $f : X \to (\text{Com}(Z), \mathcal{T}_V)$ is a countable intersection of dense open sets in $X$. That is, on $\text{Com}(Z)$, $\mathcal{T}_V$ is categorically related to $\mathcal{T}_u$ and $\mathcal{T}_l$. Hence, $f : X \to (\text{Com}(Z), \mathcal{T}_V)$ is generically continuous on $X$ if, furthermore, $X$ is a Baire space.

Fort’s theorems can be extended in several cases, see [1, 3, 10, 14, 15, 26]. In particular, the following theorem was proved in [26]:

**Theorem 1.2.** ([26]) Let $X$ be a complete metric space, $(Z, d)$ a metric space and $f : X \to \text{Cld}(Z)$ metric upper semicontinuous or metric lower semicontinuous (that is, $f : X \to (\text{Cld}(Z), \mathcal{T}_{ml})$ is continuous or $f : X \to (\text{Cld}(Z), \mathcal{T}_{mu})$ is continuous). Then $f : X \to (\text{Cld}(Z), \mathcal{T}_{dU})$ is generically continuous on $X$.

Those results can be applied to many fields. In [11], Hildenbrand pointed out that many questions in economic theory can be reduced to generic continuity of some maps in this case. The papers [5, 13, 16, 28, 27] gave some applications of the generic properties of solutions to optimal theory.

Another important case is to show that the usual topology $\mathcal{T}$ on the set $\mathbb{R}$ of real numbers is categorically related to the upper topology $\mathcal{T}_u$ generated by $\{(a, +\infty) : a \in \mathbb{R}\}$ and the lower topology $\mathcal{T}_l$ generated by $\{(-\infty, a) : a \in \mathbb{R}\}$. In fact, in [8], Fort gave a rather general theorem which states as follows:

**Theorem 1.3.** ([8]) Let $Y$ be a set and $\mathcal{T}_1, \mathcal{T}_2$ two topologies on $Y$ satisfying condition (a): there exist sequences $U_1, U_2, \cdots$ and $K_1, K_2, \cdots$ of subsets of $Y$ such that

(i) $U_n \subset K_n$ for every $n$;

(ii) if $p \in U \in \mathcal{T}_2$, then there exists $n$ such that $p \in U_n \subset K_n \subset U$;

(iii) if $q \in U_n$, then there exists $V \in \mathcal{T}_1$ such that $q \in V$ and $V \setminus K_n \in \mathcal{T}_1$.

Then, $\mathcal{T}_2$ is categorically related to $\mathcal{T}_1$.

Fort in his paper [8] gave many applications to mathematical analysis of Theorem 1.3. In particular, the following one was given:

**Corollary 1.1.** ([8]) On $\mathbb{R}$, $\mathcal{T}$ is categorically related to $\mathcal{T}_u$ and $\mathcal{T}_l$.

In [25], a generalized result of Theorem 1.3 was presented.

Generally speaking, when a map from $X$ to $Y$ is defined to be upper semicontinuous or lower semicontinuous, $Y$ is necessarily a poset (usually with a natural order). This observation is the main motivation of this paper. In fact, in the above examples, $\mathbb{R}$ has a natural linear order, both $\text{Cld}(Z)$ and $\text{Com}(Z)$ are with the inclusion order and the inverse inclusion order. All topologies above on $\mathbb{R}$, $\text{Cld}(Z)$ and $\text{Com}(Z)$ are connected with those natural orders. Hence, in the present paper, we hope to give some generalized theorems by considering $Y$ to be a poset with the topology $\mathcal{T}_d$ induced by a metric $d$ and another weaker topology $\uparrow\mathcal{T}_d$ generated by the topology and the order. We will give some conditions on the order and the metric such that $\mathcal{T}_d$ is categorically related to $\uparrow\mathcal{T}_d$. Thus above theorems become corollaries of our results.
The paper is organized as follows. In Sections 2 and 3, we give our main theorems in the paper. Some corollaries for \( Y \) being hyperspaces will be obtained in Section 4. An application to dynamical systems is also presented.

2. Maps to poset with metric

Let \( (Y, \leq) \) be a partial order set (poset for short) and \( T \) a topology on \( Y \). Then we can define the upper topology \( \uparrow T \) of \( T \) on \( Y \) which is generated by \( \{ \uparrow U : U \in T \} \) as a subbase, where \( \uparrow U := \{ y \in Y : y \geq x \text{ for some } x \in U \} \). Moreover, let \( T^\downarrow := \{ U \in T : U = \uparrow U \} \). It is verified straightforwardly that \( T^\downarrow \) is a topology on \( Y \) which is called the weaker upper topology of \( T \) on \( Y \). In fact, we have the following easy lemma which proof is omitted:

**Lemma 2.1.** For every topology \( T \) on a poset \( Y \), \( T^\downarrow \subset \uparrow T \); and \( T^\downarrow = \uparrow T \) if and only if \( \uparrow T \subset T \).

Although \( T^\downarrow \neq \uparrow T \) in general, they coincide in most of the examples which are considered in this paper. Dually, we can define the lower topology \( \downarrow T \) and the weaker lower topology \( T^\downarrow \) of \( T \) on \( Y \).

A metric \( d \) on \( (Y, \leq) \) is said to satisfy **Conditions (\( \beta \))** if the following three conditions hold:

(i) \( \uparrow T_d \subset T_d \), that is, \( T^\downarrow_d = \uparrow T_d \).

(ii) For \( a, b, c, d \in Y \), if \( b \leq a \) and \( d \leq c \), then

\[
d(b, d) \leq \max\{d(a, d), d(b, c)\}.
\]

(iii) For every \( y \in Y \), \( r > 0 \) and \( A \subset Y \), if \( A \subset \uparrow B_d(y, r) \setminus B_d(y, r) \), then \( \uparrow A \cap B_d(y, r) = \emptyset \).

A metric \( d \) on \( (Y, \leq) \) is said to satisfy **Conditions (\( \beta' \))** if

(i) \( \uparrow T_d \subset T_d \), that is, \( T^\downarrow_d = \uparrow T_d \), and

(ii) for every sequence \( \{y_n\} \) in \( Y \) and \( \varepsilon_n \in \left(0, \frac{1}{2^n}\right)\), \( \{y_n\} \) is a converging sequence in \( (Y, d) \) if

\[
\text{cl}_{T_d} B_d(y_{n+1}, \varepsilon_{n+1}) \subset \uparrow B_d(y_n, \varepsilon_n).
\]

In the rest part of section, we give and prove two main theorems.

**Main Theorem 1.** Let \( (X, d) \) be a Baire metric space and \( (Y, \leq) \) a poset with a metric \( d \) satisfying Condition (\( \beta \)). Then \( f : X \to (Y, T_d) \) is generically continuous for every continuous map \( f : X \to (Y, \uparrow T_d) \).

To prove the Main Theorem 1, we need the following lemma which can be easily proved using general method in metric spaces.

**Lemma 2.2.** Let \( (X, d) \) be a metric space and \( \varepsilon > 0 \). For every open cover \( \mathcal{U} \) of \( X \), there exists a family \( \mathcal{W} \) of open sets in \( X \) such that
\(\gamma > 0\), let \(U \ni x \in X\) be the family of all open neighborhoods of \(x\). Define

\[
\omega_f(x) := \inf \{ \text{diam}_d f(U) : U \in \mathcal{U}(x) \}.
\]

It is trivial to verify that the map \(f : X \to (Y, d)\) is continuous at \(x\) if and only if \(\omega_f(x) = 0\). Now, for every \(\gamma > 0\), let \(U_\gamma := \{ x \in X : \omega_f(x) < 5\gamma \} \). Then \(U_\gamma\) is open. To show that \(G(f, T_d)\) is a countable intersection of dense open sets in \(X\), it suffices to verify that \(U_\gamma\) is dense in \(X\) for every \(\gamma > 0\). Assume, on the contrary, there exist \(\gamma > 0\) and a non-empty open set \(U\) in \(X\) satisfying:

**Statement N**: For every \(x \in U\) and \(W \in \mathcal{U}(x)\), there exists \(x(x, W) \in W \cap U\) such that

\[
d(f(x), f(x(W))) \geq 2\gamma.
\]

We first show the following

**Fact**: For every non-empty open set \(V \subset U\) and \(\varepsilon \in (0, \gamma)\), there exists a family \(\mathcal{W} = \bigcup \{ W(x) : x \in V \} \) of open sets in \(V\) such that

(i) mesh\(_d\) \(\mathcal{W} < \varepsilon\).

