On quasisymmetry of quasiconformal mappings

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\textbf{ABSTRACT}

Suppose that \( f : D \rightarrow D’ \) is a quasiconformal mapping, where \( D \) and \( D’ \) are domains in \( \mathbb{R}^n \), and that \( D \) is a broad domain. We show that for every arcwise connected subset \( A \) in \( D \), the weak quasisymmetry of the restriction \( f|A : A \rightarrow f(A) \) implies its quasisymmetry. As a consequence, we see that the answer to one of the open problems raised by Heinonen from 1989 is affirmative, under the additional condition that \( A \) is arcwise connected.

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1. Introduction and main results

Quasisymmetric mappings originate from the work of Beurling and Ahlfors [2], who defined them as the boundary values of quasiconformal self-mappings of the upper half-plane onto the real line. Since then this concept has proved to be very useful, and it has played a significant role, e.g., in the work of Sullivan [14]. The general definition of quasisymmetry is due to Tukia and Väisälä, who introduced the general class of quasisymmetric mappings in [15], and it has been studied by numerous authors thereafter, see for example [1,4,5,9,10,16,18].

In this paper, we study the quasisymmetry of quasiconformal mappings in \( \mathbb{R}^n \). Motivation for this study arises from one of Heinonen’s open problems together with the main result, namely, Theorem 3.1 of [7]. We now recall a result of Heinonen, which is a generalization of a result of Väisälä [20, Theorem 2.20].

**Theorem A.** (See [7, Theorem 6.1].) Suppose that \( f : D \to D' \) is a \( K \)-quasiconformal mapping, where \( D \) and \( D' \) are bounded domains in \( \mathbb{R}^n \), and that \( D \) is \( \varphi \)-broad. If \( A \subset D \) is such that \( f(A) \) is \( b \)-LLC\(_2 \) with respect to \( \delta_{D'} \) in \( D' \), then the restriction \( f|_A : A \to f(A) \) is weakly \( H \)-quasisymmetric in the metrics \( \delta_D \) and \( \delta_{D'} \) with \( H \) depending only on the data

\[
\omega = \left(n, K, b, \varphi, \frac{\delta_D(A)}{d_D(x_0)}, \frac{\delta_{D'}(f(A))}{d_{D'}(f(x_0))}\right),
\]

where \( x_0 \) is some fixed point in \( A \) and \( d_D(x_0) \) (resp. \( \delta_D(A) \)) denotes the distance from \( x_0 \) to the boundary \( \partial D \) of \( D \) (resp. \( \delta_D \)-diameter of \( A \)).

The concepts used in the above result, such as broad domains, LLC\(_2 \) sets and weak quasisymmetry, are introduced in Section 2.

As a converse to Theorem A, Heinonen and Nääki [11] further obtained the result below.

**Theorem B.** (See [11, Lemma 8.3].) Suppose that \( f : D \to D' \) is a \( K \)-quasiconformal mapping, where \( D \) and \( D' \) are proper domains in \( \mathbb{R}^n \), and that \( D' \) is \( \varphi \)-broad. If \( A \subset D \) is arcwise connected and \( f|_A : A \to f(A) \) is weakly \( H \)-quasisymmetric in the metrics \( \delta_{D'} \) and \( \delta_D \), then \( A \) is \( b \)-LLC\(_2 \) with respect to \( \delta_D \) in \( D \), where \( b \) depends only on the data

\[
\varpi = (n, K, \varphi, H).
\]

In [20], Väisälä proved that every weak quasisymmetry \( f : X \to Y \) is quasisymmetric provided that both \( X \) and \( Y \) are HTB metric spaces and that \( X \) is arcwise connected (see [15] for the definition of HTB spaces). Heinonen pointed out in [7] that this amenable HTB-criterion is not automatically satisfied as there are domains which are LLC\(_2 \) with respect to \( \delta_D \), but which are not HTB. Hence, Heinonen asked whether the condition
“weakly” in Theorem A is redundant or not (see the paragraph next to the statement of [7, Theorem 6.5]). In this paper, we study this problem. Our result is as follows.

**Theorem 1.** Suppose that \( f : D \to D' \) is a \( K \)-quasiconformal mapping, where \( D \) and \( D' \) are domains in \( \mathbb{R}^n \), and that \( D \) is \( \varphi \)-broad. For an arcwise connected set \( A \) in \( D \), if the restriction \( f|_A : A \to f(A) \) is weakly \( H \)-quasisymmetric in the metrics \( \delta_D \) and \( \delta_{D'} \), then \( f|_A : A \to f(A) \) is \( \eta \)-quasisymmetric in the metrics \( \delta_D \) and \( \delta_{D'} \) with \( \eta \) depending only on the data

\[
\mu = (n, K, H, \varphi).
\]

The next result easily follows from Theorems 1, A and B together with Remark 2 given in Section 2.

**Corollary 1.** Suppose that \( D \) and \( D' \) are bounded domains in \( \mathbb{R}^n \), that \( f : D \to D' \) is a \( K \)-quasiconformal mapping, where \( D \) is a \( \varphi \)-broad domain, and that \( A \) is an arcwise connected subset of \( D \). Then the following statements are equivalent.

1. \( f|_A : A \to f(A) \) is \( \eta \)-quasisymmetric in the metrics \( \delta_D \) and \( \delta_{D'} \);
2. \( f(A) \) is \( b \)-LLC\(_2 \) with respect to \( \delta_{D'} \) in \( D' \);
3. \( f|_A : A \to f(A) \) is weakly \( H \)-quasisymmetric in the metrics \( \delta_D \) and \( \delta_{D'} \),

where \( b, H \) and \( \eta \) depend on each other and the data

\[
u = \left( n, K, \varphi, \frac{\delta_D(A)}{d_D(x_0)}, \frac{\delta_{D'}(f(A))}{d_{D'}(f(x_0))} \right),
\]

and \( x_0 \) is a fixed point in \( A \).

**Remark 1.**

1. The equivalence of (1) and (2) in Corollary 1 shows that the answer to Heinonen’s problem mentioned as above is affirmative with an additional assumption that the set \( A \) is arcwise connected;
2. The equivalence of (2) and (3) in Corollary 1 shows that the converse of Theorem A is also true when the set \( A \) is arcwise connected.

The quasisymmetry of quasiconformal mappings in Theorem 1 can be exploited to generalize the main result, Theorem 3.1, of [7]. This application will be discussed elsewhere.

This paper is organized as follows. In Section 2, we shall introduce necessary notations, and recall some preliminary results. Our main result, Theorem 1, will be proved in Section 3. The proof is mainly based on the quasiconformal invariance property of the conformal modulus of a curve family.
2. Preliminaries

2.1. Notation

Throughout the paper, we always assume that $D$ and $D'$ are domains in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to D'$ includes the assumption that $f$ is a homeomorphism from $D$ onto $D'$. Also we use $B(x_0, r)$ to denote the open ball \( \{ x \in \mathbb{R}^n : |x - x_0| < r \} \) centered at $x_0$ with radius $r > 0$. Similarly, for the closed balls and spheres, we use the notations $\overline{B}(x_0, r)$ and $S(x_0, r)$, respectively. In particular, we use $B$ to denote the unit ball $B(0,1)$ and $S$ its boundary.

For convenience, in what follows, we always assume that $x, y, z, \ldots$ are points in $D$ and the primes $x', y', z', \ldots$ denote the images of $x, y, z, \ldots$ in $D'$ under $f$, respectively. Also we assume that $\alpha, \beta, \gamma, \ldots$ are curves in $D$ and the primes $\alpha', \beta', \gamma', \ldots$ denote the images of $\alpha, \beta, \gamma, \ldots$ in $D'$ under $f$, respectively. For a set $A$ in $D$, $A'$ denotes the image of $A$ in $D'$ under $f$.

