MOBIUS INVARIANT $Q_p$ SPACES: RESULTS, TECHNIQUES AND QUESTIONS

HASI WULAN
Department of Mathematics, Shantou University
Shantou, Guangdong 515063, China

Abstract. This paper gives a survey of the M"{o}bius invariant $Q_p$-spaces and summarizes some results, techniques and related questions in recent 10 years.

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1. Background

In 1961, John and Nirenberg [40] introduced the space of functions of bounded mean oscillation (abbreviated BMO) and they applied it to smoothness problems in partial differential equations. Their results show that BMO functions cannot become unbounded too rapidly and that BMO is situated between $L^\infty$ and $L^p$ for all $p < \infty$. Later on, in 1972, Fefferman and Stein [33] obtained the principal result on the connection between BMO and conjugate harmonic functions. This concludes that BMOA consists exactly of all sums of pairs of analytic functions, one of which has a bounded real part and another one a bounded imaginary part. Fefferman’s famous equation, $(H^1)^* = \text{BMOA}$, describes a deep relation between BMO and the Hardy space. Here BMOA denotes the set of analytic functions $f(z)$ in the unit disc $\Delta$ on the complex plane whose boundary values $f(e^{i\theta}) \in \text{BMO}$, that is,

$$\|f\|_{\text{BMO}} := \sup_I \frac{1}{|I|} \int_I |f(e^{i\theta})| - \frac{1}{|I|} \int_I f(e^{it})dt|d\theta < \infty,$$

where the supremum is taken over all intervals $I$ on the unit circle $\partial \Delta$ and $|I|$ denotes the arc length of the interval $I$. BMOA may be defined, equivalently, as a family of
functions $f$ in BMO whose Poisson extensions to $\Delta$ are analytic. These facts show that BMO functions play a very important role in function theory and it led quite naturally to increased study of these functions from the point of real variable theory and complex function theory views in recent thirty years and more. The references connected with BMO are big; some of them are [18], [23], [34] and [74].

As a slight development of BMO-theory, $Q_p$ spaces introduced in [7] and [16] ten years ago have now attracted considerable attention since they contain many well-known function spaces including BMOA and the Bloch space. Moreover, the theory of $Q_p$ spaces has been applied in decompositions and interpolation problems, in the corona construction, in $\bar{\partial}$-equations, in composition operators and wavelet basis. For more materials see [12], [27], [31] and [68]. This paper summarizes some interesting results based on the interests of the author and should not be viewed as a complete survey. Results, techniques and related questions on $Q_p$ spaces are organized as a body in each section.

The author wishes to dedicate this work to the memory of Matts Essén.

Let us now go to the details. For $a \in \Delta$, let $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ be the Möbius transformation of $\Delta$ and let, for $0 < r < 1$, $\Delta(a, r) = \{ z \in \Delta : |\varphi_a(z)| < r \}$ be the pseudo-hyperbolic disk with center $a$ and radius $r$. Let $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$ be Green’s function of $\Delta$ with logarithmic singularity at $a$ and let $dm(z)$ be the Euclidean area element on $\Delta$. Let $H(\Delta)$ denote the set of analytic functions in $\Delta$. A function $f \in H(\Delta)$ is said to belong to the space $Q_p$ for $0 \leq p < \infty$, if

$$\|f\|_{Q_p} = \left\{ \sup_{a \in \Delta} \int_{\Delta} |f'(z)|^2 (g(z, a))^p \, dm(z) \right\}^{\frac{1}{2}} < \infty. \quad (1.2)$$

Note that $Q_1 = \text{BMOA}$ ([18],[34]) and $Q_0$ is the classical Dirichlet space $D$ with

$$\|f\|_D = \left\{ \int_{\Delta} |f'(z)|^2 dm(z) \right\}^{\frac{1}{2}},$$

and that for all $p$, $1 < p < \infty$, $Q_p$ is the Bloch space as follows (cf. [7]).

$$B = \{ f : f \in H(\Delta), \quad \|f\|_B = \sup_{z \in \Delta} (1 - |z|^2)|f'(z)| < \infty \}. \quad (1.3)$$

As a part of $Q_p$ space for $p > 1$ we note that the Bloch space has the following local case respect with the Dirichlet space:

$$B_1 = \{ f : f \in H(\Delta), \quad \sup_{a \in \Delta} \int_{\Delta(a, r)} |f'(z)|^2 \, dm(z) < \infty \text{ for some } r \in (0, 1) \}. \quad (1.4)$$

Under the norm $\|f\|_{Q_p} = |f(0)| + \|f\|_{Q_p}$, $Q_p$ is a Banach space. It is clear that for all $p \in (0, \infty)$, $Q_p$ space is Möbius invariant, that is, if $f \in Q_p$ then $f \circ \varphi_a \in Q_p$ for all $a \in \Delta$. In a natural way, the Bloch space $B = B_1$ is the largest Möbius-invariant
subspace in the linear space of analytic functions on $\Delta$ equipped with the topology of uniform convergence on compacta (cf. [1] and [52]). The smallest Möbius-invariant subspace in our scale is the Dirichlet space $\mathcal{D}$ since $Q_p$ has the nesting property that $\mathcal{D} \subsetneq Q_{p_1} \subsetneq Q_{p_2} \subsetneq \text{BMOA} \subsetneq \mathcal{B}$ for $0 < p_1 < p_2 < 1$ (cf. [16]).

2. Basic properties of $Q_p$ spaces

As mentioned in Section 1 the Bloch space $\mathcal{B}$ and the Dirichlet space $\mathcal{D}$ are the largest and the smallest Möbius-invariant subspaces in all $Q_p$ spaces, respectively. Let us now study the connections between $Q_p$ space and some classical spaces. For $0 < p \leq \infty$, the Hardy space $H^p$ consists of those functions $f \in H(\Delta)$ for which

$$\|f\|_{H^p} \overset{\text{def}}{=} \sup_{0 < r < 1} M_p(r, f) < \infty,$$

where, for $0 < r < 1$,

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

We refer to [25] for the theory of Hardy spaces.

**Theorem 2.1 ([30]).** Let $p \in (0, 1)$. Then

(i) $H^\infty \setminus \bigcup_{0 < q < 1} Q_q \neq \emptyset$.

(ii) $\cap_{0 < q < 1} Q_q \setminus H^\infty \neq \emptyset$.

Let $\mathcal{A}$ denote the space of all analytic functions in $\Delta$ which are continuous on $\overline{\Delta}$. It is clear that $\mathcal{A} \subset H^\infty$. We know that $\mathcal{A} \subset \text{VMOA}$, the subspace of BMOA for which

$$\lim_{|a| \to 1} \int_{\Delta} |f'(z)|^2 g(z, a) \, dm(z) = 0,$$

and the inclusion is strict, i.e. $\mathcal{A} \not\subset \text{VMOA}$. However, we have that $\mathcal{A} \not\subset \bigcup_{0 < p < 1} Q_p$. See [51].

For $p \in (-1, \infty)$, let $\mathcal{D}_p$ be the Dirichlet type space of functions $f \in H(\Delta)$ with

$$\|f\|_{\mathcal{D}_p}^2 = \int_{\Delta} |f'(z)|^2 (1 - |z|^2)^p \, dm(z) < \infty. \quad (2.1)$$
Theorem 2.2 ([8]). Let $0 < p < 1$ and $f \in H(\Delta)$. Then $f \in Q_p$ if and only if
\[
\sup_{w \in \Delta} \| f \circ \varphi_w - f(w) \|_{D_p} < \infty.
\]

It tells us that $Q_p$ is a Möbius bounded subspace of $D_p$ for $0 < p < 1$. Since this result is valid for $p \in [1, \infty)$ as well, BMOA and the Bloch space $B$ are Möbius bounded in the Hardy space $H^2$ and the weighted Bergman space, respectively. See [18] and [15]. Because of the definition of $D_p$ for $p \in (-1, 0)$ and the relation between $Q_p$ and $D_p$, there is nothing to do in $Q_p$ if $-1 < p < 0$ since it consists of complex constants only. This follows easily from the fact that polynomials $z^{n+1}$ are never in $Q_p$ if $-1 < p < 0$ for all $n = 0, 1, 2, \ldots$.