(ii) \(\mathcal{W}\) is pairwise disjoint.

(iii) \(\bigcup \mathcal{W}\) is dense in \(V\).

(iv) \(\mathcal{W}\) is a refinement of \(\mathcal{U}\), that is, for every \(W \in \mathcal{W}\), there exists \(U \in \mathcal{U}\) such that \(W \subset U\).

**Proof of the Main Theorem 1.** For \(x \in X\), let \(\mathcal{U}(x)\) be the family of all open neighborhoods of \(x\). Define

\[
\omega_f(x) := \inf \{ \text{diam}_d f(U) : U \in \mathcal{U}(x) \}.
\]

It is trivial to verify that the map \(f : X \to (Y, d)\) is continuous at \(x\) if and only if \(\omega_f(x) = 0\). Now, for every \(\gamma > 0\), let \(U_\gamma := \{ x \in X : \omega_f(x) < 5\gamma \} \). Then \(U_\gamma\) is open. To show that \(G(f, T_d)\) is a countable intersection of dense open sets in \(X\), it suffices to verify that \(U_\gamma\) is dense in \(X\) for every \(\gamma > 0\). Assume, on the contrary, there exist \(\gamma > 0\) and a non-empty open set \(U\) in \(X\) satisfying:

**Statement N**: For every \(x \in U\) and \(W \in \mathcal{U}(x)\), there exists \(x(x, W) \in W \cap U\) such that

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**Fact**: For every non-empty open set \(V \subset U\) and \(\varepsilon \in (0, \gamma)\), there exists a family \(\mathcal{W} = \bigcup \{ W(x) : x \in V \} \) of open sets in \(V\) such that

(i) mesh\(_d\) \(\mathcal{W} < \varepsilon\).

(ii) \(\mathcal{W}\) is pairwise disjoint.

(iii) \(\bigcup \mathcal{W}\) is dense in \(V\).

(iv) \(\mathcal{W}\) is a refinement of \(\mathcal{U}\), that is, for every \(W \in \mathcal{W}\), there exists \(U \in \mathcal{U}\) such that \(W \subset U\).

Now, let

\[
V(x) := \{ x' \in U(x) : d(f(x), f(x')) > \gamma \}.
\]

We show that \(V(x)\) is an open dense set in \(U(x)\). For \(x' \in V(x)\), choose \(\gamma' \in (\gamma, d(f(x), f(x')))\). By Condition (\(\beta\))(i), \(f(x') \in \uparrow B_d(f(x), \gamma') \subset T_d \subset T_d\). It follows that there exists \(\delta \in (0, d(f(x'), f(x)) - \gamma')\) such that \(B_d(f(x'), \delta) \subset \uparrow B_d(f(x), \gamma')\). Thus

\[
B_d(f(x'), \delta) \subset \uparrow B_d(f(x), \gamma') \setminus B_d(f(x), \gamma').
\]

It follows from Condition (\(\beta\))(iii) that \(\uparrow B_d(f(x'), \delta) \cap B_d(f(x), \gamma') = \emptyset\). Since \(f : X \to (Y, T_d)\) is continuous, there exists \(G(x') \in \mathcal{U}(x')\) such that \(G(x') \subset U(x)\) and \(f(x'') \in \uparrow B_d(f(x'), \delta)\) for every \(x'' \in G(x')\). Hence, \(x'' \in U(x)\) and \(d(f(x'), f(x')) \geq \gamma' > \gamma\) for every \(x'' \in G(x')\). It follows from the definition of \(V(x)\) that \(G(x') \subset V(x)\). Therefore, \(V(x)\) is open. Moreover, \(V(x)\) is dense in \(U(x)\). Otherwise, the set \(U(x) \setminus \text{cl}_T V(x)\) is a non-empty open subset of \(U\). And \(d(f(x), f(x')) \leq \gamma\) and \(d(f(x), f(x'')) \leq \gamma\) for every pair \(x', x'' \in U(x) \setminus \text{cl}_T V(x)\). Hence, \(d(f(x''), f(x')) \leq 2\gamma\). It contradicts the choice of \(\gamma\) and \(U\).

Let

\[
V := \bigcup \{ V(x) : x \in U \}.
\]

Then \(V\) is a dense open set in \(U\) and \(\{ V(x) : x \in U \} \) is an open cover of \(V\). By Lemma 2.2, there exists a family \(\mathcal{W}\) of open sets in \(V\) such that (i)-(iii) in Fact and the following (iv)' hold:
which contradicts Formula (2).

Moreover, by (i) \( W \) is pairwise disjoint.

Moreover, for every \( V \), there exist \( W \) satisfying (i)-(v) in Fact. Let \( W = \bigcup \{ W(x) : x \in U \} \). From (iv)', Formula (1) and the definition of \( V(x) \) it follows that \( W \) also satisfies (iv)-(v). We have completed the proof of Fact.

Using inductive method, we can define a sequence \( \{ W_n \} \) of families of open sets in \( X \) such that, for every \( n \),

(i)_n \( \text{mesh}_d W_n < \frac{1}{n+1} \).

(ii)_n \( W_n \) is pairwise disjoint.

(iii)_n \( W_n = \bigcup \{ W_n(x) : x \in W_{n-1} \} \), where \( W_0 = U \) and \( W_{n-1} = \bigcup W_{n-1} \).

(iv)_n \( W_n \) is a dense subset of \( W_{n-1} \).

(v)_n \( W_{n+1}(x') \subset W_n(x) \) if \( x' \in W_n(x) := \bigcup W_n(x) \) and \( x \in W_{n-1} \).

(vi)_n \( d(f(x), f(x')) > \gamma \) if \( x \in W_{n-1} \) and \( x' \in W_n(x) \).

(vii)_n \( f(x') \in \mathcal{B}_d(f(x), \frac{\gamma}{n+1}) \) if \( x \in W_{n-1} \) and \( x' \in W_n(x) \).

In fact, for \( V = U \), the fact exists \( W = \bigcup \{ W(x) : x \in U \} \) of open sets in \( U \). From (i)-(v) in Fact. Let \( W'_1 := W \) satisfies (i)_1-(vii)_1. Suppose \( W_n \) has been defined and satisfies (i)_n-(vii)_n.

For every \( V \in W_n \) and \( \varepsilon = \frac{\gamma}{n+1} \), by Fact, there exists a family \( W_{n+1}(V) = \bigcup \{ W_{n+1}(V)(x) : x \in V \} \) of open sets in \( V \) satisfying (i)-(v) in Fact. Let

\[ W_{n+1} := \bigcup \{ W(V) : V \in W_n \}. \]

Moreover, for every \( x \in W_n \), there exists a unique \( V \in W_n \) such that \( V \in V \). Define \( W_{n+1}(x) := W(V)(x) \). Then \( W_{n+1} = \bigcup \{ W_{n+1}(x) : x \in W_n \} \) satisfies (i)_n-(vii)_n, completing the inductive definition.

Since \( X \) is a Baire space, its non-empty open subspace \( U \) is also a Baire space. By (iv)_n, \( \{ U = W_0 \supset W_1 \supset W_2 \supset \cdots \} \) is a sequence of dense open sets in \( U \). Thus, \( \bigcap_{n=1}^{\infty} W_n \neq \emptyset \). Let us choose \( x_\infty = \bigcap_{n=1}^{\infty} W_n \). Then, for every \( n \), by (iii)_n, there exist \( x_{n-1} \in W_{n-1} \) and \( W^n \in W_n(x_{n-1}) \) such that \( x_\infty \in W^n \). Then \( x_n \in W^n \) for every \( n \). In fact, otherwise, we have \( x_n \in W_n(x'_{n-1}) \) for some \( x'_{n-1} \in W_{n-1} \). By (ii)_n, \( W^n \cap W_n(x'_{n-1}) = \emptyset \). By (v)_n, \( W^{n+1} \subset W_{n+1}(x_n) \subset W_n(x'_{n-1}) \). It follows that \( W^{n+1} \cap W^n = \emptyset \). A contradiction occurs. Hence, \( x_n, x_\infty \in W^n \subset W_{n-1}(x_{n-1}) \). It follows from (vi)_n and (vii)_n that, for every \( n \),

\[ d(f(x_n), f(x_{n-1})) > \gamma; \]  
\[ f(x_\infty) \in \mathcal{B}_d(f(x_{n-1}), \frac{\gamma}{n}). \]  

Moreover, by (i)_n, \( \text{diam}_d W^n < \frac{\gamma}{n+1} \). It follows that \( \lim x_n = x_\infty \), and consequently, by the continuity of \( f : X \to (Y, \mathcal{T}_d) \) at \( x_\infty \), for every \( \varepsilon > 0 \),

\[ f(x_{n-1}) \in \mathcal{B}_d(f(x_\infty), \varepsilon) \]  
holds for large enough \( n \).