2.2. Conformal modulus of a curve family

Suppose that $G$ is a domain in $\mathbb{R}^n$, that $E$ and $F$ are two disjoint continua in $G$, and that $\text{Mod}(E, F; G)$ denotes the usual conformal modulus of the family of all curves joining $E$ and $F$ in $G$. For a family of curves $\Gamma$ in $G$, we always use $\text{Mod}(\Gamma)$ to denote the conformal modulus of $\Gamma$ [17]. The following result is useful for us.

**Theorem C.** (See [17, Theorem 7.1] and [19, Lemma 2.9].) (1) There is a decreasing homeomorphism $\varrho_n : (0, \infty) \to (0, \infty)$ with the following property: if $\Gamma$ is a family of paths, each of which meets a set $E$ in $\mathbb{R}^n$ and has length at least $\lambda$, then

\[
\text{Mod}(\Gamma) \leq \varrho_n \left( \frac{\lambda}{\text{diam}(E)} \right).
\]

(2) Suppose that a family of the curves $\Gamma$ lie in a Borel set $E \subset \mathbb{R}^n$ and that $\ell(\gamma) \geq r > 0$ for every locally rectifiable $\gamma \in \Gamma$. Then

\[
\text{Mod}_p(\Gamma) \leq \frac{m(E)}{r^p},
\]

where $2 \leq p \leq n$ and $m(E)$ denotes the volume of $E$.

**Theorem D.** (See [17, Section 7.5].) For $x \in \mathbb{R}^n$ and $0 < a < b < \infty$, let $A$ denote the spherical ring $\overline{B}(x, b) \setminus \overline{B}(x, a)$, $E = S(x, a)$, $F = S(x, b)$ and $\Gamma_A = \Gamma(E, F; A)$ the family of the curves in $A$ connecting $E$ and $F$. Then

\[
\text{Mod}(\Gamma_A) = \omega_{n-1} \left( \frac{b}{a} \right)^{1-n},
\]

where $\omega_{n-1}$ denotes the $(n - 1)$-dimensional surface area of $S$. 

2.3. Internal metric, linearly locally connected sets, broad domains and (weak) quasisymmetric mappings

For $x, y$ in $D$, the internal metric $\delta_D$ in $D$ is defined by

$$\delta_D(x, y) = \inf\{\text{diam}(\alpha) : \alpha \subset D \text{ is a rectifiable arc joining } x \text{ and } y\}.$$ 

**Definition 1.** Suppose that $A \subset D$ and $b \geq 1$ is a constant. We say that $A$ is $b$-LLC$_2$ (resp. $b$-LLC$_2$ with respect to $\delta_D$) in $D$ if for all $x \in A$ and $r > 0$, the points in $A \setminus \overline{B}(x, br)$ (resp. $A \setminus \overline{B}_{\delta_D}(x, br)$) can be joined in $D \setminus \overline{B}(x, r)$ (resp. $D \setminus \overline{B}_{\delta_D}(x, r)$), where

$$\overline{B}_{\delta_D}(x, r) = \{z \in D : \delta_D(z, x) < r\}.$$ 

If $A = D$, then we say that $D$ is $b$-LLC$_2$ (resp. $b$-LLC$_2$ with respect to $\delta_D$).

**Definition 2.** Let $\varphi : (0, \infty) \to (0, \infty)$ be a decreasing homeomorphism. We say that $D$ is $\varphi$-broad if for each $t > 0$ and each pair $(C_0, C_1)$ of continua in $D$ with $C_0 \cap C_1 = \emptyset$, the condition $\delta_D(C_0, C_1) \leq t \min\{\text{diam}(C_0), \text{diam}(C_1)\}$ implies

$$\text{Mod}(C_0, C_1; D) \geq \varphi(t),$$

where $\delta_D(C_0, C_1)$ denotes the $\delta_D$-distance between $C_0$ and $C_1$.

Broad domains were introduced in [20]. It was later proved that a simply connected planar domain is broad if and only if it is John [13, Section 8]. Further, Gehring and Martio proved that each uniform domain in $\mathbb{R}^n$ is broad, see [6, Lemma 2.6].

The notion of a broad domain also goes under the term Löwner space, which was introduced by Heinonen and Koskela [10] in their study of quasiconformal mappings between metric spaces. Heinonen’s monograph [8] provides a comprehensive account of this topic and related results. See for example [1,3,12,16] for more discussions in this direction.

**Definition 3.** Let $(X, d)$ and $(X', d')$ be two metric spaces, and let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. We say that an embedding $f : X \to X'$ is $\eta$-quasisymmetric, or briefly $\eta$-QS, in the metrics $d$ and $d'$ if $d(a, x) \leq td(a, y)$ implies

$$d'(a', x') \leq \eta(t)d'(a', y')$$

for all $a, x, y \in X$, and if there is a constant $\nu \geq 1$ such that $d(a, x) \leq d(a, y)$ implies

$$d'(a', x') \leq \nu d'(a', y'),$$

then $f$ is said to be weakly $\nu$-quasisymmetric, or briefly weakly $\nu$-QS, in the metrics $d$ and $d'$.
Obviously, quasisymmetry implies weak quasisymmetry.

For convenience, we say that the quantity $\frac{d(a, x)}{d(a, y)}$ is the quasisymmetric ratio of points $a, x$ and $y$ with respect to the metric $d$.

**Remark 2.** It follows from [3] that $f^{-1}$ is $\eta_1$-quasisymmetric if $f$ is $\eta$-quasisymmetric, where $\eta_1(t) = \frac{1}{\eta^{-1}(1/t)}$. It also follows from [13, Lemma 3.9] that QS mappings preserve broad domains.

### 3. Linear local connectedness and quasisymmetry

This section consists of two subsections. In the first subsection, we establish a result on quasisymmetric ratios, which will be used later on. The aim of the second subsection is to prove Theorem 1.

#### 3.1. Main lemma

The aim of this subsection is to state and prove the following main lemma which will play a key role in the proof of Theorem 1.

**Lemma 1.** Suppose that $f : D \to D'$ is a $K$-quasiconformal mapping, that $D$ is $\varphi$-broad, and that $A \subset D$ is arcwise connected such that the restriction $f|_A : A \to A'$ is weakly $H$-QS in the metrics $\delta_D$ and $\delta_{D'}$. For $a, x, y \in A$, if $\delta_D(a, x) \leq \delta_D(a, y)$, then

$$\frac{\delta_{D'}(a', x')}{\delta_{D'}(a', y')} \leq \psi \left( \frac{\delta_D(a, x)}{\delta_D(a, y)} \right),$$

where $\psi : (0, 1] \to (0, \infty)$ is an increasing homeomorphism which depends only on the data

$$\sigma = (n, K, H, \varphi, \varrho_n)$$

and $\varrho_n$ is the decreasing homeomorphism from Theorem C.

To prove Lemma 1, we need the following basic result.

**Lemma 2.** Under the assumptions of Lemma 1, we see that for $z_1, z_2, z_3 \in A$, if $\delta_D(z_1, z_3) \leq c\delta_D(z_1, z_2)$, where $c$ is a constant, then

$$\delta_{D'}(z_1', z_3') \leq \mu_7(c)\delta_{D'}(z_1', z_2'),$$

where $\mu_7(c) = H\mu_6^{1+\log_3 c}$ and

$$\mu_6 = \max \left\{ 8HK, 8H \exp \left( 2 \frac{2K \omega_{n-1}}{\varphi(45/4)} \right)^{n-1} + 8H, 2H(H + 1) \left( \varrho_n^{-1} \frac{\varphi(8)}{K} + 1 \right) \right\}.$$
Fig. 1. The arc $\beta$ in $A$ and its partition.