Theorem 2.3 ([4]). Let $p \in (0, \infty)$ and $f \in H(\Delta)$. Then $f \in Q_p$ if and only if
\[
\sup_{w \in \Delta} \sup_{0 < r < 1} \int_0^{2\pi} | f_w(r e^{i\theta}) - f(w) |^2 d\theta < \infty,
\]
where $f_w(z) = f \circ \varphi_w(z)$ and $g(z) = 1 + \sum_{n=1}^{\infty} n^{\frac{1-p}{2}} z^n$.

It means that $f \in H(\Delta)$ belongs to $Q_p$ if and only if the Hadamard product of $f \circ \varphi - f(\varphi(0))$ and $g(z) = 1 + \sum_{n=1}^{\infty} n^{\frac{1-p}{2}} z^n$, where $\varphi$ ranges over all Möbius transformations mapping $\Delta$ onto $\Delta$, is a bounded subset of the Hardy space $H^2$. Note that we obtain a characterization of BMOA-functions if $p = 1$.

Anderson, Clunie and Pommerenke [3] proved that if $f, g \in B$, then $f * g \in B$ and
\[
\| f * g \|_B \leq C \| f \|_B \cdot \| g \|_B.
\]

Using a result in [44] and Fefferman’s duality theorem $(H^1)^* = \text{BMOA}$, we can prove that if $f \in \text{BMOA}$ and $g \in B$ then $f * g \in \text{BMOA}$ and
\[
\| f * g \|_{\text{BMOA}} \leq C \| f \|_{\text{BMOA}} \cdot \| g \|_B.
\]

A natural question arises whether this result remains true for $0 < p < 1$. Here we give a partial answer to it.

Theorem 2.4 ([5]). Let $0 < p < 1$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\Delta)$ with $a_n \geq 0$ for all $n$. If $f \in Q_p$ and $g \in B$ then $f * g \in Q_p$ and
\[
\| f * g \|_{Q_p} \leq C \| f \|_{Q_p} \cdot \| g \|_B.
\]

Note that the proof of Theorem 2.4 is based on a coefficient characterization of $Q_p$ functions which will be stated as Theorem 4.3 in Section 4. Our first question on $Q_p$ spaces is the following.
Question 2.1. Could the assumption \( a_n \geq 0 \) for all \( n \) in Theorem 2.4 be dropped?

For \( 1 \leq q < \infty \) and \( 0 < \alpha \leq 1 \), we say that \( f \in H(\Delta) \) is in the mean Lipschitz space \( \Lambda(q, \alpha) \) if

\[
M_q(r, f') = O((1 - r)^{\alpha - 1}), \quad r \in (0, 1).
\]

(2.2)

A related result due to Mateljevic and Pavlovic [44]:

\[
\int_0^1 (1 - r)M_\infty(r, f')^2 \, dr < \infty \quad \Rightarrow \quad f \in \text{BMOA} = Q_1.
\]

In the case \( 0 < p \leq 1 \) we have

Theorem 2.5 ([5]). For \( 0 < p \leq 1 \) and \( f \in H(\Delta) \), the following implication holds:

\[
\int_0^1 (1 - r)^p M_\infty(r, f')^2 \, dr < \infty \quad \Rightarrow \quad f \in Q_p.
\]

(2.3)

Furthermore, the exponent \( p \) in the left hand side of (2.3) cannot be increased, i.e., for given \( \epsilon > 0 \) there exists a function \( f \notin Q_p \) such that

\[
\int_0^1 (1 - r)^{p+\epsilon} M_\infty(r, f')^2 \, dr < \infty.
\]

To consider the relationship between \( Q_p \) and the mean Lipschitz spaces we need to study the question of whether or not (2.3) remains true if we substitute \( M_\infty(r, f') \) by \( M_q(r, f') \). Theorem 3.3 in [5] asserted that for \( 0 < p < 1 \) and \( 2 < q < \infty \) if \( 1 - 2/q \geq p \) then the following implication holds:

\[
\int_0^1 (1 - r)^p M_q(r, f')^2 \, dr < \infty \quad \Rightarrow \quad f \in Q_p.
\]

(2.4)

For \( 1 - 2/q \geq p \) the condition \( \int_0^1 (1 - r)^p M_q(r, f')^2 \, dr < \infty \) implies that

\[
M_q(r, f') = o((1 - r)^{-1+1/q}), \quad r \to 1,
\]

so that \( f \) belongs to the mean Lipschitz space \( \Lambda\left(q, \frac{1}{q}\right) \).

Consequently, Theorem 3.3 in [5] can also be considered as a complement to the following result.

Theorem 2.6 ([11], [30]). If \( 2 < q < \infty \) and \( 1 - \frac{2}{q} < p \) then \( \Lambda\left(q, \frac{1}{q}\right) \nsubseteq Q_p \).

Furthermore, the lower bound \( 1 - \frac{1}{q} \) is sharp in sense that \( \Lambda\left(q, \frac{1}{q}\right) \nsubseteq Q_{1-\frac{1}{q}} \) for \( q > 2 \).

3. \( Q_p \) spaces and Carleson measures
In his work on interpolation by bounded analytic functions on $\Delta$, L. Carleson \cite{21, 22} obtained the following well-known result:

Let $\mu$ be a positive measure on $\Delta$ and $0 < p < \infty$. Then

$$\left( \int_{\Delta} |f(z)|^p d\mu(z) \right)^{1/p} \leq C\|f\|_{H^p}$$

(3.1)

holds for all $f \in H^p$ if and only if there exists a constant $C > 0$ such that

$$\mu(S(I)) \leq C|I|$$

(3.2)

for all Carleson box

$$S(I) = \{ z \in \Delta : z/|z| \in I, 1 - |I| \leq |z| < 1 \}.$$

Here $\mu$ is called a Carleson measure on $\Delta$. We remark that Carleson measure must be finite. Here and elsewhere constants are denoted by $C$, which is positive and may be different from one occurrence to the next.

It is easy to see that the validity of (3.1) for given $p \in (0, \infty)$ is equivalent to its validity for $p = 2$ (cf. \cite{25}). Carleson measure, which pre-shadows Fefferman’s theorem on the duality of $H^1$ and BMO, has been developed a crucial role in the theory of function spaces.

A well-known characterization of BMOA in terms of Carleson measure is that an analytic function $f \in \text{BMOA}$ if and only if the differential form $d\mu_f(z) = |f'(z)|^2(1 - |z|^2)dm(z)$ is a Carleson measure on $\Delta$ \cite{34}. This result leads us to consider whether the space $Q_p$ is characterized by Carleson measure or some analogues. For future considerations we now proceed into some details.

A positive Borel measure $\mu$ defined on $\Delta$ is said an $\alpha$-Carleson measure if

$$\sup \{ \mu(S(I))/|I|^\alpha : I \subset \partial \Delta \} < \infty, \quad 0 < \alpha < \infty.$$  

(3.3)

If $\alpha = 1$, we obtain the classical Carleson measure (cf. \cite{34}). A useful characterization of Carleson measure shows that $\mu$ is an $\alpha$-Carleson measure if and only if (cf. \cite{11} and \cite{61})

$$\sup_{w \in \Delta} \int_{\Delta} \left( \frac{1 - |w|^2}{|1 - \overline{w}z|^2} \right)^\alpha d\mu(z) < \infty.$$  

(3.4)

The relationships between $Q_p$-spaces and $\alpha$-Carleson measures have been described as follows. Similar results ($p \geq 1$) for BMOA and $B$ are known. For BMOA we refer to work of Fefferman \cite{32}.