By Condition (\( \beta \)) (ii), Formulae (3) and (4) imply

\[ d(f(x_\infty), f(x_{n-1})) < \varepsilon \]  
holds for large enough \( n \),

which contradicts Formula (2).

\[ \square \]

**Corollary 2.1.** Let \( X \) be a regular Hausdorff space with a Baire metrizable dense \( G_\delta \)-subspace and \( Y \) a poset with a metric \( d \) satisfying Conditions (\( \beta \)). Then \( f : X \to (Y, \mathcal{T}_d) \) is generically continuous for every continuous map \( f : X \to (Y, \mathcal{T}_d) \). In particular, if \( X_0 \) is a completely metrizable space, then for every compactification \( X_0 \) of it, every poset \( Y \) with a metric \( d \) satisfying Conditions (\( \beta \)), \( f : X \to (Y, \mathcal{T}_d) \) is generically continuous if \( f : X \to (Y, \mathcal{T}_d) \) is continuous.

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Main Theorem 2. If $(Y, \leq)$ is a poset with a metric $d$ satisfying Condition (β′), then $T_d$ is categorically related to $\nuparrow T_d$. That is, for every topological space $X$ and every continuous map $f : X \to (Y, \nuparrow T_d)$, the set $G(f, T_d)$ is a countable intersection of open dense sets in $X$.

Proof. Let $X$ be a topological space and $f : X \to (Y, \nuparrow T_d)$ a continuous map. If $G(f, T_d)$ is not a countable intersection of open dense sets in $X$, then, repeat the same argument in the proof of the Main Theorem 1, we can define $\gamma > 0$ and a non-empty open set $U$ in $X$ satisfying Statement (N). Moreover, by induction on $n \in \mathbb{N}$, we can define a sequence $\{x_n\}$ in $U$ and a sequence $\{\varepsilon_n\}$ such that

(i) $0 < \varepsilon_n \leq \frac{1}{2^n}$.
(ii) $d(f(x_{n+1}), f(x_n)) > \gamma$.
(iii) $\forall x \in U, \varepsilon_{n+1} \in \nuparrow T_d(f(x_n), \varepsilon_n)$.

In fact, choose $x_1 \in U$ and let $\varepsilon_1 := \frac{1}{2}$. For $n \in \mathbb{N}$, suppose $x_n \in U$ and $\varepsilon_n$ have been defined. Then $f(x_n) \in \nuparrow T_d(f(x_n), \varepsilon_n)$. It follows from the continuity of $f : X \to (Y, \nuparrow T_d)$ that there exists $U(x_n) \in U(x_n)$ such that $f(x) \in \nuparrow T_d(f(x_n), \varepsilon_n)$ for $x \in U(x_n)$. Using Statement (N), let $x_{n+1} := z(x_n, U(x_n))$. Then $x_{n+1} \in \nuparrow T_d(f(x_n), \varepsilon_n) \in \nuparrow T_d$. By Condition (β′)(i), there exists $\varepsilon_{n+1} \in (0, \frac{1}{2^{n+1}})$ such that (iii) above holds. Then $x_{n+1} \in U$ and $\varepsilon_{n+1}$ satisfy (i)-(iii).

By Condition (β′)(ii), (iii) above implies that $\{f(x_n)\}$ is a converging sequence in $(Y, d)$, which contradicts (ii) above. \qed

3. Maps to domain with metric

As first, we give some basic knowledge on domain theory. For further details, please refer to [9]. Let $(Y, \leq)$ be a poset. A set $D$ in $Y$ is called directed if for every pair $d_1, d_2 \in D$, there exists $d \in D$ such that $d_1, d_2 \leq d$. The poset $Y$ is called a directed complete poset, or dcpo for short, if every directed subset $D$ of $Y$ has a supremum $\sup D$ in $Y$. For $x, y \in Y$, we say that $x$ is way-below to $y$ and denoted by $x \ll y$ if for every directed set $D$ in $Y$, if $\sup D$ exists and $y \leq \sup D$, then there exists $d \in D$ such that $x \leq d$. Let $\downarrow x := \{y \in P : y \ll x\}$ and $\nuparrow x := \{y \in P : x \ll y\}$. The poset $Y$ is called a domain if $Y$ is a dcpo, and for every $x \in Y$, $\downarrow x$ is directed and $\sup \downarrow x = x$. When $Y$ if both a domain and a complete lattice, we call $Y$ a continuous lattice.

As in [9], the lower topology on a poset $Y$, generated by $\{Y \setminus \nuparrow x : x \in Y\}$, is denoted by $\omega Y$. A subset $U$ of $Y$ is called Scott open if (i) $U = \nuparrow U$ and (ii) for any directed subset $D$ of $Y$ for which $\sup D$ exists, $\sup D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of $Y$ form a topology, and we call this topology the Scott topology on $Y$ and denote it by $\sigma Y$. The common refinement $\sigma Y$ and $\omega Y$ is called the Lawson topology on $Y$ and is denoted by $\Lambda Y$. Let $\Sigma Y := (Y, \sigma Y)$ and $\Lambda Y := (Y, \Lambda Y)$.

Using [9, Theorem III 4.5] and [29, Corollary 3.10 and Theorem 4.5], we get the following

Lemma 3.1. Let $Y$ be a continuous lattice such that $\Lambda Y$ is separable metrizable. Then $\Lambda Y$ can be topologically embedded into the Hilbert cube $I^\omega$ via an order embedding.

The following is another main theorem in this paper:

Main Theorem 3. Let $Y$ be a domain and $d$ a metric on the set $Y$ satisfying the following conditions:

(i) $d(l_2, l_3) \leq d(l_1, l_3)$ if $l_1 \leq l_2 \leq l_3$ in $Y$, and
(ii) if $D$ is directed in $Y$ and $l_\infty = \sup D$, then $\lim_{i \in D} d(l, l_\infty) = 0$.

Then, the topology $T_d$ is categorically related to $\sigma Y$. Thus, for every space $X$ and every continuous map $f : X \to \Sigma Y$, the set $G(f, T_d)$ is a countable intersection of dense open sets in $X$. Hence, $f : X \to (Y, T_d)$ is generically continuous if, furthermore, $X$ is a Baire space.

Proof. Otherwise, similar to the proof of the Main Theorem 1, there exist $\gamma > 0$ and a non-empty set $U$ in $X$ satisfying Statement N in the proof of the Main Theorem 1.
4. Applications to hyperspace

The first step, choose \( x_1 \in U \). Since \( f(x_1) \) is directed and \( \cup f(x_1) = f(x_1) \), by the assumption (ii), there exists \( l_1 \in Y \) such that \( l_1 \not\ll f(x_1) \), and \( d(l_1, f(x_1)) < \frac{1}{2} \). Note that \( U \) is a neighborhood of \( f(x_1) \) in \( \Sigma Y \) and hence \( U := f^{-1}(\langle l_1 \rangle) \subset U(x_1) \) since \( f : X \to \Sigma Y \) is continuous.

The second step, define \( x_2 := x(x_1, U_1) \) in Statement N. Then \( l_1 \not\ll f(x_2) \) and there exists \( l'_2 \in L \) such that \( l'_2 \not\ll f(x_2) \) and \( d(l'_2, f(x_2)) < \frac{1}{2} \). Since \( \cup f(x_2) \) is directed, we can find \( l_2 \in \cup f(x_2) \) such that \( l_1, l'_2 \leq l_2 \). Then, by our assumption (i),

\[
d(l_2, f(x_2)) < \frac{1}{2^2}, \quad l_1 \leq l_2, \quad d(f(x_1), f(x_2)) \geq 2\gamma \quad \text{and} \quad l_2 \not\ll f(x_2).
\]

By induction, we can define a sequence \( \{x_n\} \) in \( U \) and a sequence \( \{l_n\} \) in \( Y \) such that

\[
d(l_n, f(x_n)) < \frac{1}{2^n}, \quad l_1 \leq l_2 \leq \cdots \leq l_n \leq \cdots,
\]

\[
d(f(x_{n-1}), f(x_n)) \geq 2\gamma \quad \text{and} \quad l_n \not\ll f(x_n).
\]

It follows from the assumption (ii) that \( \{l_n\} \) is a converging sequence in \((Y, d)\) and hence so is \( \{f(x_n)\} \). But this contradicts \( d(f(x_{n-1}), f(x_n)) \geq 2\gamma \) for every \( n \).