**Proof.** If $\delta_D(z_1, z_2) \geq \delta_D(z_1, z_3)$, then the assumption "$f|_A$ being weakly $H$-QS in the metrics $\delta_D$ and $\delta_D'$" implies

$$\delta_D'(z'_1, z'_2) \leq H\delta_D'(z'_1, z'_2). \tag{3.1}$$

For the other case, that is, $\delta_D(z_1, z_2) < \delta_D(z_1, z_3)$, we take $\beta$ to be an arc joining $z_1$ and $z_3$ in $A$. We partition $\beta$ with the aid of a finite sequence $\{v_i\}_{i=1}^{s+2}$ of points in $\beta$ as follows.

If $\delta_D(z_1, z_3) < 3\delta_D(z_1, z_2)$, then we let $s = 1$ and $x_s = z_1$.

If $\delta_D(z_1, z_3) \geq 3\delta_D(z_1, z_2)$, let $s$ be the number which satisfies

$$z_3 \in D \setminus B_{\delta_D}(z_1, 3^{s-1}\delta_D(z_1, z_2)) \quad \text{and} \quad z_3 \in B_{\delta_D}(z_1, 3^s\delta_D(z_1, z_2)).$$

Obviously, $s \geq 2$. It is possible that $\delta_D(z_1, z_3) = 3^{s-1}\delta_D(z_1, z_2)$. We let $x_1 = z_1$, and let $x_2, \ldots, x_s$ be points such that for each $i \in \{1, \ldots, s-1\}$, $x_{i+1}$ denotes the last point in $\beta$ along the direction from $z_1$ to $z_3$ such that (see Fig. 1)

$$\delta_D(x_1, x_{i+1}) = 3^{i-1}\delta_D(z_1, z_2). \tag{3.2}$$

Obviously,

$$3^{s-1}\delta_D(z_1, z_2) \leq \delta_D(z_1, z_3) < 3^s\delta_D(z_1, z_2). \tag{3.3}$$

Hence, we have chosen points $\{x_1, \ldots, x_s\}$ from $\beta$ including the case $s = 1$, and now, we still need to pick up another point, denoted by $y_{s+1}$, in $\beta$, which is the first point in $\beta[z_3, x_s]$ along the direction from $z_3$ to $x_s$ such that (see Fig. 1)

$$\delta_D(y_{s+1}, z_3) = \frac{1}{8}\delta_D(x_s, z_3). \tag{3.4}$$
Then we see that
\[
\delta_D(x_s, y_{s+1}) \leq \delta_D(x_s, z_3) + \delta_D(y_{s+1}, z_3) = 9\delta_D(y_{s+1}, z_3),
\]
and for each \( w \in \beta[y_{s+1}, z_3] \),
\[
\delta_D(y_{s+1}, w) \leq \delta_D(y_{s+1}, z_3) + \delta_D(z_3, w) \leq \delta_D(y_{s+1}, z_3) + \frac{1}{8}\delta_D(x_s, z_3) \\
\leq 2\delta_D(y_{s+1}, z_3).
\]

Moreover,
\[
\delta_D(x_1, z_3) \geq \frac{3}{4}\delta_D(x_s, z_3).
\]

This inequality is obvious if \( s = 1 \), and if \( s \geq 2 \), (3.2) and (3.3) imply
\[
\delta_D(x_1, z_3) \geq \delta_D(x_s, z_3) - \delta_D(x_1, x_s) \geq \delta_D(x_s, z_3) - \frac{1}{3}\delta_D(x_1, z_3),
\]
from which the inequality easily follows.

Next, we have the following useful inequalities related to \( y_{s+1} \). First, we deduce from (3.4), (3.6) and the choice of \( y_{s+1} \) that for \( w \in \beta[y_{s+1}, z_3] \),
\[
\delta_D(x_1, w) \geq \delta_D(x_1, z_3) - \delta_D(w, z_3) \geq \frac{3}{4}\delta_D(x_s, z_3) - \delta_D(w, z_3) \\
\geq 5\delta_D(y_{s+1}, z_3) \\
\geq \frac{5}{2}\delta_D(y_{s+1}, w),
\]
and then, if \( s \geq 2 \), we infer from (3.3) and (3.4) that
\[
\delta_D(x_s, y_{s+1}) \leq \delta_D(x_1, x_s) + \delta_D(x_1, z_3) + \delta_D(y_{s+1}, z_3) \\
= \delta_D(x_1, x_s) + \delta_D(x_1, z_3) + \frac{1}{8}\delta_D(x_s, z_3) \\
\leq \frac{9}{8}(\delta_D(x_1, x_s) + \delta_D(x_1, z_3)) \\
\leq \frac{45}{4}\delta_D(x_1, x_s).
\]

We now take \( v_i = x_i \) for each \( i \in \{1, \ldots, s\} \), where \( s \geq 1 \), \( v_{s+1} = y_{s+1} \) and \( v_{s+2} = z_3 \). In this way, we get the desired partition of \( \beta \) (see Fig. 1).

The following two claims are inequalities on the image of the partition of \( \beta \) under \( f \), which will be needed later on.

**Claim 1.** If \( s = 1 \), then \( \delta_D(v'_1, v'_3) \leq \mu_6 \delta_D(v'_1, z'_2) \).
Suppose on the contrary that
\[\delta_{D'}(v'_1, v'_3) > \mu_0\delta_{D'}(v'_1, z'_2).\]

Under this assumption, we shall get a contradiction by using the quasiconformal invariance property of the conformal modulus of a curve family.

Let \(\chi\) be an arc joining \(v_1\) and \(z_2\) in \(A\) (see Fig. 2), and let \(y_1\) be the first point in \(\chi\) along the direction from \(v_1\) to \(z_2\) satisfying (see Fig. 2)
\[\delta_D(v_1, y_1) = \frac{1}{2}\delta_D(v_1, z_2). \tag{3.9}\]

First, we apply the assumption “\(D\) being \(\varphi\)-broad” to get a lower bound for the conformal modulus \(\text{Mod}(\chi[v_1, y_1], \beta[v_2, v_3]; D)\). For this, we need to show that the curves \(\chi[v_1, y_1]\) and \(\beta[v_2, v_3]\) are disjoint.

For each \(u \in \chi[v_1, y_1]\), by the assumption on the weak quasisymmetry of \(f\) and the choice of \(y_1\), we have
\[\delta_{D'}(v'_1, u') \leq H\delta_{D'}(v'_1, z'_2), \tag{3.10}\]
and for each \(v \in \beta[v_2, v_3]\), it follows from (3.4) together with the choice of \(y_1\) and \(v_2\) that
\[\begin{align*}
\delta_D(u, v) &\geq \delta_D(v_1, v_3) - \delta_D(v_3, v) - \delta_D(v_1, u) \\
&\geq \frac{7}{8}\delta_D(v_1, v_3) - \frac{1}{2}\delta_D(v_1, z_2) \geq \frac{3}{8}\delta_D(v_1, v_3) = 3\delta_D(v_2, v_3) \\
&\geq 3\delta_D(v, v_3). \tag{3.11}
\end{align*}\]