**Theorem 3.1** (\cite{11}). Let $0 < p < 1$, and let $f \in H(\Delta)$. Then $f \in Q_p$ if and only if $|f'(z)|^2(1 - |z|^2)^p dm(z)$ is a $p$-Carleson measure.
Theorem 3.2 ([10]). Let $0 < p < 1$, $n \in \mathbb{N}$ and let $f \in H(\Delta)$. Then $f \in Q_p$ if and only if
\[
\sup_{w \in \Delta} \int_{\Delta} |f^{(n)}(z)|^2 (1 - |\varphi_w(z)|^2)^p (1 - |z|^2)^{2n-2} \, dm(z) < \infty.
\]

It means that $f \in Q_p$ if and only if $|f^{(n)}(z)|^2(1 - |z|^2)^{2n+p-2} \, dm(z)$ is a $p$-Carleson measure on $\Delta$. Theorem 3.1, together with (3.4), tells us that Green's function $g(z, a)$ in the definition (1.2) can be replaced by the expression $1 - |\varphi_a(z)|^2$. The reason is that $|f'(z)|^2$ is subharmonic in $\Delta$. In fact, we have more general case as follows.

Theorem 3.3. Let $u \geq 0$ be subharmonic in $\Delta$ and $0 < p < \infty$. Then the following are equivalent.

(i) $\sup_{a \in \Delta} \int_{\Delta} u(z)(g(z, a))^p \, dm(z) < \infty$;

(ii) $\sup_{a \in \Delta} \int_{\Delta} u(z)(1 - |\varphi_a(z)|^2)^p \, dm(z) < \infty$.

We note that Theorem 3.3 is not true if $u$ is not subharmonic; see [14].

Now let us return back to the Carleson inequality (3.1) and review some related results. For $0 < p \leq q < \infty$, Duren [24] gave an extension of (3.1) for $q/p$-Carleson measure $\mu$. Luecking [43] considered the derivatives of $H^p$-functions to describe $q/p + nq$-Carleson measure; that is, $\mu$ is $q/p + nq$-Carleson measure if and only if there exists a constant $C$ such that for all $f \in H^p$
\[
\left( \int_{\Delta} |f^{(n)}(z)|^q d\mu(z) \right)^{1/q} \leq C \|f\|_{D_p}.
\]

On the other hand, Carleson measure can be defined as follows (cf. [55]).

Definition. Let $0 < p < \infty$. A positive Borel measure $\mu$ on $\Delta$ is said $p$-Carleson measure if there exists a constant $C$ such that for all $f \in D_p$
\[
\left( \int_{\Delta} |g(z)|^2 d\mu(z) \right)^{1/2} \leq C \|f\|_{D_p}. \tag{3.5}
\]

Note that if $1 \leq p < \infty$ the above two definitions of Carleson measure are same and for $0 \leq p < 1$ they are different. For subset $E$ of the unit circle $\partial \Delta$, let $\tau(E)$ denote the corresponding subset in the interval $[-\pi, \pi]$ obtained by the natural identification of this interval with the line. At this case $0 \leq p < 1$, (3.5) is equivalent to
\[
\mu \left( \bigcup_{j=1}^n (S(I_j)) \right) \leq C \Gamma_p \left( \bigcup_{j=1}^n \tau \left( \bigcup_{j=1}^n I_j \right) \right)
\]
whenever $I_1, \ldots, I_n$ are disjoint arcs on $\partial \Delta$. Here
\[
\Gamma_p(I) \approx \begin{cases} |I|^p, & 0 < p < 1, \\ (\log |I|^{-1})^{-1}, & p = 0. \end{cases}
\]
We note that some results by Stegenga (see [55], Lemma 2.2 and Theorem 2.3) show that, for \( 0 < p < 1 \), the estimate that \( \mu(S(I)) \leq C|I|^p \) for all \( S(I) \) is not enough to obtain (3.5) for all \( f \in D_p \). Wu Zhijian considered the Carleson measure on a more general Dirichlet space; see [60].

Using Hadamard product of \( f \in D_p \) and balancing functions \( g \) for any given \( p \in (0, \infty) \), we have the following theorem which shows that Carleson measure can be not only described by \( D_1 = H_2 \) but by functions \( f \in D_p, 0 < p < 1 \). Therefore, the theorem can be viewed as a generalization of the well-known Carleson’s result above.

**Theorem 3.4 ([4]).** Let \( 0 < p < \infty \). A positive Borel measure \( \mu \) on \( \Delta \) is a Carleson measure if and only if there exists a constant \( C \) such that

\[
\int_{\Delta} |f \ast g(z)|^2 d\mu(z) \leq C(|f(0)| + \|f\|_{D_p})^2
\]

for all \( f \in D_p \), where \( g(z) = 1 + \sum_{n=1}^{\infty} n^{\frac{1-p}{2}} z^n \).

Similar to (3.1) and results above we hope to describe the positive Borel measure \( \mu \) such that

\[
\int_{\Delta} |f'(z)|^2 d\mu(z) \leq C\|f\|_{Q_p}^2, \quad f \in Q_p, \tag{3.6}
\]

or

\[
\int_{\Delta} |f(z)|^2 d\mu(z) \leq C(|f(0)| + \|f\|_{Q_p})^2, \quad f \in Q_p. \tag{3.7}
\]

In the case \( p > 1 \), Arazy-Fisher-Peetre [1] gave a result on (3.6). Our question about the Carleson inequality is stated as follows.

**Question 3.1.** If (3.6) or (3.7) holds for all \( Q_p \) functions, \( 0 < p \leq 1 \), what characterization the measure has?

The Carleson inequality with BMO-functions can be stated as follows.

**Theorem 3.5 ([61]).** Let \( 0 < p \leq q < \infty \) and \( n \in \mathbb{N} \). A positive Borel measure \( \mu \) defined on \( \Delta \) is a \( q/p + nq \)-Carleson measure if and only if there exists a constant \( C \) such that, for all \( f \in \text{BMOA} \),

\[
\sup_{w \in \Delta} \int_{\Delta} |f^{(n)}(z) - f^{(n)}(w)|^q \frac{(1 - |w|^2)^{\frac{q}{p} + \frac{2}{3}nq}(1 - |z|^2)^{\frac{2n}{q}}}{|1 - wz|^2(\frac{2}{p} + nq)} d\mu(z) \leq C\|f\|_{q/\text{BMOA}}^q. \tag{3.8}
\]

Note that for \( n = 0 \) and \( p = q \) see [73].

**Question 3.2.** Is it possible to describe \( \alpha \)-Carleson measure by \( Q_p \) functions for \( 0 < p < 1 \) similar to (3.8)?

4. Taylor coefficients of \( Q_p \) functions
We start with a well-known unpublished result of Fefferman for a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) to belong to BMOA in terms of its Taylor coefficients \( a_n \). That is, if

\[
\sup_k \frac{1}{k} \sum_{n=0}^{\infty} \left( \frac{\min(n,k)}{n} \sum_{m=0}^{\infty} |a_{n-m+1}| \right)^2 < \infty, \tag{4.1}
\]

then \( f \in \text{BMOA} \). Conversely, if \( f \in \text{BMOA} \) with \( a_n \geq 0 \), then (4.1) holds.