We show that Corollary 1.1 is also a corollary of the Main Theorem 3.

**Proof of Corollary 1.1.** Note that \( \mathbb{R} \), as a poset and a topological space, is isomorphic and homeomorphic to \((0, 1)\) and hence we can use \((0, 1)\) to replace \( \mathbb{R} \). Moreover, \((0, 1) \subset \mathbb{I} \) is a subposet and a subspace of \( \mathbb{I} \). The domain \( \mathbb{I} \) with the usual metric satisfies all conditions in the Main Theorem 3. Hence, the usual topology on \( \mathbb{I} \) is categorically related to the upper topology. Similarly, the usual topology on \( \mathbb{I} \) is categorically related to the lower topology. It follows that they also hold for \((0, 1)\). \( \square \)

**Corollary 3.1.** Let \( Y \) be a continuous lattice such that \( \Lambda Y \) is separable metrizable. Then \( \Lambda Y \) is categorically related to \( \sigma Y \).

**Proof.** By Lemmas 3.1, 3.1 and [9, Proposition II-2.1], there exists a one-to-one map \( \phi : Y \to \mathbb{I}^\mathbb{N} \) such that \( \phi \) is a topological imbedding and persevering directed supremum. It is trivial to verify that the admissible metric

\[
\rho(a, b) := \sum_{n=1}^{\infty} \frac{1}{2^n} |a_n - b_n|, \quad \text{for} \ a, b \in \mathbb{I}^\mathbb{N}
\]

on \( \mathbb{I}^\mathbb{N} \) satisfies (i) and (ii) in the Main Theorem 3. It follows that the admissible metric \( \rho|\phi(Y) \) on \( \phi(Y) \) satisfies (i) and (ii) in the Main Theorem 3. It induces an admissible metric \( d \) on \( \Lambda Y \) satisfies (i) and (ii) in the Main Theorem 3. By the Main Theorem 3, this corollary holds. \( \square \)

4. Applications to hyperspace

For general theory on hyperspace, readers can refer to [2].

As said in Section 1, \( \text{Cld}(Z) \) and \( \text{Com}(Z) \) are the family of all non-empty closed sets and that of all non-empty compact sets in a space \( Z \), respectively. It is well-know that \( \text{Com}(Z) \subset \text{Cld}(Z) \) if \( Z \) is Hausdorff.

For \( A \subset Z \), let

\[
A^- := \{C \in \text{Cld}(Z) : C \cap A \neq \emptyset \}, \quad A^+ := \{C \in \text{Cld}(Z) : C \subset A \}
\]

and

\[
A^* := \{C \in \text{Cld}(Z) : C \cap A = \emptyset \} = (Z \setminus A)^+.
\]

Now we define topologies on \( \text{Cld}(Z) \) as follows:

- **Lower topology** \( \mathcal{T}_L \): generated by \( \{V^- : V \text{ is open in } Z\} \) as a subbase.
- **Upper topology** \( \mathcal{T}_U \): generated by \( \{V^+ : V \text{ is open in } Z\} \) as a base.
- **Weakly upper topology** \( \mathcal{T}_{wU} \): generated by \( \{C^* : C \text{ is compact and closed in } Z\} \) as a base.
Vietoris topology $T_V$: generated by $\{V^-, V^+: V \text{ is open in } Z\}$ as a subbase.

Fell topology $T_F$: generated by $\{V^-, C^*: V \text{ is open and } C \text{ is compact and closed in } Z\}$ as a subbase.

Let $(Z, d)$ be a metric space. We can always assume that $d$ is bounded since $d$ can be replaced by $\min\{d, 1\}$ if necessary. Define the Hausdorff metric on $\text{Cld}(Z)$ as follows:

$$d_H(C, D) := \inf\{\varepsilon : C \subset B_d(D, \varepsilon) \text{ and } D \subset B_d(C, \varepsilon)\}, \quad C, D \in \text{Cld}(Z).$$

It is straightforward to verify that $d_H$ is a metric on $\text{Cld}(Z)$. For $A \in \text{Cld}(Z)$ and $r > 0$, let

$$B_{d^{-H}}(A, r) := \{B \in \text{Cld}(Z) : A \subset B_d(B, r') \text{ for some } r' \in (0, r)\},$$

$$B_{d^+}(A, r) := \{B \in \text{Cld}(Z) : B \subset B_d(A, r') \text{ for some } r' \in (0, r)\}.$$

Then

$$B_{d^{-H}}(A, r) = B_{d^{-H}}(A, r) \cap B_{d^+}(A, r).$$

It is worth noting that the meaning of symbol $B_{d^{-H}}(A, r)$ differs from that of symbol $B_{d}(A, r)$. Now we define topologies on $\text{Cld}(Z)$ as follows:

**Metric lower topology** $T_{ml}$: generated by $\{B_{d^{-H}}(A, r) : r > 0\}$ as a neighborhood base at $A$.

**Metric upper topology** $T_{mu}$: generated by $\{B_{d^+}(A, r) : r > 0\}$ as a neighborhood base at $A$.

**Hausdorff metric topology** $T_{dH}$: the topology induced by $d_H$.

Some basic properties of those topologies are summarized as follows:

**Lemma 4.1.** (1) $T_u, T_{wu} \subset T_V; T_l, T_{wu} \subset T_F$.

(2) $T_{wu} \subset T_u$ and $T_F \subset T_V$.

(3) $T_u = T_{wu}$ and $T_V = T_F$ if $Z$ is a compact Hausdorff space.

(4) $T_{dH}$ is the topology generated by $T_{mu} \cup T_{ml}$, $T_l \subset T_{ml}$ and $T_{mu} \subset T_u$ if $(Z, d)$ is a metric space.

(5) $T_{ml} = T_l$, $T_{mu} = T_u$ and $T_V = T_{dH}$ on $\text{Com}(Z)$ if $(Z, d)$ is a metric space. Therefore, $T_{ml}, T_{mu}$ and $T_{dH}$ on $\text{Com}(Z)$ are independent on choice of admissible metric $d$ on $Z$.

**Proposition 4.1.** Let $(Z, d)$ be a bounded metric space. Then

1. $\uparrow T_{dH} = T_{d^+} = T_{mu}$ in $(\text{Cld}(Z), \supset)$.
2. $\uparrow T_{dH} = T_{d^{-H}} = T_{ml}$ in $(\text{Cld}(Z), \subset)$.
3. The metric $d_H$ satisfies $(\beta)$ in both posets $(\text{Cld}(Z), \supset)$ and $(\text{Cld}(Z), \subset)$.

**Proof.**

(1). To show that $\uparrow T_{dH} = T_{mu}$ in $(\text{Cld}(Z), \supset)$, it suffices to verify that

$$\uparrow B_{dH}(A, r) = B_{d^+}(A, r)$$

in $(\text{Cld}(Z), \supset)$ for $A \in \text{Cld}(Z)$ and $r > 0$. For every $B \in \uparrow B_{dH}(A, r)$, there exists $C \in \text{Cld}(Z)$ such that $C \supset B$ and $d_H(A, C) < r' < r$ for some $r' > 0$. Then, $B \subset C \subset B_d(A, r')$. It follows that $B \in B_{d^+}(A, r)$. Conversely, for every $B \in B_{dH}(A, r)$, there exists $r' \in (0, r)$ such that $B \subset B_d(A, r')$. Let $C := A \cup B$. Then, $C \supset B$ and $C \in B_{dH}(A, r)$. Thus, $B \in \uparrow B_{dH}(A, r)$.

By Lemmas 2.1 and 4.1(4), the above conclusion implies that $\uparrow T_{dH} = T_{dH}^\uparrow$ in $(\text{Cld}(Z), \supset)$.