So the assumption on the weak quasisymmetry of \(f\) implies
\[\delta_{D'}(v'_1, v'_3) \leq \delta_{D'}(v'_1, v') + \delta_{D'}(v', v'_3) \leq (H + 1)\delta_{D'}(v'_1, v'). \tag{3.12}\]
Now, \((3.10)\) and \((3.12)\) show that
\[
\delta_{D'}(u', v') \geq \delta_{D'}(v'_1, v') - \delta_{D'}(v'_1, u') \\
\geq \frac{1}{H + 1} \delta_{D'}(v'_1, v'_3) - H \delta_{D'}(v'_1, z'_2) \\
> \left( \frac{\mu_6}{H + 1} - H \right) \delta_{D'}(v'_1, z'_2).
\]
Hence \(\chi[v_1, y_1] \cap \beta[v_2, v_3] = \emptyset\). Further, the combination of \((3.4)\) and \((3.9)\) guarantees that
\[
\frac{\delta_D(v_1, v_3)}{\min\{\text{diam}(\chi[v_1, y_1]), \text{diam}(\beta[v_2, v_3])\}} \leq 8,
\]
we see from the assumption “\(D\) being \(\varphi\)-broad” that
\[
\text{Mod}(\chi[v_1, y_1], \beta[v_2, v_3]; D) \geq \varphi\left(\frac{\delta_D(v_1, v_3)}{\min\{\text{diam}(\chi[v_1, y_1]), \text{diam}(\beta[v_2, v_3])\}}\right) \geq \varphi(8),
\]
which is the desired bound.
Since \((3.10)\) implies
\[
\text{diam}(\chi'[v'_1, y'_1]) \leq 2H \delta_{D'}(v'_1, z'_2)
\]
and the quasiconformal invariance property of the conformal modulus of a curve family shows
\[
\text{Mod}(\chi[v_1, y_1], \beta[v_2, v_3]; D) \leq K \text{Mod}(\chi'[v'_1, y'_1], \beta'[v'_2, v'_3]; D'),
\]
we infer from Theorem C and \((3.13)\) that
\[
\varphi(8) \leq \text{Mod}(\chi[v_1, y_1], \beta[v_2, v_3]; D) \leq K \text{Mod}(\chi'[v'_1, y'_1], \beta'[v'_2, v'_3]; D') \\
\leq K \varrho_n \left( \frac{(\frac{\mu_6}{H + 1} - H) \delta_{D'}(v'_1, z'_2)}{\text{diam}(\chi'[v'_1, y'_1])} \right) < \varphi(8),
\]
where \(\varrho_n\) is from Theorem C. This obvious contradiction shows that the claim is true.

Claim 2. If \(s \geq 2\), then \(\delta_{D'}(v'_i, v'_{i+1}) \leq \mu_6 \delta_{D'}(v'_1, v'_i)\) for each \(i \in \{2, \ldots, s + 1\}\).

Suppose on the contrary that there exists an \(i \in \{2, \ldots, s + 1\}\) such that
\[
\delta_{D'}(v'_i, v'_{i+1}) > \mu_6 \delta_{D'}(v'_1, v'_i).
\]
If \(i = s + 1\), then we get
\[
\delta_{D'}(v'_{s+1}, v'_{s+2}) \geq \delta_{D'}(v'_1, v'_{s+2}) - \delta_{D'}(v'_1, v'_{s+1}) \geq (\mu_6 - 1) \delta_{D'}(v'_1, v'_{s+1}),
\]

whence, again, the assumption on the weak quasisymmetry of \( f \) implies

\[
\delta_D(v_{s+1}, v_{s+2}) \geq \delta_D(v_1, v_{s+1}),
\]

which contradicts (3.7). Hence we see that \( i \in \{2, \ldots, s\} \). In the following, by using the quasiconformal invariance property of the conformal modulus, we shall show that it is also impossible. For this purpose, some preparation is needed.

Let \( \gamma'_i \) be an arc joining \( v'_i \) and \( v'_{i+1} \) in \( D' \) such that (see Fig. 3)

\[
\text{diam}(\gamma'_i) < \frac{5}{4} \delta_{D'}(v'_1, v'_i),
\]

and we let \( w'_i \) be the last point in \( \beta'[v'_i, v'_{s+2}] \) along the direction from \( v'_i \) to \( v'_{s+2} \) such that (see Fig. 3)

\[
\delta_{D'}(v'_1, w'_i) = \frac{\mu_6}{3H} \delta_{D'}(v'_1, v'_i).
\]

Obviously,

\[
\delta_{D'}(v'_1, v'_{i+1}) > 3H \delta_{D'}(v'_1, w'_i).
\]

Let us leave the proof of the claim for a moment and determine the position of \( w'_i \) in \( \beta'[v'_i, v'_{s+2}] \). Our result is as follows.

**Subclaim 1.** \( w'_i \in \beta'[v'_i, v'_{i+1}] \).

To prove this subclaim, we divide the arguments into the case where \( i \leq s - 1 \) and the case where \( i = s \). We first consider the case \( i \leq s - 1 \). Then, once again, by the assumption on the weak quasisymmetry of \( f \) together with (3.16), we have

\[
\delta_D(v_1, v_{i+1}) > \delta_D(v_1, w_i),
\]

and so the choice of \( v_{i+1} \) shows that \( w'_i \in \beta'[v'_i, v'_{i+1}] \).
On the other hand, for the case \( i = s \), (3.15) leads to
\[
\delta_D(v_{s+1}', w_s') \geq \delta_D'(v_s', w_{s+1}') - 2\delta_D'(v_s', v_s') - \delta_D'(v_s', w_s') > H\delta_D'(v_s', w_s'),
\]
whence we infer from the assumption on the weak quasisymmetry of \( f \) that
\[
\delta_D(v_{s+1}, w_s) > \delta_D(v_1, w_s),
\]
which together with (3.7) shows that \( w_s' \in \beta'[v_s', v_{s+1}'] \). Hence Subclaim 1 is also true in this case. The proof of the subclaim is complete.

We shall now present a proof of Claim 2. We shall reach a contradiction by obtaining a lower bound and an upper bound for the conformal modulus \( \text{Mod}(\gamma_i, \beta[w_i, v_{s+2}]; D) \). For a lower bound of this quantity, we need an inequality: If \( s \geq 2 \), then for each \( i \in \{2, \ldots, s\} \),
\[
\delta_D(v_i, v_{i+1}) \leq \frac{45}{4} \min\{\delta_D(v_1, v_i), \delta_D(v_{i+1}, v_{s+2})\}. \tag{3.17}
\]
This inequality easily follows from (3.5) and (3.8) together with the following inequalities: For \( s \geq 3 \) and \( i \in \{2, \ldots, s - 1\} \),
\[
\delta_D(v_i, v_{i+1}) \leq \delta_D(v_1, v_i) + \delta_D(v_1, v_{i+1}) = 4\delta_D(v_1, v_i),
\]
and further
\[
\delta_D(v_i, v_{i+1}) \leq 4\delta_D(v_1, v_i) = \frac{4}{3}\delta_D(v_1, v_{i+1}) \leq \frac{2}{3}\delta_D(v_{i+1}, v_{s+2}),
\]
where in the third inequality, the following estimate is used:
\[
\delta_D(v_{i+1}, v_{s+2}) \geq \delta_D(v_1, v_{s+2}) - \delta_D(v_1, v_{i+1}) \geq 2\delta_D(v_1, v_{i+1}).
\]
Since for each \( w' \in \gamma_i' \) and \( z' \in \beta'[w_i', v_{s+2}'] \), we have from (3.14) and the choice of \( w_i' \) that
\[
\delta_D'(w', z') \geq \delta_D'(v_i', z') - \delta_D'(v_i', w') \geq \left(\frac{\mu_6}{3H} - \frac{5}{4}\right)\delta_D'(v_i', v_i'), \tag{3.18}
\]
we see that \( \gamma_i \cap \beta[w_i, v_{s+2}] = \emptyset \), whence it follows from the assumption “\( D \) being \( \varphi \)-broad” that
\[
\text{Mod}(\gamma_i, \beta[w_i, v_{s+2}]; D) \geq \varphi \left(\frac{\delta_D(\gamma_i, \beta[w_i, v_{s+2}])}{\min\{\text{diam}(\gamma_i), \text{diam}(\beta[w_i, v_{s+2}])\}}\right).
\]
Moreover, (3.17) and Subclaim 1 show that
\[
\delta_D(\gamma_i, \beta[w_i, v_{s+2}]) \leq \delta_D(v_i, v_{i+1}) \leq \frac{45}{4} \min\{\text{diam}(\gamma_i), \text{diam}(\beta[w_i, v_{s+2}])\},
\]
where \( \gamma_i \cap \beta[w_i, v_{s+2}] = \emptyset \) and \( \varphi \) is defined as
\[
\varphi(\delta) = \begin{cases} 
\frac{\delta}{\min\{\text{diam}(\gamma_i), \text{diam}(\beta[w_i, v_{s+2}])\}} & \text{if } \delta > \frac{\mu_6}{3H} - \frac{5}{4}, \\
1 & \text{otherwise},
\end{cases}
\]
and
\[
\text{diam}(\gamma_i) = \max\{\text{diam}(\gamma_i), \text{diam}(\beta[w_i, v_{s+2}])\}.
\]
and so we obtain a lower bound as follows:

$$\text{Mod}(\gamma_i, \beta[w_i, v_{s+2}]; D) \geq \varphi\left(\frac{45}{4}\right).$$  \hspace{1cm} (3.19)

For an upper bound for \(\text{Mod}(\gamma_i, \beta[w_i, v_{s+2}]; D)\), we still need to construct two families of curves in \(D'.\) Let (see Fig. 4):

$$\Upsilon'_{1i} = \{\eta'_{1i} : \eta'_{1i} \subset \mathbb{B}(v'_i, (\frac{\mu_6}{4H})^{\frac{3}{2}} \delta D'(v'_i, v'_i))\}$$

and

$$\Upsilon'_{2i} = \{\eta'_{2i} : \eta'_{2i} \cap \mathbb{S}(v'_i, (\frac{\mu_6}{4H})^{\frac{3}{2}} \delta D'(v'_i, v'_i)) \neq \emptyset\},$$

where \(\eta'_{ji}\) \((j = 1, 2)\) denote the curves joining \(\gamma'_i\) and \(\beta'[w'_i, v'_{s+2}]\) in \(D'.\)

Now, the quasiconformal invariance property of the conformal modulus of a curve family implies

$$\text{Mod}(\gamma_i, \beta[w_i, v_{s+2}]; D) \leq K \text{Mod}(\gamma'_i, \beta'[w'_i, v'_{s+2}]; D'),$$

whence it follows from Theorems C and D together with (3.18) that

$$\text{Mod}(\gamma_i, \beta[w_i, v_{s+2}]; D) \leq K \left(\text{Mod}(\Upsilon'_{1i}) + \text{Mod}(\Upsilon'_{2i})\right)$$

$$\leq K \omega_{n-1} \left(\left(\frac{\mu_6}{4H}\right)^{\frac{3}{2}} + \left(\frac{1}{2} \log \frac{4\mu_6}{25H}\right)^{1-n}\right)$$

$$< \varphi\left(\frac{45}{4}\right),$$
where in the second inequality, the fact used is \( \eta_{2i} \cap S(v_i', \frac{5}{3} \delta_D'(v_1', v_i')) \neq \emptyset \) for each \( \eta_{2i} \in \mathcal{Y}_{2i} \), which easily follows from (3.14). This obviously contradicts (3.19). Hence Claim 2 is true.

Let us now finish the proof of the lemma. Let \( \mu_7(c) = H\mu_6^{1+\log_3 c} \). Then, since \( \mu_6 < \mu_7(c) \), Claim 1 guarantees that Lemma 2 is true when \( s = 1 \). For the case \( s \geq 2 \), we see from the assumption on the weak quasisymmetry of \( f \) and Claim 2 that
\[
\delta_{D'}(z'_1, z'_2) \geq \frac{1}{H} \delta_{D'}(v'_1, v'_2) \geq \frac{1}{H\mu_6} \delta_{D'}(v'_1, v'_3) \geq \ldots \geq \frac{1}{H\mu_6^s} \delta_{D'}(v'_1, v'_{s+2}),
\]
and so
\[
\delta_{D'}(z'_1, z'_3) \leq H\mu_6^s \delta_{D'}(z'_1, z'_2) \leq H\mu_6^{1+\log_3 c} \delta_{D'}(z'_1, z'_2),
\]
since (3.3) and the assumption \( \delta_D(z_1, z_3) \leq c\delta_D(z_1, z_2) \) imply that \( s - 1 \leq \log_3 c \), which shows that the lemma is also true in this case. Hence the proof of Lemma 2 is complete. \( \square \)

Now, we are ready to prove our main lemma.

**The proof of Lemma 1.** For a proof, we let \( a, x \) and \( y \in A \) with \( \delta_D(a, x) \leq \delta_D(a, y) \). In the rest of this subsection, we write
\[
s = \frac{\delta_D(a, y)}{\delta_D(a, x)}.
\]
Obviously, \( s \geq 1 \). Let
\[
\mu_8 = \max \left\{ 6^3 K, H^2 \exp \left( \frac{8K\omega_{n-1}}{\varphi(6)} \right)^{\frac{1}{n-1}} \right\}.
\]
With the aid of \( \mu_8 \), we divide the discussions into two cases: \( s \leq 4\mu_8^2 \) and \( s > 4\mu_8^2 \). For each case, we shall construct an increasing homeomorphism. The desired homeomorphism follows from this.

Suppose first that \( s \leq 4\mu_8^2 \). Then by the assumption \( \|f|_A \) being weakly \( H\)-QS, we have
\[
\frac{\delta_{D'}(a', x')}{\delta_{D'}(a', y')} \leq H \leq 4H\mu_8^2 \frac{\delta_D(a, x)}{\delta_D(a, y)}.
\]
In this case we let
\[
\psi_1(p) = 4H\mu_8^2 p \tag{3.20}
\]
for \( p \in [\frac{1}{4\mu_8}, 1] \).
Suppose next that \( s > 4\mu^2_8 \). In this case, we shall exploit the quasiconformal invariance property of the conformal modulus together with Theorems C and D to get an increasing homeomorphism. First, we do some preparation.

Let \( \alpha \) be an arc joining \( a \) and \( y \) in \( A \) (see Fig. 5). We give the following partition to \( \alpha \). Let \( x_1 = a \), and let \( x_2, \ldots, x_{t+1} \) be points such that for each \( i \in \{1, \ldots, t\} \), \( x_{i+1} \) denotes the last point in \( \alpha \) along the direction from \( a \) to \( y \) such that (see Fig. 5)

\[
\delta_D(x_i, x_{i+1}) = 6^i\delta_D(a, x) \quad \text{and} \quad \delta_D(x_{t+1}, y) < 6\delta_D(x_t, x_{t+1}). \quad (3.21)
\]

It is possible that \( x_{t+1} = y \).