The following Theorem 4.1 extends the above result to \( Q_p \) functions.

**Theorem 4.1 ([11]).** Let \( p \in (0,1) \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\Delta) \). If

\[
\sup_k \frac{1}{k} \sum_{n=0}^{\infty} (n+1)^{1-p} \left( \frac{\min(n,k)}{n+1} \sum_{m=0}^{\infty} \frac{|a_{n-m+1}|}{(m+1)^{1-p}} \right)^2 < \infty, \tag{4.2}
\]

then \( f \in Q_p \). Conversely, if \( f \in Q_p \) with \( a_n \geq 0 \), then (4.2) holds.

**Theorem 4.2 ([5]).** Let \( 0 < p < \infty \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\Delta) \). Then \( f \in Q_p \) if and only if

\[
\sup \sum_{w \in \Delta} \sum_{n=0}^{\infty} (1-|w|^2)^p \left[ \sum_{m=0}^{n} \frac{(m+1) a_{m+1} \Gamma(n-m+p) \Gamma(n-m+1)}{\Gamma(n-m+1)} \right] < \infty. \tag{4.3}
\]

By choosing \( w = 0 \), \( p = 2 \) and \( p = 1 \), respectively, we obtain the characterizations of \( D_p \), \( B \) and BMOA. For the case of series with positive coefficients, by Theorem 4.2 we immediately get the following result.

**Theorem 4.3 ([5]).** Let \( 0 < p < \infty \) and let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\Delta) \) with \( a_n \geq 0 \) for all \( n \). Then, \( f \in Q_p \) if and only if

\[
\sup \sum_{0 \leq r < 1} \sum_{n=0}^{\infty} (1-r^2)^p \left( \sum_{k=0}^{n} (k+1) a_{k+1} (n-k+1)^{p-1} r^{n-k} \right)^2 < \infty.
\]

**Question 4.1.** Could Theorem 4.3 remain true without the assumption \( a_n \geq 0 \)?

If \( f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in H(\Delta) \) has Hadamard gaps, that is, \( n_{k+1}/n_k \geq \lambda > 1 \) for all \( k = 1, 2, \ldots \), then \( f \in B \) if and only if \( \{a_k\} \) is bounded (cf. [48]) and \( f \in \text{BMOA} \) if and only if \( f \in H^2 \), i.e. \( \sum_{k=1}^{\infty} |a_k|^2 < \infty \). By using two results in [44] and [75] Aulaskari, Xiao and Zhao gave the theorem below.

**Theorem 4.4 ([16]).** Let \( p \in (0,1) \) and let \( f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in H(\Delta) \) have Hadamard gaps. Then \( f \in Q_p \) if and only if \( f \in D_p \), that is, \( \sum_{k=1}^{\infty} n_k^{1-p} |a_k|^2 < \infty \).

Since \( \Lambda(2, \frac{1}{2}) \subset \Lambda\left(q, \frac{1}{p}\right) \) for all \( q > 2 \) (see [20]), we finish this section by exhibiting a simple sufficient condition on the Taylor coefficients of a function \( f \) which implies its membership in all \( Q_p \) spaces.
Theorem 4.5 ([5]). Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\Delta) \). If \( |a_n| = O(\frac{1}{n}) \), then \( f \in \cap_{0<p<\infty} Q_p \).

Example. As an immediate consequence we deduce that \( \log \frac{1}{1-z} \in \cap_{0<p<\infty} Q_p \).

5. The boundary value characterization

The boundary value definition of BMOA given in (1.1) is actually equivalent to

\[
\|f\|_{\text{BMO}(\partial \Delta)} := \left\{ \sup_{I \subset \partial \Delta} \frac{1}{|I|^p} \int_I \int_I |f(z) - f(w)|^2 |dz| |dw| \right\}^{1/2} < \infty. \tag{5.1}
\]

Thus, \( Q_1 = \text{BMO} = \text{BMO}(\partial \Delta) \cap H^2 \). For \( p \in (0,1) \), it is possible to characterize \( Q_p \) using boundary values. Basing on a deep result in [55, Lemma 5.1], Essén and Xiao proved the following theorem.

Theorem 5.1 ([30]). Let \( 0 < p < 1 \) and \( f \in H^2 \). Then \( f \in Q_p \) if and only if

\[
\sup_{I \subset \partial \Delta} \frac{1}{|I|^p} \int_I \int_I \frac{|f(z) - f(w)|^2}{|z-w|^{2-p}} |dz| |dw| < \infty. \tag{5.2}
\]

It is not possible to consider the case \( p > 1 \) for Theorem 5.1 since the Bloch functions in general have no boundary values. However, (5.1) and (5.2) lead quite naturally to introduce a boundary counterpart of \( Q_p \) in which the analytic property of functions is not needed. For \( p \in (-\infty, \infty) \) let

\[
K_p(f, I) = \left\{ \int_I \int_I \frac{|f(z) - f(w)|^2}{|z-w|^{2-p}} |dz| |dw| \right\}^{1/2}. \tag{5.3}
\]

Define the class \( Q_p(\partial \Delta) \) which consists of all functions \( f \in L^2(\partial \Delta) \) for which

\[
\|f\|^2_{Q_p(\partial \Delta)} = \sup_{I \subset \partial \Delta} \{ |I|^{-p} K_p(f, I)^2 \} < \infty. \tag{5.4}
\]

It is clear that this space corresponds with BMO, that is, \( Q_2(\partial \Delta) = \text{BMO}(\partial \Delta) \).

Theorem 5.2 ([69]). Let \( p \in (-\infty, \infty) \). Then \( Q_p(\partial \Delta) \) is nondecreasing in \( p \). In particular:

(i) If \( p \in (-\infty, -1] \), then \( Q_p(\partial \Delta) = \mathbb{C} \).

(ii) If \( -1 < p_1 \neq p_2 \leq 1 \), then \( Q_{p_1}(\partial \Delta) \neq Q_{p_2}(\partial \Delta) \) and \( Q_{p_1}(\partial \Delta) \neq \text{BMO}(\partial \Delta) \).

(iii) If \( p \in (1, \infty) \), then \( Q_p(\partial \Delta) = \text{BMO}(\partial \Delta) \).

There is a close connection between \( Q_p(\partial \Delta) \) and the Sobolev space \( \mathcal{L}^2_p(\partial \Delta) \) of all measurable functions on \( \partial \Delta \) for which

\[
\|f\|_{\mathcal{L}^2_p(\partial \Delta)} = \left\{ \int_{\partial \Delta} \int_{\partial \Delta} \frac{|f(z) - f(w)|^2}{|z-w|^{2-p}} |dz| |dw| \right\}^{1/2} < \infty. \tag{5.5}
\]

We know that \( \mathcal{L}^2_p(\partial \Delta) = \mathbb{C} \) for \( p \leq -1 \) and \( \mathcal{L}^2_p(\partial \Delta) = L^2(\partial \Delta) \) for \( p \in [1, 2] \).
Theorem 5.3 ([69]). Let \( p \in (0, \infty) \). Then \( f \in Q_p(\partial \Delta) \) if and only if

\[
\sup_{a \in \Delta} \| f \circ \varphi_a \|_{L^2_p(\partial \Delta)} < \infty.
\]

It shows that \( Q_p(\partial \Delta) \) is identified as the Möbius invariant subspace of the Sobolev space \( L^2_p(\partial \Delta) \). Moreover, if \( f \in H(\Delta) \) we have that \( f \in Q_1(\partial \Delta) \) if and only if

\[
\sup_{a \in \Delta} \int_{\Delta} |f'(z)|^2 (1 - |\varphi_a(z)|^2) \log \frac{1}{1 - |\varphi_a(z)|^2} \, dm(z) < \infty. \tag{5.6}
\]

Comparing the definition of \( Q_1 \) and (5.6), we deduce that \( H(\Delta) \cap Q_1(\partial \Delta) \subseteq Q_1 = \text{BMOA} \).