(2). As explained in (1), to show that $\uparrow T_{dH} = T_{lu}$ in $(\text{Cld}(Z), \subset)$, it suffices to verify that

$$\uparrow B_{dH}(A, r) = B_{d^{-H}}(A, r)$$

in $(\text{Cld}(Z), \subset)$ for $A \in \text{Cld}(Z)$ and $r > 0$. It is similar to (1) to verify that $\uparrow B_{dH}(A, r) \subset B_{d^{-H}}(A, r)$. To show the reverse inclusion, consider $B \in B_{d^{-H}}(A, r)$. Then, there exists $r' \in (0, r)$ such that $A \subset B_d(B, r')$. Let

$$C := \overline{B_d}(A, r') \cap B.$$
Then, $C$ is closed, $C \subset B$ and $C \subset B_d(A, r''')$ for some $r''' \in (r', r)$. Moreover, for every $a \in A$, there exists $b \in B$ such that $d(a, b) < r'$. Thus, $b \in C$. It follows that $C$ is non-empty and $A \subset B_d(C, r'')$. Hence $d_H(A, C) < r$. So, $B \in \mathcal{T}_{\delta d}(A, r)$.

Similar to (1), we have that $\gamma\mathcal{T}_{\delta d} = \mathcal{H}_{\delta d}$ in $(\text{Cld}(Z), \subset)$.

(3). We only show that the metric $d_H$ satisfies $(\beta)$ in the poset $(\text{Cld}(Z), \subset)$. The proof of second statement that the metric $d_H$ satisfies $(\beta)$ in $(\text{Cld}(Z), \supset)$ is similar.

$(\beta(i))$: $\gamma\mathcal{T}_{\delta d} \subset \mathcal{T}_{\delta d}$. It follows from (2).

$(\beta(ii))$: For $A, B, C, D \in \text{Cld}(Z)$, if $B \subset A, D \subset C$, then

$$d_H(B, D) \leq \max\{d_H(A, D), d_H(B, C)\}.$$

It is trivial.

$(\beta(iii))$: For every $A \in \text{Cld}(Z)$, $r > 0$ and $F \subset \text{Cld}(Z)$, if $F \subset \gamma\mathcal{T}_{\delta d}(A, r) \\backslash B_{\delta d}(A, r)$, then $\gamma F \cap B_{\delta d}(A, r) = \emptyset$. For every $E \in \gamma F$, there exists $F \in F$ such that $F \subset E$. Since $F \subset \gamma\mathcal{T}_{\delta d}(A, r) \\backslash B_{\delta d}(A, r)$, there exists $B \in \text{Cld}(Z)$ such that

$$d_H(A, B) < r, \quad B \subset F \quad \text{and} \quad d_H(A, F) \geq r.$$

It follows that $F \not\subset B_{\delta d}(A, r')$ for every $r' < r$. So, $E \not\subset B_{\delta d}(A, r')$ for every $r' < r$. Hence, $E \not\subset B_{\delta d}(A, r)$. □

By Corollary 2.1 and Proposition 4.1, Theorem 1.2 can be generalized as follows:

**Corollary 4.1.** Let $X$ be a regular Hausdorff space with a Baire metrizable dense $G_\delta$-subspace and $(Z, d)$ a bounded metric space. Then for every continuous map $f : X \rightarrow (\text{Cld}(Z), T_{\text{mu}})$ (every continuous map $f : X \rightarrow (\text{Cld}(Z), \mathcal{H}_{\delta d}))$, $f : X \rightarrow (\text{Cld}(Z), \mathcal{T}_{\delta d})$ is generically continuous. In particular, if $X_0$ is a completely metrizable space, then for every compactification $X$ of $X_0$, every bounded metric space $(Z, d)$ and every continuous map $f : X \rightarrow (\text{Cld}(Z), T_{\text{mu}})$ (every continuous map $f : X \rightarrow (\text{Cld}(Z), \mathcal{H}_{\delta d}))$ is generically continuous.

**Remark 1.** Note that $(\text{Cld}(Z), \mathcal{T}_{\delta d})$ may not be separable and hence may not be with any countable network. The above corollary shows that the Main Theorems are neither corollaries of Theorem 1.3 nor an extension of Theorem 1.3 in [25] since in Fort’s and Wilhelm’s results, $(Y, T_2)$ must be with a countable network.

We need the following proposition. Some results in this proposition may already exist in literatures. For example, see [2] and [22]. Proofs of others are straightforward and thus are omitted.

**Proposition 4.2.** Let $Z$ be a subspace of a Hausdorff space $\tilde{Z}$. Then $\text{Com}(Z)$ is a subset of both the sets $\text{Cld}(Z)$ and $\text{Cld}(\tilde{Z})$. Moreover, we have:

(1) The subspace topology on $\text{Com}(Z)$ of $(\text{Cld}(Z), T)$ is the same as the one of $(\text{Cld}(\tilde{Z}), T)$ for every $\mathcal{T} \in \{T_\mathcal{H}, T_{\mathcal{U}}, T_{\mathcal{V}}\}$.

(2) If $\tilde{Z}$ is metrizable, then the space $(\text{Com}(Z), \mathcal{T}_{\delta d})$ is a subspace of the space $(\text{Com}(\tilde{Z}), \mathcal{T}_{\delta d})$ for every admissible bounded metric $d$ on $X$ and admissible bounded metric $\tilde{d}$ on $\tilde{X}$.

(3) It is possible that the subspace topology on $\text{Com}(Z)$ of $(\text{Cld}(Z), T)$ is not the same as the one of $(\text{Cld}(\tilde{Z}), T)$ for $\mathcal{T} \in \{T_\mathcal{H}, T_{\mathcal{V}}\}$.

(4) The subspace topology on $\text{Com}(Z)$ of $(\text{Cld}(Z), T_{\mathcal{U}})$ (or $(\text{Cld}(Z), T_{\mathcal{V}})$, respectively) is the same as the one of $(\text{Cld}(\tilde{Z}), T_{\mathcal{U}})$ (or $(\text{Cld}(\tilde{Z}), T_{\mathcal{V}})$, respectively) if $\tilde{Z}$ is compact.

(5) Suppose $Z$ is a locally compact Hausdorff space and $\tilde{Z}$ is a compactification of $Z$. Define $\phi : \text{Com}(Z) \rightarrow \text{Com}(\tilde{Z})$ as

$$\phi(C) := C \cup (\tilde{Z} \setminus Z).$$

Then $\phi$ is well-defined and both $\phi : (\text{Com}(Z), T_{\mathcal{U}}) \rightarrow (\text{Com}(\tilde{Z}), T_{\mathcal{U}})$ and $\phi : (\text{Com}(Z), T_{\mathcal{V}}) \rightarrow (\text{Com}(\tilde{Z}), T_{\mathcal{V}})$ are embeddings. □
**Proposition 4.3.** Let \((Z,d)\) be a bounded complete metric space. Then, the Hausdorff metric \(d_H\) on \(\text{Com}(Z)\) satisfies \((\beta')\) in the poset \((\text{Com}(Z), \supseteq)\).

**Proof.** It has been proved in Proposition 4.1 that \(d_H\) satisfies \((\beta')\)(i) in the poset \((\text{Cld}(Z), \supseteq)\) and hence it does in \((\text{Com}(Z), \supseteq)\). To show that \(d_H\) satisfies \((\beta')\)(ii) in the poset \((\text{Com}(Z), \supseteq)\), let \(\{K_n\}\) be a sequence of compact sets in \((Z,d)\) and \(\{\varepsilon_n\}\) a sequence satisfying \(0 < \varepsilon_n \leq \frac{1}{2^n}\) and

\[
\text{cl} \tau_d B_{d_H}(K_{n+1}, \varepsilon_{n+1}) \subset \text{cl} B_{d_H}(K_n, \varepsilon_n).
\]

(5)

Note that \(B_{d_H}(K_{n+1}, \varepsilon_{n+1}), \text{cl} \tau_d B_{d_H}(K_{n+1}, \varepsilon_{n+1})\) and \(\text{cl} B_{d_H}(K_n, \varepsilon_n)\) are subsets of \(\text{Com}(Z)\). We show that the above formula can imply that

\[
\text{cl} \tau_d B_{d_H}(K_{n+1}, \varepsilon_{n+1}) \subset B_{d_H}(K_n, \varepsilon_n).
\]

(6)

It is worth noting that the sets in Formula (6) are not necessarily compact in \(Z\). Now we verify Formula (6). For every \(x \in \text{cl} \tau_d B_{d_H}(K_{n+1}, \varepsilon_{n+1})\), we at first verify

\[
K_{n+1} \cup \{x\} \in \text{cl} \tau_d B_{d_H}(K_{n+1}, \varepsilon_{n+1}).
\]

(7)

For every \(\varepsilon > 0\), we can choose \(y \in B_d(x, \varepsilon) \cap B_{d_H}(K_{n+1}, \varepsilon_{n+1})\). Then \(K_{n+1} \cup \{y\} \in B_{d_H}(K_{n+1}, \varepsilon_{n+1})\) and \(d_H(K_{n+1} \cup \{x\}, K_{n+1} \cup \{y\}) < \varepsilon\). Hence, Formula (7) holds. It follows from Formula (5) that \(\{x\} \subset K_{n+1} \cup \{x\} \subset L\) for some \(L \in B_{d_H}(K_n, \varepsilon_n)\). Thus, Formula (6) holds.