For this partition, first, we get a lower bound for \( t \) in terms of \( s \). From this partition, we have

\[
\sum_{i=2}^{t+1} \delta_D(x_{i-1}, x_i) = \frac{6}{3} \left( \delta_D(x_t, x_{t+1}) - \delta_D(a, x) \right),
\]

whence

\[
\delta_D(a, y) \leq \sum_{i=2}^{t+1} \delta_D(x_{i-1}, x_i) + \delta_D(x_{t+1}, y) < \frac{36}{3} \delta_D(x_t, x_{t+1}),
\]

and so

\[
\sum_{i=2}^{t+1} \delta_D(x_{i-1}, x_i) \geq \left( \frac{1}{6} - \frac{3}{10\mu^2_8} \right) \delta_D(a, y),
\]
since \( s = \frac{\delta_D(a,y)}{\delta_D(a,x)} > 4\mu^2 \). Thus we deduce from

\[
\sum_{i=2}^{t+1} \delta_D(x_{i-1}, x_i) = \frac{6}{5}(6^t - 1)\delta_D(a,x)
\]

that

\[
t > \log_6(4s) - 2. \tag{3.22}
\]

Then the following estimates regarding this partition are also useful. It follows from (3.21) that for each \( q < i \in \{1, \ldots, t\} \),

\[
delta_D(x_q, x_i) \leq \sum_{j=q}^{i-1} \delta_D(x_j, x_{j+1}) = \frac{6}{5}(6^j - 1)\delta_D(a,x) < \frac{1}{5}\delta_D(x_i, x_{i+1}), \tag{3.23}
\]

and thus for \( i \geq 3 \) and \( j \in \{1, \ldots, i-2\} \),

\[
\frac{6}{5}\delta_D(x_{i-1}, x_i) \geq \delta_D(x_{i-1}, x_i) + \delta_D(x_j, x_{i-1}) \\
\geq \delta_D(x_j, x_i) \\
\geq \delta_D(x_{i-1}, x_i) - \delta_D(x_j, x_{i-1}) \\
\geq \frac{4}{5}\delta_D(x_{i-1}, x_i). \tag{3.24}
\]

In order to continue the proof, we still need to pick up three points from \( \alpha[x_i, x_{i+1}] \) for each \( i \in \{2, \ldots, t\} \) as follows. Let \( y_i \) (resp. \( u_i \)) denote the first point in \( \alpha \) along the direction from \( x_i \) to \( x_{i+1} \) (resp. from \( y_i \) to \( x_{i+1} \)) such that (see Fig. 5)

\[
delta_D(x_i, y_i) = \frac{1}{\mu_5^2}\delta_D(x_{i-1}, x_i) \quad \text{(resp. } \delta_D(x_i, u_i) = \frac{6}{\mu_5^2}\delta_D(x_{i-1}, x_i)) \tag{3.25}
\]

and let \( w_i \) be the first point in \( \alpha \) along the direction from \( u_i \) to \( x_{i+1} \) such that (see Fig. 5)

\[
\delta_D(u_i, w_i) = \frac{1}{\mu_5^2}\delta_D(x_{i-1}, x_i). \tag{3.26}
\]

We see that

\[
\alpha[x_i, y_i] \cap \alpha[u_i, w_i] = \emptyset.
\]

This can be seen from the following estimate: For \( w \in \alpha[x_i, y_i] \) and \( z \in \alpha[u_i, w_i] \), the choice of \( y_i \) and \( w_i \) implies that

\[
\delta_D(w, z) \geq \delta_D(x_i, u_i) - \delta_D(u_i, z) - \delta_D(x_i, w) \geq \frac{4}{\mu_5^2}\delta_D(x_{i-1}, x_i). \tag{3.27}
\]
Thus the assumption “$D$ being $\varphi$-broad” together with (3.25) and (3.26) leads to

$$\text{Mod}(\alpha[x_i, y_i], \alpha[u_i, w_i]; D) \geq \varphi \left( \frac{\delta_D(\alpha[x_i, y_i], \alpha[u_i, w_i])}{t_i} \right) \geq \varphi(6),$$

(3.28)

where $t_i = \min\{\text{diam}(\alpha[x_i, y_i]), \text{diam}(\alpha[u_i, w_i])\}$.

Let $\Gamma_i$ denote the family of all curves connecting $\alpha[x_i, y_i]$ and $\alpha[u_i, w_i]$ in $D$. Then

$$\text{Mod}(\Gamma_i) = \text{Mod}(\alpha[x_i, y_i], \alpha[u_i, w_i]; D).$$

To decompose $\Gamma_i$, we construct a finite sequence of balls in $D$ as follows. For each $i \in \{2, \ldots, t\}$, we let

$$C_i = B_{\delta_D(x_i, 1)} \mu_9 \left( x_i, \frac{1}{\mu_9} \delta_D(x_{i-1}, x_i) \right),$$

where $\mu_9 = \exp(4\mu_8)$. Then (see Fig. 6)

$$\Gamma_i = \Gamma_{0i} \cup \Gamma_{1i} \cup \Gamma_{2i},$$

where

$$\Gamma_{0i} = \{ \gamma \in \Gamma_i : \gamma \subset B_i \cap C_i \},$$

$$\Gamma_{1i} = \{ \gamma \in \Gamma_i : \gamma \subset B_i - B_i \cap C_i \},$$

$$\Gamma_{2i} = \left\{ \gamma \in \Gamma_i : \gamma \cap S \left( x_i, \frac{\mu_8}{\mu_9^2} \delta_D(x_{i-1}, x_i) \right) \neq \emptyset \right\},$$

$$B_i = B \left( x_i, \frac{\mu_8}{\mu_9^2} \delta_D(x_{i-1}, x_i) \right).$$

Thus, we see that

**Claim 3.** $\Gamma_{1i} = \emptyset$ (see Fig. 6).

For a proof of this claim, it suffices to show that for each $\gamma$ in $\Gamma_i$, $\gamma \subset B_i$ implies $\gamma \subset C_i$. Let $z$ denote the endpoint of $\gamma$ in $\alpha[x_i, y_i]$. Then for $x \in \gamma$,

$$\delta_D(x_i, x) \leq \delta_D(x_i, z) + \delta_D(z, x).$$

(3.29)

Since the choice of $y_i$ implies

$$\delta_D(x_i, z) \leq \delta_D(x_i, y_i) = \frac{1}{\mu_9} \delta_D(x_{i-1}, x_i)$$

and the assumption $\gamma \subset B_i$ ensures

$$\delta_D(z, x) \leq \text{diam}(\gamma) \leq \frac{2\mu_8}{\mu_9} \delta_D(x_{i-1}, x_i),$$
we see from (3.29) that for each $x \in \gamma$,

$$
\delta_D(x_i, x) \leq \delta_D(x_i, y_i) + \text{diam}(\gamma) \leq \left(\frac{1 + 2\mu_8}{\mu_9^2}\right)\delta_D(x_{i-1}, x_i) < \frac{1}{\mu_9}\delta_D(x_{i-1}, x_i),
$$

which implies $\gamma \subset C_i$, and so Claim 3 is proved.

It follows from Claim 3 that

$$
\Gamma_i = \Gamma_{0i} \cup \Gamma_{2i}.
$$

Further, we need

Claim 4. For each $\gamma \in \Gamma_{2i}$, $\gamma \cap S(x_i, \frac{1}{\mu_9^2}\delta_D(x_{i-1}, x_i)) \neq \emptyset$.

Since the choice of $y_i$ gives

$$
\alpha[x_i, y_i] \setminus \{y_i\} \subset B_{\delta_D}(x_i, \frac{1}{\mu_9^2}\delta_D(x_{i-1}, x_i)),
$$

we know that

$$
\alpha[x_i, y_i] \setminus \{y_i\} \subset B(x_i, \frac{1}{\mu_9^2}\delta_D(x_{i-1}, x_i)).
$$

Hence the claim easily follows from the assumption $\gamma \in \Gamma_{2i}$.

Based on Claim 4, we infer from (3.28) and Theorem D that

$$
\varphi(6) \leq \text{Mod}(\Gamma_i) \leq \text{Mod}(\Gamma_{0i}) + \text{Mod}(\Gamma_{2i}) \leq \text{Mod}(\Gamma_{0i}) + \frac{1}{4}\varphi(6).
$$
Now, the quasiconformal invariance property of the conformal modulus of a curve family implies

\[ \text{Mod}(\Gamma_0') \geq \frac{1}{K} \text{Mod}(\Gamma_0) \geq \frac{3}{4K} \varphi(6). \]  

(3.30)

This will help us to get an upper bound for the conformal modulus of a subfamily of $\Gamma_0'$ below.