Concerning the dyadic BMO [35] it is also possible to consider the space \( Q^d_p(\partial \Delta) \), dyadic counterpart of \( Q_p(\partial \Delta) \), and to study its properties; see [39]. Similar to the well-known work of Fefferman, Essén, Janson, Peng and Xiao defined in [26] an analogue of \( Q_p(\partial \Delta) \) on \( \mathbb{R}^n \), \( n \geq 1 \), with extension to half spaces \( \mathbb{R} \times \mathbb{R}^n_+ \). They extended the standard Carleson characterization of BMO in terms of wavelet coefficients to the full range of \( Q_p(\partial \Delta) \) spaces and gave a number of properties of \( Q_p(\partial \Delta) \). Using Jones’s constructive solution of the \( \partial \)-equation [41], Nicolau-Xiao proved

Theorem 5.4 ([46]). Let \( p \in (0,1) \). If \( |g(z)|^2 (1 - |z|^2)^p \, dm(z) \) is a \( p \)-Carleson measure, then there exists a function \( f \) defined on \( \Delta = \Delta \cup \partial \Delta \) such that \( \partial f = g \) on \( \Delta \) and \( f \in Q_p(\partial \Delta) \cap L^\infty(\partial \Delta) \).

6. \( Q_K \)-spaces

We know that for special choices of \( p \) the space \( Q_p \) gives \( Q_0 = D \), \( Q_1 = \text{BMOA} \) and \( Q_p = B \) for \( 1 < p < \infty \), respectively. Is there a more general structure behind these facts? To consider a more general case, we let \( K : [0, \infty) \to [0, \infty) \) be a right-continuous and nondecreasing function and define \( Q_K \) as follows.

Definition 6.1. \( f \in H(\Delta) \) belongs to the space \( Q_K \) if

\[
\| f \|_{Q_K}^2 = \sup_{a \in \Delta} \int_{\Delta} |f'(z)|^2 K(g(z,a)) \, dm(z) < \infty. \tag{6.1}
\]

Modulo constants, \( Q_K \) is a Banach space under the norm defined in (6.1). For \( 0 < p < \infty \), \( K(t) = t^p \) gives the space \( Q_p \). Choosing \( K(t) = 1 \), we get the Dirichlet space \( D \). For a fixed \( r \), \( 0 < r < 1 \), we choose \( K_0(t) = 1 \) if \( t \geq \log(1/r) \) and \( K_0(t) = 0 \) if \( 0 < t < \log(1/r) \) then we obtain by (1.4) that \( Q_{K_0} = B \).

Most results listed in this section and a general theory of \( Q_K \) spaces can be found in [29] and [66]. We say that the space \( Q_K \) is trivial if \( Q_K \) contains only constant functions. Whether our space \( Q_K \) is trivial or not depends on the integral

\[
\int_0^{1/e} K(\log(1/\rho)) \rho \, d\rho = \int_1^\infty K(t) e^{-2t} \, dt. \tag{6.2}
\]
Proposition 6.1. **If the integral (6.2) is divergent, then the space** \( Q_K \) **is trivial.**

The convergence of (6.1) is related to the growth order of \( K \). The log-order of \( K(r) \) is defined as
\[
\rho = \lim_{r \to \infty} \frac{\log^+ \log^+ K(r)}{\log r},
\]
where \( \log^+ x = \max\{\log x, 0\} \). If \( 0 < \rho < \infty \), the log-type of \( K(r) \) is defined as
\[
\sigma = \lim_{r \to \infty} \frac{\log^+ K(r)}{r^\rho}.
\]

Proposition 6.2. **Let \( \rho \) and \( \sigma \) be the log-order and the log-type of** \( K \).

(i) **If** \( \rho > 1 \), **then the space** \( Q_K \) **is trivial.**

(ii) **If** \( \rho = 1 \) **and** \( \sigma > 2 \), **then the space** \( Q_K \) **is trivial.**

Note that in the critical case \( \rho = 1 \) and \( \sigma = 2 \), \( Q_K \) may be trivial or nontrivial.

From now on, we assume that the (kernel) function \( K : [0, \infty) \to [0, \infty) \) is right-continuous and nondecreasing and that the integral (6.2) is convergent.

Theorem 6.1. \( Q_K \subset B \) **for all increasing functions** \( K \).

Theorem 6.2. **We assume that** \( K(1) > 0 \) **and define** \( K_1(r) = \inf(K(r), K(1)) \). Then** \( Q_K = Q_{K_1} \).

Corollary 6.1. **Let** \( 0 < p < \infty \). **Assume that** \( K(r) \approx r^p, \ r \to 0 \). **Then** \( Q_K = Q_p \).

Theorem 6.3. \( Q_K = B \) **if and only if**
\[
\int_0^1 K(\log(1/r))(1 - r^2)^{-2} r dr < \infty.
\]

Choosing \( K(r) = r^p \), we obtain that (6.5) holds for \( p > 1 \) and that (6.5) fails for \( 0 < p \leq 1 \), that is,

Corollary 6.2 ([17], [16]). \( Q_p = B \) **when** \( p > 1 \) **and** \( Q_p \subsetneq B \) **when** \( 0 < p \leq 1 \).

The next theorem gives a general method to construct a space which is between two different spaces \( Q_{K_1} \) and \( Q_{K_2} \).

Theorem 6.4. **Let** \( K_1 \leq K_2 \) **in** \( (0, 1) \). **Assume furthermore that** \( K_1(r)/K_2(r) \to 0 \) **as** \( r \to 0 \) **and that the integral in (6.5) is divergent when** \( K = K_2 \). **Then** \( Q_{K_2} \subsetneq Q_{K_1} \).

An immediate consequence of Theorem 6.4 is following

Corollary 6.3.

i) \( Q_p \subsetneq Q_q \), \( 0 \leq p < q < 1 \)  \( \text{(cf. [16])} \);

ii) \( D \subset Q_K \). **Furthermore, we have** \( D = Q_K \) **if and only if** \( K(0) > 0 \);

iii) **Let** \( K_0(t) = \frac{t}{|\log t|}, \ 0 < t \leq \frac{1}{e} \) **and** \( K_0(t) = t, t > \frac{1}{e} \). **Then** \( BMOA \subsetneq Q_{K_0} \subsetneq B \).

We know that \( \log \frac{1}{1-\varepsilon} \in Q_p \) **for all** \( p \in (0, \infty) \). **Indeed we have more as follows.**
**Theorem 6.5.** \( \log \frac{1}{1-z} \in Q_K \) if and only if
\[
\int_0^1 (1 - r^2)^{-1} K(\log(1/r))rdr < \infty. \tag{6.6}
\]

### 7. Geometric properties of \( Q_p \) functions

For analytic univalent function \( f \) in \( \Delta \), Pommerenke [49] proved that \( f \in B \implies f \in BMOA \), which easily implies a result of Baernstein about univalent Bloch functions: *If \( g(z) \neq 0 \) is an analytic univalent function in \( \Delta \), then \( \log g \in BMOA \); see [17].* Recently, Pommerenke’s result mentioned above was generalized to \( Q_p \) spaces for all \( p, 0 < p < \infty \); see [8]. In the present section, we study the condition of the \( q \)-valent functions belonging to \( Q_K \). As corollaries, we give an extended version of Pommerenke’s theorem, which is also a slight improvement of all results above and the proof presented is independently developed. All results listed here can be found in [65].