Let

\[
K := \bigcap_{n=1}^{\infty} \text{cl} \tau_d B_{d_H}(K_n, \varepsilon_n).
\]

It is sufficient to verify that \(K\) is in \(\text{Com}(Z)\) and \(d_H(K, K) \to 0\).

Firstly, we verify that \(K\) is compact. Since \(K\) is closed in the complete metric space \((Z,d)\), it suffices to verify that \(K\) is totally bounded in the metric space \((Z,d)\). For every \(\varepsilon > 0\), choose \(n\) such that \(\varepsilon_n \leq \frac{\varepsilon}{2}\). Since \(K_n\) is compact, there exists a finite set \(F \subset K_n\) such that \(K_n \subset \bigcup \{B(y, \varepsilon_n) : y \in F\}\). It follows that \(K \subset \bigcup \{B(y, 2\varepsilon_n) : y \in F\}\). Hence

\[
K \subset \bigcup \{B(y, 2\varepsilon_n) : y \in E\} \subset \bigcup \{B(y, \varepsilon) : y \in E\}
\]

for some finite set \(E \subset K\).

Therefore, \(K\) is total bounded.

Secondly, \(K \neq \emptyset\). For every \(n\), we define a finite family \(A_n\) of closed sets in \((Z,d)\) satisfying

(i) \(\bigcup A_n = \text{cl} \tau_d B_{d_H}(K_n, \varepsilon_n)\), and

(ii) mesh\(_d\) \(A \leq 4\varepsilon_n\).

In fact, as we have shown in first part, there exists a finite set \(F \subset K_n\) such that

\[
\text{cl} \tau_d B_{d_H}(K_n, \varepsilon_n) \subset \bigcup \{\text{cl} \tau_d B_{d_H}(y, 2\varepsilon_n) : y \in F\}.
\]

Then \(A_n := \{\text{cl} \tau_d B_{d_H}(K_n, \varepsilon_n) \cap \text{cl} \tau_d B_{d_H}(y, 2\varepsilon_n) : y \in F\}\) satisfies our requirements.

Using the sequence \(\{A_n\}\), we can define a tree \(T\) as follows:

\[
T := \{(A_1, A_2, \ldots, A_n) : A_i \in A_i, A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset, n = 1, 2, \ldots\}\.
\]

By Formula (6), the height of \(T\) is infinite. But every level is finite. It follows from König’s Infinity Lemma (see [17, Theorem 6 in Chapter VI]) that there exists an infinite branch, that is, there exists a sequence \(\{A_n\}\) such that \(A_n \in A_n\) and \(A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset\) for every \(n\). It follows from \((Z,d)\) being complete and diam\(_d\) \(A_n \leq 4\varepsilon_n \leq \frac{1}{2^n}\) for every \(n\) that \(K \supset \bigcap_{n=1}^{\infty} A_n \neq \emptyset\).

Thirdly, \(d_H(K, K) \to 0\). Note that \(K \subset B_{d_H}(K_n, \varepsilon_n) \subset B_{d_H}(K_n, \frac{1}{2^n})\) for every \(n\). It suffices to verify that, for every \(\varepsilon > 0\), there exists \(N\) such that \(K_n \subset B_d(K, \varepsilon)\) for \(n > N\). Otherwise, without loss of generality, we can assume that for \(\varepsilon = 2\) and for every \(n\), \(K_n \not\subset B_d(K, 2)\). Now, define

\[
K' := K_n \setminus B_d(K, 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}).
\]
Then $K'_n \in \text{Com}(Z)$ for every $n$. Moreover, we show that
\begin{equation}
\text{cl}_{\mathcal{F}_u} B_d(K'_{n+1}, \frac{1}{2^n}) \subset B_d(K'_n, \frac{1}{2^{n-1}}).
\end{equation}
In fact, for every $y \in \text{cl}_{\mathcal{F}_u} B_d(K'_{n+1}, \frac{1}{2^n})$, there exists $y_{n+1} \in K'_{n+1}$ such that
\begin{equation}
d(y, y_{n+1}) \leq \frac{1}{2^n}.
\end{equation}
Since $y_{n+1} \in K'_{n+1}$, we have that $y_{n+1} \in B_d(K'_n, \frac{1}{2^n})$ and
\begin{equation}
d(y_{n+1}, K) \geq 1 + \frac{1}{2} + \cdots + \frac{1}{2^n}.
\end{equation}
Thus, there exists $y_n \in K_n$ such that
\begin{equation}
d(y_n, y_{n+1}) < \frac{1}{2^n}.
\end{equation}
By Formulae (10) and (11), we have that $d(y_n, K) \geq 1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$. It follows that $y_n \in K'_n$. Hence, $d(y_{n+1}, K'_n) < \frac{1}{2^n}$. By Formula (9), we have that $d(y, K'_n) < \frac{1}{2^{n-1}}$. This shows that Formula (8) holds.

Using the second statement, we have that
\begin{align*}
K' &:= \bigcap_{n=1}^{\infty} \text{cl}_{\mathcal{F}_u} B_d(K'_{n+1}, \frac{1}{2^n}) \neq \emptyset.
\end{align*}

But, by the definition of $K'_n$, we have $K' \subset K$ and $K \cap K' = \emptyset$, a contradiction.

\begin{corollary}
Let $Z$ be a metrizable space. Then, on $\text{Com}(Z)$, $\mathcal{T}_V$ is categorically related to $\mathcal{T}_u$.
\end{corollary}

\begin{proof}
By Lemma 4.1 (5) and Proposition 4.2 (1)(2), we can assume that $(Z, d)$ is a complete metric space. Then, by Proposition 4.3, the Hausdorff metric $d_H$ on $\text{Com}(Z)$ satisfies $(\beta')$ in the poset $(\text{Com}(Z), \supseteq)$.

It follows from the Main Theorem 2, Lemmas 4.2(4) and 4.1(5) that $\mathcal{T}_V$ is categorically related to $\mathcal{T}_u$ on $\text{Com}(Z)$.
\end{proof}

\begin{corollary}
Let $Z$ be a separable locally compact metrizable space. Then, on $\text{Com}(Z)$, $\mathcal{T}_F$ is categorically related to $\mathcal{T}_{wu}$.
\end{corollary}

\begin{proof}
Since $Z$ is a separable locally compact metrizable space, there exists a compact metrizable space $\tilde{Z}$ such that $|\tilde{Z} \setminus Z| \leq 1$. By Proposition 4.2 (5), $\phi \circ f : X \to \text{Com}_u(\tilde{Z})$ is continuous. Hence,
\begin{align*}
G_0 &= \{x \in X : \phi \circ f : X \to \text{Com}_{\mathcal{T}_{wu}}(\tilde{Z}) \text{ is continuous at } x\}
\end{align*}
is a countable intersection of dense open sets in $X$. Since $\phi : (\text{Com}(Z), \mathcal{T}_F) \to \text{Com}_{\mathcal{T}_{wu}}(\tilde{Z})$ is an embedding, $G(f, \mathcal{T}_F) = G_0$. Thus, $G(f, \mathcal{T}_F)$ is a countable intersection of dense open sets in $X$.
\end{proof}

\begin{remark}
For a space $X$, a Hausdorff space $Z$ and a map $f_0 : X \to Z$, we can define a map $f : X \to \text{Com}(Z)$ by $f(x) := \{f_0(x)\}$. Evidently, $f : X \to (\text{Com}(Z), \mathcal{T}_u)$ is continuous if and only if $f : X \to (\text{Com}(Z), \mathcal{T}_u)$ is continuous and only if the map $f_0 \circ X \to Z$ is continuous. And, they can imply that the map $f : X \to (\text{Com}(Z), \mathcal{T}_{wu})$ is continuous. But, the converse is not true. Indeed, for any Hausdorff space $X$ without isolated point, let $Z = X$, as a set, with the discrete topology. Then the map $f_0 = \text{id}_X : X \to Z$ is a counterexample. This ideal will be used in Examples 2 and 4 below.
\end{remark}

\begin{proposition}
Let $(Z, d)$ be a compact metric space. Then, the Hausdorff metric $d_H$ on $\text{Com}(Z)$ satisfies $(\beta')$ in the poset $(\text{Com}(Z), \supseteq)$.
\end{proposition}
Thus, the uniqueness of equilibrium state in dynamical systems. Theory and dynamical systems [24]. In this section, we present an application of our main theorems to pressure as a generalization of topological entropy was first introduced by Ruelle [20] in 1973 for an expansive system. Then, by Proposition 4.4, the Hausdorff metric \( \overline{\mathcal{B}}_d(K,\varepsilon) \) is compact. Hence, \( k = \lim_{i \to \infty} k_{n+i} \) exists. Moreover, for every \( i \geq 0 \),

\[
d(k_{n+i}, k) \leq d(k_{n+i+1}, k_{n+i+1}) + d(k_{n+i+1}, k_{n+i+2}) + \cdots \leq \varepsilon + \varepsilon + \varepsilon + \cdots = \varepsilon.
\]

Thus, \( k \in K \) and \( d(k, k_n) \leq \delta_n \). This completes the proof of Fact.