The following discussion still needs a lower bound on the inner distance between $\alpha'[x_i, y_i']$ and $\alpha'[u_i, w_i']$. We first prove some elementary inequalities. It follows from (3.24) and (3.25) that

\[ \delta_D(x_i, y_i) = \frac{1}{\mu_5} \delta_D(x_{i-1}, x_i) \geq \frac{5}{6\mu_5} \delta_D(x_1, x_i) \geq \frac{5}{\mu_5} \delta_D(a, x), \]

whence Lemma 2 implies

\[ \delta_D'(x'_1, x'_i) \leq \mu_7(\frac{6}{5\mu_5^2}) \delta_D'(x'_i, y'_i), \]  

(3.31)

and the assumption on the weak quasisymmetry of $f$ shows

\[ \delta_D'(a', x') \leq H \delta_D'(x'_1, x'_i). \]  

(3.32)

Now, we are ready to establish the lower bound. For all $w \in \alpha[x_i, y_i]$ and $z \in \alpha[u_i, w_i]$, since

\[ \delta_D(x_i, w) \leq \frac{1}{\mu_5} \delta_D(x_{i-1}, x_i) \]

and

\[ \delta_D(w, y_i) \leq \delta_D(x_i, w) + \delta_D(x_i, y_i) \leq 2 \delta_D(x_{i-1}, x_i), \]

we infer from (3.27) that

\[ \delta_D(w, z) > \max\{\delta_D(x_i, w), \delta_D(w, y_i)\}. \]

Then (3.31) and (3.32) show

\[ \delta_D'(w', z') \geq \frac{1}{H} \max\{\delta_D'(x'_i, w'), \delta_D'(w', y'_i)\} \geq \frac{1}{2H} \delta_D'(x'_i, y'_i) \]

\[ \geq \frac{1}{2H^2\mu_7(\frac{6}{5\mu_5^2}) \delta_D'(a', x'), \]  

(3.33)

which is the required lower bound.
In order to apply Theorems C and D in the proof, we decompose $\Gamma'_{0i}$ in the following way (see Fig. 7):

$$\Gamma'_{0i} = \Gamma'_{0i_1} \cup \Gamma'_{0i_2},$$

where

$$\Gamma'_{0i_1} = \{ \gamma' \in \Gamma'_{0i} : \gamma' \subset B(x'_i, \mu_8 \delta_{D'}(a', y')) \}$$

and

$$\Gamma'_{0i_2} = \{ \gamma' \in \Gamma'_{0i} : \gamma' \cap S(x'_i, \mu_8 \delta_{D'}(a', y')) \neq \emptyset \}.$$

Set

$$B'_{1i} = B'_i \cap C'_i \cap B(x'_i, \mu_8 \delta_{D'}(a', y')).$$

At present, we shall obtain a relationship between the curve family $\Gamma'_{0i_2}$ and the sphere $S(x'_i, H^2 \delta_{D'}(a', y'))$. Since (3.34) implies

$$\delta_D(a, y) \geq \delta_D(x_t, y) - \delta_D(x_1, x_t) \geq \frac{4}{5} \delta_D(x_t, x_{t+1}),$$

and so for each $i \in \{2, \ldots, t\}$ and $w \in \alpha[x_i, y_i]$, the choice of $y_i$, (3.23) and (3.24) guarantee

$$\delta_D(a, y) \geq \frac{4}{5} \delta_D(x_i, x_{i+1}) \geq 4 \delta_D(x_1, x_i) \geq \frac{16}{5} \delta_D(x_{i-1}, x_i) \geq \frac{16}{5} \mu_8^2 \delta_D(x_i, w).$$
Hence by the assumption on the weak quasisymmetry of \( f \), we have
\[
\delta_{D'}(a', y') \geq \frac{1}{H} \delta_{D'}(a', x'_i) \geq \frac{1}{H^2} \delta_{D'}(x'_i, w'),
\]
which implies that
\[
\alpha'[x'_i, y'_i] \subset \overline{B}(x'_i, H^2 \delta_{D'}(a', y')),
\]
and thus, we see that for each \( \gamma' \in \Gamma'_{0i_2} \) (see Fig. 7),
\[
\gamma' \cap S(x'_i, H^2 \delta_{D'}(a', y')) \neq \emptyset.
\]

Now, we are ready to apply Theorems C and D to get an increasing homeomorphism. It follows from (3.30) and Theorem D that
\[
\frac{3}{4K} \varphi(6) \leq \text{Mod}(\Gamma'_{0i_1}) \leq \text{Mod}(\Gamma'_{0i_1}) + \text{Mod}(\Gamma'_{0i_2}) \leq \text{Mod}(\Gamma'_{0i_1}) + \frac{1}{8K} \varphi(6),
\]
whence
\[
\text{Mod}(\Gamma'_{0i_1}) \geq \frac{5}{8K} \varphi(6),
\]
and so by (3.33) and Theorem C,
\[
\frac{5}{8K} \varphi(6) \leq \text{Mod}(\Gamma'_{0i_1}) \leq \left( \frac{2H^2 \mu_7 \left( \frac{6}{5} \mu_5^2 \right)}{\delta_{D'}(a', x')} \right)^n m(B'_{1i}),
\]
from which we get
\[
m(B'_{1i}) \geq \frac{5 \varphi(6)}{8K \left( 2H^2 \mu_7 \left( \frac{6}{5} \mu_5^2 \right) \right)^n (\delta_{D'}(a', x'))^n}. \tag{3.35}
\]

Since we see from (3.23) and (3.34) that for each \( i \in \{2, \ldots, t\} \),
\[
\delta_D(x_1, x_i) \leq \delta_D(x_1, x_{i-1}) + \delta_D(x_{i-1}, x_i) \leq \frac{6}{5} \delta_D(x_{i-1}, x_i) \leq \frac{1}{4} \delta_D(a, y),
\]
so again, the assumption on the weak quasisymmetry of \( f \) implies that for each \( w' \in \overline{B}'_{1i} \),
\[
\delta_{D'}(x'_1, w') \leq \delta_{D'}(x'_1, x'_i) + \delta_{D'}(x'_i, w') \leq (H + \mu_8) \delta_{D'}(a', y') < 2\mu_8 \delta_{D'}(a', y'),
\]
which assures the inclusion
\[
B'_{1i} \subset \overline{B}(x'_1, 2\mu_8 H \delta_{D'}(a', y')).
\]
The disjointness of the balls \( \{C_i\}_{i=2}^t \) is needed now, which is indicated in the following claim.

**Claim 5.** For any \( i \neq j \in \{2, \ldots, t\} \), \( C_i \cap C_j = \emptyset \).

For all \( i \neq j \in \{2, \ldots, t\} \), (3.23) and (3.24) guarantee that
\[
\delta_D(x_i, x_j) \geq \frac{4}{5} \max \{\delta_D(x_{i-1}, x_i), \delta_D(x_{j-1}, x_j)\} \geq \frac{2}{5} \mu_9 \max \{\delta_D(C_i), \delta_D(C_j)\},
\]
from which the claim follows.