A function \( f \) analytic in the unit disk is said to be \( q \)-valent if the equation \( f(z) = w \) has never more than \( q \) solutions. Let \( n(w) = n(w, f) \) be the number of roots of this equation in \( \Delta \) counted according to their multiplicity and let
\[
p(R) = (2\pi)^{-1} \int_0^{2\pi} n(R e^{i\phi})d\phi.
\]

If \( \int_0^R p(\rho)d(\rho^2) \leq qR^2, \ R > 0 \) or \( p(R) \leq q, \ R > 0 \) for a positive number \( q \), we say that \( f \) is areally mean \( q \)-valent or circumferentially mean \( q \)-valent, respectively (cf. [36]). If \( f \) is univalent, then \( f \) is areally and circumferentially mean 1-valent.

**Theorem 7.1.** Let \( f \) be an areally mean \( q \)-valent function in \( \Delta \). If
\[
\int_0^1 \left( \log \frac{1}{1-r} \right) (1 - r)^{-1} K(\log(1/r))rdr < \infty, \tag{7.1}
\]
then \( f \in B \) if and only if \( f \in Q_K \).

If \( K(t) = t^p \), it is clear that the integral in (7.1) is convergent for \( p > 0 \), and we obtain

**Corollary 7.1.** An areally mean \( q \)-valent function \( f \) in \( B \) is in \( Q_p \) for all \( p \in (0, \infty) \).

An immediate consequence of Theorem 7.1 is the following theorem:

**Corollary 7.2 ([8]).** Let \( f \) be an analytic function in \( \Delta \) such that
\[
\int_{|w-w_0|<1} n(w, f)dm(w) \leq A < \infty \tag{7.2}
\]
for all \( w_0 \in \mathbb{C} \). Then \( f \in B \) if and only if \( f \in Q_p \) for all \( p \in (0, \infty) \).
Theorem 7.2. Let $f$ be a circumferentially mean $q$-valent and non-vanishing function in $\Delta$. If (7.1) holds, then $\log f \in Q_K$.

Corollary 7.3. Let $f$ be an analytic univalent function in $\Delta$. Then

(i) $f \in B$ if and only if $f \in BMOA$ (cf. [49]);

(ii) if furthermore $f(z) \neq 0$ in $\Delta$, then $\log f \in BMOA$ (cf. [17]).

Note that a fact that $\log \frac{1}{1-z}$ is univalent and comparing (6.5) of Theorem 6.3, (6.6) of Theorem 6.5 with (7.1) of Theorem 7.1, it is natural to conjecture the following.

Question 7.1. Does there exist a much weaker assumption than (7.1) on $K$ such that $f \in B$ belongs to $Q_K$ if we also assume that $f$ is univalent or $q$-valent?

Using the logarithmic capacity Hayman and Pommenrenke [37] gave a sufficient geometric condition discovered independently by D. A. Stegenga [56] on the domain $G$ which contains the image of a function analytic on the unit disc which guarantees that the boundary function belongs to BMO. In the converse direction, they proved that if $G$ does not satisfy this thick complement condition, then universal covering maps from $\Delta$ onto $G$ do not belong to BMOA. Thus there is no weaker omitted set condition which will always guarantee that $f \in BMOA$. Their result is now stated as follows.

Theorem 7.3 ([37]). The domain $G \subset \mathbb{C}$ has the property that every function $f(z)$ analytic in $\Delta$ with values in $G$ belongs to BMOA if and only if there exist constants $R$ and $\delta > 0$ such that

$$\text{cap}(E \cap \{|w-w_0| \leq R\}) > \delta \quad (w_0 \in G)$$

where $E = \mathbb{C} \setminus G$ and $\text{cap}$ denotes the logarithmic capacity.

Since BMOA is a subspace of the Bloch space $B$, the range condition for membership in BMOA, which is sufficient but not necessary, is a partial parallel to Bloch’s theorem. That is, it implies that if there exists an $R$ such that $\{ |w-w_0| \leq R \} \cap (\mathbb{C} \setminus G)$ is nonempty for every $w_0$, then $f \in B$ whenever $f(\Delta) \subset G$. But for $B$ more is known: $f \in B$ if and only if the Riemann surface of $f$ contains no large schlicht disks. This leads to the problem of finding a geometric condition on the Riemann surface of $f$ which is both necessary and sufficient for $f$ to belong to BMOA. Such a condition has been found in [54]. Roughly, the characterizing condition requires that each point on the surface lie within some fixed distance of a substantial portion of the ideal boundary. For $f \in H(\Delta)$, $a \in \Delta$ and $r > 0$, the component containing $a$ of the inverse image under $f$ of the open disk with center $f(a)$ and radius $r$ is denoted $\Omega_a(r)$; the harmonic measure of $\partial \Omega_a(r) \cap \partial \Delta$ with respect to $\Omega_a(r)$, evaluated at $a$, is denoted $\omega_a(r)$.

Theorem 7.4 ([54]). Let $f$ be an analytic function in $\Delta$. Then

(i) $f \in B$ if and only if there is a constant $r$ such that $\partial \Omega_a(r) \cap \partial \Delta \neq \phi$ for all $a \in \Delta$;
(ii) $f \in BMOA$ if and only if there are constants $r, \delta > 0$ and $\omega_a(r) \geq \delta$ for all $a \in \Delta$.

A Jordan curve in $\mathbb{C}$ is said quasi-conformal if

$$\max_{w \in \Gamma(w_1, w_2)} \frac{|w_2 - w| + |w - w_1|}{|w_2 - w_1|}$$

is bounded on $\Gamma$ and quasi-smooth if $L(w_1, w_2)/|w_1 - w_2|$, $w_1, w_2 \in \Gamma$, is bounded on $\Gamma$, where $\Gamma(w_1, w_2)$ is the shorter arc of $\Gamma$ between the points $w_1$ and $w_2$ on $w_1$ and $w_2$ on $\Gamma$ and $L(w_1, w_2)$ is the length of this arc.

**Theorem 7.5 ([49]).** Let $g$ be analytic and univalent in $\Delta$ and $\Gamma = \partial g(\Delta)$ be a Jordan curve. Consider the representations of the form

$$f(z) = b \log g'(z) \quad (b \in \mathbb{C}, b \neq 0).$$

Then (i) $f \in BMOA \iff \Gamma$ is quasi-smooth; (ii) $f \in B \iff \Gamma$ is quasi-conformal.

**Question 7.2.** Let $p \in (0,1)$. Give a geometric description of $Q_p$-functions.

8. Composition operators on $Q_p$ and $Q_K$-spaces

Let $\phi$ be an analytic self-map of the unit disk $\Delta$. The equation $C_\phi(f) = f \circ \phi$ defines a composition operator $C_\phi$ on the space of holomorphic functions in $\Delta$. A fundamental problem in the study of composition operators is to determine when $C_\phi$ maps boundedly or compactly between various Banach spaces in $H(\Delta)$ and the study of such operators has attracted the most attention. Related spaces include the Hardy space, the Bergman space, the Bloch space, the Dirichlet space, the Besov space, BMOA and so on. The first paper considering composition operator $C_\phi$ on $Q_p$ spaces is [57] in which the boundedness criteria of composition operators from $B$ to $Q_p$ are shown. However, they did not find a compactness condition to make the composition operator $C_\phi$ from $B$ to $Q_p$ to be compact. By employing some ideas in [19], J. Xiao [70] provided a function-theoretic characterization of when the operator $C_\phi: B \rightarrow Q_p$ is compact. Z. Lou [42] extends the work of Bourdon, Cima and Matheson [19] for compactness of $C_\phi$ on $Q_1$=BMOA to $Q_p$ spaces. However, these compactness conditions (cf. Theorem 3.1 in [19], Theorems 3.1 and 6.4 in [42]) not only depend on the map $\phi$ but also contain a requirement that all functions in $Q_p$ are required to satisfy an integration which is convergent. A natural idea is to drop the requirement and to find a new one.