As a direct corollary, we have that \( K \) is a non-empty compact set in \( (Z, d) \) and \( K \subset \overline{\mathcal{B}}_d(K,\delta_n) \).

Conversely, for every \( \varepsilon > 0 \), since \( K \) is compact, there exists a finite set \( F \subset \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \overline{\mathcal{B}}_d(K_m,\delta_m) \) such that \( K \subset K \). Thus, \( K \subset K \). It follows that \( K \subset K \). Therefore, \( d(H(K_n, K)) \rightarrow 0 \).

This shows that \( d(H) \) on \( \text{Com}(Z) \) satisfies \( (\beta') \) in the poset \( (\text{Com}(Z), \subset) \).

**Corollary 4.4.** Let \( Z \) be a separable metrizable space. Then, on \( \text{Com}(Z) \), \( \mathcal{T}_V \) is categorically related to \( \mathcal{T}_I \).

**Proof.** By Lemma 4.1 (5) and Proposition 4.2 (1)(2), we can assume that \( (Z, d) \) is a compact metric space. Then, by Proposition 4.4, the Hausdorff metric \( d(H) \) on \( \text{Com}(Z) \) satisfies \( (\beta') \) in the poset \( (\text{Com}(Z), \subset) \). It follows from the Main Theorem 2 and Lemma 4.2(4) that \( \mathcal{T}_V \) is categorically related to \( \mathcal{T}_I \) on \( \text{Com}(Z) \). □

5. An Application to Dynamical Systems

The concept of entropy, introduced into the realm of dynamical systems more than sixty years ago, has become an important ingredient in the characterization of the complexity of dynamical systems. Topological pressure as a generalization of topological entropy was first introduced by Ruelle [20] in 1973 for an expansive dynamical system and later by Walters [23] for the general case. The theories related to the topological pressure, variational principle and equilibrium states play a fundamental role in statistical mechanics, ergodic theory and dynamical systems [24]. In this section, we present an application of our main theorems to uniqueness of equilibrium state in dynamical systems.

In this section, we always assume that \( (X, d) \) is a compact metric space, \( f \) is a continuous self-map on \( X \), \( \mathcal{B}(X) \) is the Borel \( \sigma \)-algebra of \( X \) and \( C(X, \mathbb{R}) \) is the Banach algebra of real-valued continuous functions on \( X \) equipped with the supreme norm:

\[
\|\phi\| := \max_{x \in X} |\phi(x)|, \quad \phi \in C(X, \mathbb{R}).
\]

Let \( \mathcal{M}(X) \) denote the set of all probability measures on \( X \). And let \( \mathcal{M}_f(X) \) denote the set of all \( f \)-invariant on \( X \). On \( \mathcal{M}(X) \), we can define a metric as follows. Choose a countable dense sets \( \{\phi_n\} \) of \( (C(X, \mathbb{R}), \| \cdot \|) \) with \( \phi_n \neq 0 \) for every \( n \). Let

\[
D(\mu, \nu) := \sum_{n=1}^{\infty} \frac{\int \phi_n d\mu - \int \phi_n d\nu}{2^n \|\phi_n\|}, \quad \mu, \nu \in \mathcal{M}(X).
\]

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Then $(\mathcal{M}(X), D)$ is a compact metric space, see [24, Theorems 6.4 and 6.5]. Moreover, $\mathcal{M}_f(X)$ is closed in $(\mathcal{M}(X), D)$.

First, we recall metric entropy of $f$. Let $\alpha$ be a finite measurable partition, $\mu \in \mathcal{M}_f(X)$ and $n \in \mathbb{N}$, let

$$H_\mu(\alpha) := -\sum_{A \in \alpha} \mu(A) \log \mu(A), \quad \alpha_n^{A_0} := \{ \bigcap_{i=0}^{n-1} f^{-i}(A_i) : A_i \in \alpha \}.$$ 

The metric entropy of $f$ with respect to $\mu \in \mathcal{M}_f(X)$ and $\alpha$ of $X$ is defined as

$$h_\mu(f, \alpha) := \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_n^{A_0}).$$

Then the metric entropy of $f$ with respect to $\mu$ is defined as

$$h_\mu(f) := \sup_{\alpha} h_\mu(f, \alpha),$$

where $\alpha$ ranges over all finite measurable partitions of $X$.

For any $\phi \in C(X, \mathbb{R})$, we consider the following quantity $P(f, \phi) := \sup_{\mu \in \mathcal{M}_f(X)} P(f, \phi)$, where $P(f, \phi) := h_\mu(f) + \int \phi \, d\mu$. By variational principle [24, Theorem 9.10], $P(f, \phi)$ is the topological pressure of $f$, we refer to [24, Definition 9.2] for the precise definition of topological pressure. A $\mu \in \mathcal{M}_f(X)$ such that $P(f, \phi)$ attains its supremum is called an equilibrium state. Let $\mathcal{M}_{f,\phi}(X)$ be the collection of all equilibrium states for $\phi$. One can show that under certain conditions (see for example [24, Theorem 8.2]) the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous. As a consequence, $\mathcal{M}_{f,\phi}(X) \neq \emptyset$. The original proof of the following theorem on uniqueness of equilibrium state is elegant but much more involved. Now we can present a new proof of this theorem using our Main Theorems.

**Theorem 5.1.** ([24, Corollary 9.15.1]) If entropy map is upper semi-continuous, then there exists a dense $G_\delta$-set $\mathcal{U} \subset C(X, \mathbb{R})$ such that any $\phi \in \mathcal{U}$, $\text{card}(\mathcal{M}_{f,\phi}(X)) = 1$.

**Proof.** At first, note that $\mathcal{M}_{f,\phi}(X)$ is compact in $\mathcal{M}_f(X)$. Define a map

$$\mathcal{M}_{f,\phi}(X) : C(X, \mathbb{R}) \to \text{Com}(\mathcal{M}_f(X)), \quad \phi \mapsto \mathcal{M}_{f,\phi}(X).$$

Then, $\mathcal{M}_{f,\phi}(X)$ is upper semi-continuous since the entropy map is upper semi-continuous. Hence, using Corollary 4.4, we have that the set $\mathcal{U}$ of continuity points of the map $\mathcal{M}_{f,\phi}(X) : C(X, \mathbb{R}) \to (\text{Com}(\mathcal{M}_f(X)), D_H)$ is a dense $G_\delta$-set in the complete metric space $C(X, \mathbb{R})$. By [21, Appendix A.3.7], the set

$$\{ \psi : \text{card}(\mathcal{M}_{f,\phi}(X)) = 1 \}$$

is dense in $C(X, \mathbb{R})$. In particular, for every $\phi \in \mathcal{U}$, $\phi$ is a limit of a sequence in this set. Thus, $\text{card}(\mathcal{M}_{f,\phi}(X)) = 1$ since the map $\mathcal{M}_{f,\phi}(X) : C(X, \mathbb{R}) \to (\text{Com}(\mathcal{M}_f(X)), D_H)$ is continuous at $\phi$. \hfill \Box

**6. Counterexamples**

**Remark 3.** In Corollary 3.1, the condition that $\Lambda Y$ is metrizable cannot be removed. In fact, let $X := I$ with the usual metric and $Z := \mathbb{I}^1$ with the pointwise order. Then $X$ is compact metric space and $Y$ is a completely distributive lattice. Define $f : X \to Y$ by

$$f(x)(t) := \begin{cases} 
0 & x = 1, t \in [0, 1], \\
1 & x \in [0, 1), t < x, \\
0 & x \in [0, 1), t \geq x.
\end{cases}$$

It is not hard to show that $f : X \to \Sigma Z$ is continuous but $f : X \to \Lambda Z$ is not continuous at any point $x \in X$. 13
Moreover, we can assume that $f$ is compact-valued, then, since $\mathcal{T}_V | \text{Com}(Z) = \mathcal{T}_d | \text{Com}(Z)$ under $Z$ being metrizable, the conclusions are true if $\mathcal{T}_d$ is replaced by $\mathcal{T}_V$. But, the conclusions in Corollary 4.1 are not true if $\mathcal{T}_d$ is replaced by $\mathcal{T}_V$ under the case that $Z$ is not metrizable. In fact, there exist two continuous maps $f : X \to (\text{Com}(Z), \mathcal{T}_a)$ and $g : X \to (\text{Com}(Z), \mathcal{T}_l)$ from a compact metric space $X$ to the hyperspaces of a compact Hausdorff space $Z$ such that $G(f, \mathcal{T}_V) = G(g, \mathcal{T}_V) = \emptyset$.