Next we consider the composition operators on the $Q_K$ spaces. Our compactness condition for $C_\phi$ from $B$ to $Q_K$ depends only on $\phi$ and our contribution can be considered as a slight improvement of the known results. A technique borrowed from [19] has been modified to make it more efficient (cf. [64], [67]). It is needed to mention that Carleson measure and its various generalizations are important tools in the study of composition operators.
Theorem 8.1 ([64], [67]). Let $\phi$ be an analytic self-map of $\Delta$. Then

(i) $C_{\phi} : B \to Q_K$ is bounded if and only if
\[
\sup_{a \in \Delta} \int_{\Delta} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} K(g(z, a)) dm(z) < \infty; \tag{8.1}
\]

(ii) $C_{\phi} : B \to Q_K$ is compact if and only if $\phi \in Q_K$ and
\[
\lim_{r \to 1^-} \sup_{a \in \Delta} \int_{|\phi| > r} \frac{|\phi'(z)|^2}{(1 - |\phi(z)|^2)^2} K(g(z, a)) dA(z) = 0. \tag{8.2}
\]

Question 8.1. Give a description of the function-theoretic characterizations of $\phi$ such that the composition operator $C_{\phi}$ from $Q_{K_1}$ to $Q_{K_2}$ to be bounded or compact.

9. Meromorphic classes $Q_p^#$ and Carleson type measures

The meromorphic counterpart of $Q_p$ is the class $Q_p^#$ consisting of meromorphic functions $f$ in $\Delta$ which are such that
\[
\sup_{a \in \Delta} \int_{\Delta} f^\#(z)^2 g(z, a)\, dm(z) < \infty,
\]
where $f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$ is the spherical derivative of $f$. We know that $Q_1^# = UBC$, the class of meromorphic functions of uniformly bounded characteristic introduced by Yamashita in [72], and for each $p \in (1, \infty)$ the class $Q_p^#$ is the class of normal meromorphic functions $N$ (cf. [7]) for which
\[
\|f\|_N = \sup \{(1 - |z|^2) f^\#(z) : z \in \Delta\} < \infty.
\]

We note that there exists a non-normal function $f_0$ such that $f_0^\#(z)^2 (1 - |z|^2)^p dm(z)$ is a $p$-Carleson measure ([14]).

Theorem 9.1 ([63]). Let $0 < p < \infty$. A meromorphic function $f \in Q_p^#$ if and only if $f \in N$ and $f^\#(z)^2 (1 - |z|^2)^p dm(z)$ is a $p$-Carleson measure.

It means that Theorem 3.1 does not remain true for the meromorphic case. Thus finding a new Carleson type measure different from the classical one is needed.

We start by replacing the usual Carleson box $S(I)$ by a subset $S(w, r)$ which we call a Carleson type region and which is defined by
\[
S(w, r) = \{ z \in \Delta : |w| \leq |z| < |w| + r(1 - |w|), |\arg(z/w)| < r(1 - |w|) \}, w \neq 0,
\]
and $S(w, r) = \{|z| < r\}$ if $w = 0$, where $0 < r \leq 1$. A Carleson type region and a classical Carleson box $S(I)$ are comparable if $r = 1$. 
**Definition 9.1.** Let \( \mu \) be a positive Borel measure on \( \Delta \) and let \( \alpha > 0 \). (i) \( \mu \) is an \( \alpha \)-Carleson measure of type I if there exists a constant \( C > 0 \) such that

\[
\mu(S(w, r)) \leq C(r(1 - |w|)^\alpha), \quad w \in \Delta,
\]

for some \( r \in (0, 1] \). In particular, if \( r = 1 \), we call \( \mu \) a classical or standard \( \alpha \)-Carleson measure. (ii) \( \mu \) is an \( \alpha \)-Carleson measure of type II if there exist constants \( C > 0 \) and \( R \in (0, 1] \) such that (9.1) holds for all \( r \in (0, R] \). If \( R = 1 \), we call \( \mu \) a standard \( \alpha \)-Carleson measure of type II. See [28] and [63].

An essential distinction between a classical \( \alpha \)-Carleson measure and an \( \alpha \)-Carleson type measure is that the former must be bounded on \( \Delta \) and that an \( \alpha \)-Carleson type measure is not bounded in general on \( \Delta \) if \( 0 < \alpha \leq 1 \). Theorem 2.2.4 in [63] asserts that for each \( \alpha \in (0, 1] \), there exists an unbounded \( \alpha \)-Carleson measure of type II on \( \Delta \).

If replace \(|f'(z)|\) in (1.4) by \( f^\#(z) \) we obtain the meromorphic counterpart of \( B_1 \), denoted by \( B^\# \). We note that the meromorphic counterpart of \( B \) is \( N \) above. In contrast to the analytic case, the situation is different and we have here \( N \subsetneq B^\# \) (cf. [63]).

**Theorem 9.2 ([63]).** Let \( f \) be a meromorphic function in \( \Delta \) and \( 0 < p \leq 1 \).

(a) \( f \in B^\# \) if and only if \( (f^\#(z))^2(1 - |z|^2)^p \, dm(z) \) is a \( p \)-Carleson measure of type I;

(b) \( f \in N \) if and only if \( f^\#(z))^2(1 - |z|^2)^p \, dm(z) \) is a \( p \)-Carleson measure of type II;

(c) \( f \in Q^\#_p \) if and only if \( f^\#(z))^2(1 - |z|^2)^p \, dm(z) \) is a standard \( p \)-Carleson measure of type II.

Similar to the analytic case, we define a Möbius invariant \( Q^\#_K \) by replacing \(|f'(z)|\) in (1.2) by \( f^\#(z) \). We know that if the integral (6.2) is divergent, then the space \( Q^\#_K \) is trivial. Again, we assume here that the integral (6.2) is convergent.

**Theorem 9.3 ([29]).** \( Q^\#_K \subset B^\# \) for all increasing functions \( K \).

We note that Theorems 6.1 and 6.2 do not remain valid in the meromorphic case and that it can occur that \( Q^\#_K \not\subset N \). We quote a theorem from [66], a result which is an immediate consequence of the main result in [14].

**Theorem 9.4 ([66]).** Let \( 0 < p < \infty \). Assume that \( K \) is bounded and that \( K(r) = O(r^p) \) as \( r \to 0 \). Then there exists a non-normal function \( f_0 \in Q^\#_K \).

What more conditions on \( K \) are required for the inclusion \( Q^\#_K \subset N \)? When are the classes \( Q^\#_{K_1} \) and \( Q^\#_{K_2} \) identical for \( K_1 \neq K_2 \)?

**Theorem 9.5 ([66]).** Assume that \( K(r) \to \infty \) as \( r \to \infty \). Then \( Q^\#_K \subset N \).
**Theorem 9.6** ([29]). Define $K_1(r) = \inf(K(r), K(1))$ with $K(1) > 0$.

i) If $K$ is bounded, then $Q^#_K = Q^#_{K_1}$.

ii) If $K$ is unbounded, then $Q^#_K = N \cap Q^#_{K_1}$.

Next, we state conditions on $K_1$ and $K_2$ which imply that $Q^#_{K_1} = Q^#_{K_2}$.

**Theorem 9.7** ([29]). Assume that $K_1$ and $K_2$ are both either bounded or unbounded and that $K_1(r) \approx K_2(r)$ as $r \to 0$. Then $Q^#_{K_1} = Q^#_{K_2}$.

**Theorem 9.8** ([29]).

i) If $K$ is unbounded and (6.5) holds, then $Q^#_K = N$;

ii) If $K$ is bounded and (6.5) holds, then $Q^#_K = \mathcal{B}^#$;

iii) In (i) and (ii), (6.5) is necessary to obtain that $Q^#_K = N$ or $Q^#_K = \mathcal{B}^#$, respectively.

We have the following analogue of Corollary 6.3 but the proof is different since we cannot use the open mapping theorem there.

**Theorem 9.9** ([29]). If $K(0) > 0$, then the spherical Dirichlet class $\mathcal{D}^# = Q^#_K$.

10. Decompositions

The well-known Fefferman-Stein decomposition theorem [33] states that an arbitrary BMO function can be represented as the sum of an $L^\infty$ function and the conjugate of another such function. An analogue of this theorem in the context of $Q_p$ spaces is due to Nicolau and Xiao by continuing study of Theorem 5.1.

**Theorem 10.1** ([46]). Let $p \in (0, 1)$ and $f \in H^2$. Then $f \in Q_p$ if and only if $f$ can be written as $f_1 + if_2$, where $f_1, f_2 \in A$, and $\Re f_1, \Re f_2 \in L^\infty(\partial \Delta) \cap Q_p(\partial \Delta)$.

I would like to mention here a little more about the paper [46] because it provides an important contribution to the recently initiated study of $Q_p$ spaces. The celebrated corona theorem of L. Carleson states that the maximal ideals formed by functions which vanish at a prescribed point in the disc are dense in the maximal ideal space of $H^\infty$. This can also be reformulated as follows: given $n$ functions $f_1, f_2, \ldots, f_n$ in $H^\infty$ such that $|f_1(z)| + |f_2(z)| + \cdots + |f_n(z)| \geq \delta$ for some $\delta > 0$, there exist functions $g_1, g_2, \ldots, g_n$ in $H^\infty$ such that $f_1g_1 + f_2g_2 + \cdots + f_ng_n \equiv 1$. In this paper, the authors proved an analogous theorem which is obtained by replacing $H^\infty$ by $Q_p \cap H^\infty$ everywhere in the above formulation. A sequence of points $\{z_n\}$ in the unit disc is said to be an interpolating sequence for $Q_p \cap H^\infty$ if for every bounded sequence $\{w_n\}$ in the complex plane there exists a function $f$ in this space such that $f(z_n) = w_n$. The authors gave a necessary and sufficient condition for such sequences and used, in the proof of necessity, Khinchin’s inequality and a reproducing formula of [53].
The authors of [58] established decomposition theorems for $Q_p$ spaces for $0 < p < 1$. For $\mathcal{B}$ and BMOA the corresponding decomposition theorems are known from the work of Rochberg and Wu [53].

**Theorem 10.2 ([58]).** Suppose $0 < p < 1$. There exists an $\eta_0 > 0$ such that for any $\eta$-lattice $\{z_j\}$ in $\Delta$ with $0 < \eta < \eta_0$, we have

(i) If $f \in Q_p$, then

$$f(z) = \sum_{j=0}^{\infty} \lambda_j \frac{(1 - |z_j|^2)^{b - \frac{p}{2}}}{(1 - \overline{z_j}z)^b}, \quad b > \frac{1 + p}{2}$$

and

$$\| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \| \leq C \| f \|_{Q_p}. \quad (10.2)$$

(ii) Conversely, if $\{\lambda_j\}$ satisfies that the measure $\sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j}$ is a $p$-Carleson measure, then $f$, defined by (10.1), is in $Q_p$ with

$$\| f \|_{Q_p} \leq C \| \sum_{j=0}^{\infty} |\lambda_j|^2 \delta_{z_j} \|.$$ 

An inner function is a function $I$ analytic on $\Delta$, having the properties $|I(z)| \leq 1$ for all $z \in \Delta$ and $|I(e^{i\theta})| = 1$ a.e. on $\partial \Delta$. Of course, any inner function belongs to BMOA. However, this fact is no longer true for $Q_p$ since $H^\infty$ is not a subspace of $Q_p$ for $p \in (0, 1)$. An outer function is a function of the form

$$O_\psi(z) = \exp \left( \int_{\partial \Delta} \frac{\zeta + z}{\zeta - z} \log \psi(\zeta) \left| \frac{d\zeta}{2\pi} \right| \right),$$

where $\psi \geq 0$ a.e. on $\partial \Delta$ and $\log \psi \in L^1(\partial \Delta)$.

Write $f = (IO_-)/(1/O_+)$, where $I$ is inner and

$$O_+(z) = \exp \left\{ \int_{\partial \Delta} \frac{\zeta + z}{\zeta - z} \log(\max(|O(\zeta)|, 1)) |d\zeta|/2\pi \right\},$$

$$O_-(z) = \exp \left\{ \int_{\partial \Delta} \frac{\zeta + z}{\zeta - z} \log(\min(|O(\zeta)|, 1)) |d\zeta|/2\pi \right\}.$$ 

Then it is easy to see that both $O_-$ and $1/O_+$ belong to $H^\infty$ and $O = O_-O_+$; see [71].

A function $f$ meromorphic on $\Delta$ is said to belong to the Nevanlinna class $\mathcal{N}$ if

$$T(1, f) = \lim_{r \to 1} T(r, f) < \infty,$$
where
\[ T(r, f) = \frac{1}{\pi} \int_0^r t^{-1} \int_{|z| < t} (f^#(z))^2 \, dm(z) \, dt, \quad 0 < r < 1, \]
is the Ahlfors-Shimizu characteristic function. The well-known Nevanlinna quotient theorem (cf. [45]) says that every function in $\mathcal{N}$ is the quotient of two functions in $H^\infty$. We know that if a meromorphic function $f$ belongs to $\mathcal{N}$, then $f = IO/J$, where $I, J$ are inner functions whose greatest common divisor is 1 and $O$ is an outer function in $\mathcal{N}$. Conversely, such a function $f = IO/J$ belongs to $\mathcal{N}$. Up to some unimodular constants, the functions $I, J, O$ are uniquely determined in this case. Based on a result in [13] we have a new decomposition theorem for $Q^\#_p$.

**Theorem 10.4 ([62]).** Let $p \in (0, \infty)$. Let $f = IO/J \in \mathcal{N}$, where $I, J$ are inner functions whose greatest common divisor is 1 and $O$ is outer having two cut-off outer functions $O_+$ and $O_-$. Let $f_1 = IO_-$ and $f_2 = J/O_+$. Then $f = f_1/f_2 \in Q^\#_p$ if and only if both $f_1$ and $f_2$ lie in $H^\infty \cap Q_p$ and
\[ \inf_{z \in \Delta} (|f_1(z)|^2 + |f_2(z)|^2) > 0. \]

**Final Remarks**

Finally, we should mention that there have been some other investigations on $Q_p$, such as the several real and complex variables cases, and generalizations to Riemann surfaces, and $Q_p$ spaces of Clifford holomorphic and harmonic functions and so on. See [2], [6], [9], [26], [38] and [47]. To avoid this survey too long, we did not mention any notation or results on the little $Q^0_p$ cases here.

We conclude this lecture notes by an important progress in the study of $Q_p$ spaces. Fefferman’s famous equation, $(H^1)^* = \text{BMOA}$, gives us a lot of happiness when it comes. It is natural to ask: What is the predual space of $Q_p$ for $0 < p < 1$ if it exists?

Recently, Wu and Xie [59] answered it by investigating Morrey spaces. Almost at the same time the answer to the question was discovered independently by Peng and Yang [50] in several real variables case.

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