Example 1. Let $X := I$ and $Z := I^I$ with the product topology. Define $f : X \to \text{Com}(Z)$ and $g : X \to \text{Com}(Z)$ as

$$f(x) := \begin{cases} Z & x = 0, \\ I^{[0,x]} \times \{0\}^{[x,1]} & x \in (0,1] \end{cases}$$
and

$$g(x) := \begin{cases} \{0\}^I & x = 1, \\ I^{[0,x]} \times \{0\}^{[x,1]} & x \in [0,1). \end{cases}$$

We show that $f : X \to (\text{Com}(Z), \mathcal{T}_a)$ is continuous at every point $x \in X$. If $x = 0$, it is trivial. If $x \in (0,1]$ and $f(x) \subseteq U$ for some open set in $Z$. We can assume that $U = \bigcup_{i \in F} U_i$ and $U_i = \bigcap_{s \in F} P_s^{-1}(\{0,b_{i,s}\})$, where $F$ is a finite set, $F_i \subseteq I$ is also a finite set for every $i \in F$ and $p_s : Z \to I$ is the projection to $I_s = I$. Moreover, we can assume that $f(x) \subseteq \bigcup_{i \in F} U_i$ for every proper subset $E$ of $F$.

Then $s > x$ for every $s \in \bigcup_{i \in F} F$. Otherwise, $s_0 \leq x$ for some $i_0 \in F$ and $s_0 \in F_n$. Choose $y_0 \in f(x) \setminus \bigcup_{i \in F \setminus \{i_0\}} U_i$. Then $y \not\in \bigcup_{i \in F \setminus \{i_0\}} U_{\{i_0\}}$ if $y(s) \geq y_0(s)$ for every $s \in I$. Now, we define $y_1 \in Z$ as $y_1(s) = y_0(s)$ for $s \not= s_0$ and $y_1(s_0) = 1$. Since $s_0 \leq x$, we have that $y_1 \in f(x)$ but $y_1 \not\in \bigcup_{i \in F} U_i$.

Now, consider a neighborhood $(0, \min\{s : i \in F, s \in F_i\})$ of $x$. Then it is trivial to verify that $f(y) \subseteq U$ for every $y \in (0, \min\{s : i \in F, s \in F_i\})$. Hence $f : X \to (\text{Com}(Z), \mathcal{T}_a)$ is continuous at every point $x \in X$.

Clearly, $f \left(\frac{1}{2}\right) = I^{[0,\frac{1}{2}]} \times \{0\}^{[\frac{1}{2},1]} \not= \emptyset$ in $\text{Com}(Z)$, $\mathcal{T}_l$. Hence $f : X \to (\text{Com}(Z), \mathcal{T}_l)$ is not continuous at $0$. For $x \in (0,1]$, $f(x - \frac{1}{2}) \to I^{[0,x]} \times \{0\}^{[x,1]} \not= \emptyset$ in $(\text{Com}(Z), \mathcal{T}_l)$. This shows that $f : X \to (\text{Com}(Z), \mathcal{T}_l)$ is not continuous at $x$.

Similarly, we can verify that $g : X \to (\text{Com}(Z), \mathcal{T}_l)$ is continuous at every point $x \in X$ but $g : X \to (\text{Com}(Z), \mathcal{T}_l)$ is not continuous at any point $x \in X$.

Remark 5. In Corollary 4.3, $(\text{Com}(Z), \mathcal{T}_F)$ can not be replaced by $(\text{Com}(Z), \mathcal{T}_V)$ even by $(\text{Com}(Z), \mathcal{T}_a)$. Let $X = Y = R$. We write the set $Q$ of rational numbers as $\{r_n : n \in N\}$. Define $f : X \to \text{Com}(Z)$ by

$$f(x) := \begin{cases} \{x\} & x \in R \setminus Q, \\ \{r_n, n\} & x = r_n. \end{cases}$$

Then $f : X \to (\text{Com}(Z), \mathcal{T}_{wu})$ is continuous. In fact, for every $x \in X = R$ and a compact set $K$ in $Y = R$, if $f(x) \cap C = \emptyset$, then $C \cap N$ is finite. Let

$$U := (R \setminus C) \setminus \{r_n : n \in C \cap N\}.$$ 

Then $U$ is a neighborhood $U$ of $x$ and $f(y) \in C^*$ for every $y \in U$. Thus, $f : X \to (\text{Com}(Z), \mathcal{T}_{wu})$ is continuous.

Now, we show that $f : X \to (\text{Com}(Z), \mathcal{T}_a)$ is not continuous at any point $x$. For $x \in X$, consider a bounded neighborhood $B(f(x), 1)$ of $f(x)$ in the space $Y$. Then $f(x) \in B(f(x), 1)$. But, for every $\delta > 0$, there exists $r_n \in (x - \delta, x + \delta)$ such that $n \not\in B(f(x), 1)$. Hence $f(r_n) \not\in B(f(x), 1)$. It follows that $f : X \to (\text{Com}(Z), \mathcal{T}_a)$ is not continuous at $x$.

Remark 6. In Corollary 4.3, the space $Z$ are assumed to satisfy three conditions: locally compact, separable and metrizable. The following three examples show that no condition in them may be dropped.

Example 2. Let $X := I$ with the usual topology and $Z := [0,1]$, as a set, with the discrete topology. Then $X$ is compact metric space and hence is a Baire space, $Z$ is locally compact and metrizable. It is not hard to verify that $f(x) := \{x\}$ is continuous from $X$ to $(\text{Com}(Z), \mathcal{T}_{wu})$ but it has no any continuous point as a map $f : X \to (\text{Com}(Z), \mathcal{T}_F)$. That is, $G(f, \mathcal{T}_F) = \emptyset$. 

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Example 3. Let $X, Z$ and $f : X \to \text{Com}(Z)$ be the same as in Example 1. Then $X$ is a compact metric space and hence a Baire space, $Z$ is a separable compact space. Since $Z$ is compact, $\mathcal{T}_u = \mathcal{T}_{wu}$ and $\mathcal{T}_V = \mathcal{T}_F$. It follows from Example 1 that $f : X \to (\text{Com}(Z), \mathcal{T}_{wu})$ is continuous at every point in $X$ but $f : X \to (\text{Com}(Z), \mathcal{T}_F)$ is not continuous at any point in $X$. That is, $G(f, \mathcal{T}_F) = \emptyset$.

Example 4. Let $C(C, \mathbb{R})$ be the family of all continuous maps from the countable and compact subspace $C := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $\mathbb{R}$ to $\mathbb{R}$. Let $X$ be the space $C(C, \mathbb{R})$ with the uniformly convergent topology, that is, $X = (C(C, \mathbb{R}), \| \cdot \|)$, and let $Z$ be the space $(C, \mathbb{R})$ with the pointwise convergent topology. Then $X$ is a complete metric space and hence a Baire space. $Z$ is a separable metric space. It is not hard to verify that the map $f(x) := \{x\}$, for every $x \in X$, is continuous at every point from $X$ to $(\text{Com}(Z), \mathcal{T}_{wu})$ but is not continuous at any point from $X$ to $(\text{Com}(Z), \mathcal{T}_F)$. That is, $G(f, \mathcal{T}_F) = \emptyset$.

References

[22] K. Sakai and Z. Yang, Hyperspaces of non-compact metrizable spaces which are homeomorphic to the Hilbert cube, Topology Appl. 127(2003), 331-342.