We see from Claim 5 and (3.35) that
\[
\frac{5 \varphi(6)}{8K \left(2H^2 \mu_7 \left(\frac{6}{5} \mu_9^2\right)\right)^{t/2}} \cdot t \leq \sum_{i=1}^{t} m(B_{i}^1) \leq \omega_{n-1}(2\mu_8H \delta_{D'}(a', y'))^n.
\]
Hence (3.22) leads to
\[
\frac{\delta_{D'}(a', x')}{\delta_{D'}(a', y')} \leq 4H^2 \mu_8 \mu_7 \left(\frac{6}{5} \mu_9^2\right) \left(\frac{16K \omega_{n-1}}{5 \varphi(6) \log_6 \frac{\delta_D(a, y)}{\delta_D(a, x)}}\right)^{\frac{1}{t}}.
\]
In this case, we let
\[
\psi_2(p) = 4H^2 \mu_8 \mu_7 \left(\frac{6}{5} \mu_9^2\right) \left(\frac{16K \omega_{n-1}}{5 \varphi(6) \log_6 \frac{1}{p}}\right)^{\frac{1}{t}} \quad (3.36)
\]
for \( p \in (0, \frac{1}{4\mu_8^2}) \).

In conclusion, we see from (3.20) and (3.36) that the homeomorphism
\[
\psi(p) = \begin{cases} 
\frac{\mu_{10}}{(\log_6 \frac{1}{p})^{\frac{1}{t}}} & \text{if } p \in (0, \frac{1}{4\mu_8^2}), \\
\frac{4\mu_8^2 \mu_{10}}{(\log_6 4\mu_8^2)^{\frac{1}{t}}} p & \text{if } p \in [\frac{1}{4\mu_8^2}, 1]
\end{cases}
\]
is the desired one, where \( \mu_{10} = \max \left\{4H \mu_8^2, 4H^2 \mu_8 \mu_7 \left(\frac{6}{5} \mu_9^2\right) \left(\frac{16K \omega_{n-1}}{5 \varphi(6)}\right)^{\frac{1}{t}}\right\} \).

### 3.2. The proof of Theorem 1

To prove this theorem, by the definition, it suffices to show that there exists an increasing homeomorphism \( \eta : (0, \infty) \rightarrow (0, \infty) \) such that the inequality
\[
\frac{\delta_{D'}(x'_1, x'_2)}{\delta_{D'}(x'_1, x'_3)} \leq \eta \left(\frac{\delta_D(x_1, x_2)}{\delta_D(x_1, x_3)}\right)
\]
Fig. 8. The arcs \( \alpha_1', \alpha_2', E_1' \) and \( E_2' \) in \( D' \).

holds for all \( x_1, x_2 \) and \( x_3 \in A \) with \( x_1 \neq x_3 \). We divide the construction into the cases where \( \delta_D(x_1, x_2) \leq \delta_D(x_1, x_3) \) and where \( \delta_D(x_1, x_2) > \delta_D(x_1, x_3) \). In each case, we shall get a homeomorphism or homeomorphisms. Then we construct the desired homeomorphism \( \eta \) from the obtained ones.

First, we suppose that \( \delta_D(x_1, x_2) \leq \delta_D(x_1, x_3) \). Then Lemma 1 shows that

\[
\frac{\delta_D'(x_1', x_2')}{\delta_D'(x_1', x_3')} \leq \psi \left( \frac{\delta_D(x_1, x_2)}{\delta_D(x_1, x_3)} \right),
\]

where \( \psi \) is the increasing homeomorphism constructed in Lemma 1. In this case, we let

\[
\eta_1(t) = \psi(t) \quad (3.37)
\]

for \( t \) in \((0, 1]\).

Next, we consider the case \( \delta_D(x_1, x_2) > \delta_D(x_1, x_3) \). Again, we divide the discussion into two cases which are as follows.

**Case 1.** \( \delta_D'(x_1', x_2') \leq H_1 \delta_D'(x_1', x_3') \), where \( H_1 = 36H^2 \).

Obviously,

\[
\frac{\delta_D'(x_1', x_2')}{\delta_D'(x_1', x_3')} \leq H_1 \frac{\delta_D(x_1, x_2)}{\delta_D(x_1, x_3)}. 
\]

In this case, we define

\[
\eta_2(t) = H_1 \psi(1)t 
\quad (3.38)
\]

for \( t > 1 \).

**Case 2.** \( \delta_D'(x_1', x_2') > H_1 \delta_D'(x_1', x_3') \).
In this case, we shall exploit the quasiconformal invariance property of the conformal modulus of a curve family to obtain a homeomorphism. For this, we need some preparation. Let \( \alpha_1' \) be an arc joining \( x_1' \) and \( x_2' \) in \( A' \) (see Fig. 8), and let \( x_4' \) be the first point in \( \alpha_1' \) along the direction from \( x_2' \) to \( x_1' \) such that (see Fig. 8)

\[
\delta_{D'}(x_4', x_2') = \frac{1}{8} \delta_{D'}(x_1', x_2').
\]

(3.39)

Then we get

\[
\delta_{D'}(x_4', x_1') \leq \delta_{D'}(x_1', x_2') + \delta_{D'}(x_4', x_2') = \frac{9}{8} \delta_{D'}(x_1', x_2').
\]

(3.40)

In what follows, two claims stated as below are needed.

Claim 6. \( \delta_D(x_4, x_2) \geq H_2 \delta_D(x_1, x_2) \) with \( H_2 = \frac{\psi^{-1}(1/9)}{1 + \psi^{-1}(1/9)} \).

Now, we prove this claim. If \( \delta_D(x_4, x_2) \geq \delta_D(x_4, x_1) \), then

\[
\delta_D(x_4, x_2) \geq \frac{1}{2} \delta_D(x_1, x_2),
\]

since \( \delta_D(x_4, x_2) + \delta_D(x_4, x_1) \geq \delta_D(x_1, x_2) \). The claim is true since \( H_2 < 1/2 \).

On the other hand, if \( \delta_D(x_4, x_2) < \delta_D(x_4, x_1) \), then by Lemma 1 together with (3.39) and (3.40), we have

\[
\frac{1}{9} \leq \frac{\delta_{D'}(x_4', x_2')}{\delta_{D'}(x_4', x_1')} \leq \psi \left( \frac{\delta_D(x_4, x_2)}{\delta_D(x_4, x_1)} \right),
\]

whence

\[
\delta_D(x_4, x_2) \geq \frac{\psi^{-1}(1/9)}{1 + \psi^{-1}(1/9)} \delta_D(x_1, x_2).
\]

The proof of Claim 6 is complete.

Let \( \alpha_2' \) be an arc joining \( x_1' \) and \( x_3' \) in \( A' \) (see Fig. 8). Then we have

Claim 7. There exists a point \( y_3' \) in \( \alpha_2' \) which satisfies

1. \( \delta_{D'}(x_1', x_3') \leq \delta_{D'}(x_1', y_3') \leq 2H \delta_{D'}(x_1', x_3'); \)
2. \( \delta_D(x_1, x_3) \leq \delta_D(x_1, y_3); \) and
3. \( \alpha_2'[x_1', y_3'] \subset \overline{M}(x_1', 2H \delta_{D'}(x_1', x_3')). \)

In order to establish the existence of \( y_3' \), we separate the discussions into two parts: \( \delta_{D'}(\alpha_2') \leq 2H \delta_{D'}(x_1', x_3') \) and \( \delta_{D'}(\alpha_2') > 2H \delta_{D'}(x_1', x_3') \). For the first part, we let \( y_3' = x_3' \). Obviously, \( y_3' \) satisfies all requirements in Claim 7.
For the remaining part, that is, $\delta_{D'}(\alpha'_2) > 2H\delta_{D'}(x'_1, x'_3)$, we see that if

$$\alpha'_2[x'_1, x'_3] \cap \mathcal{S}(x'_1, 2H\delta_{D'}(x'_1, x'_3)) \neq \emptyset,$$

we take $y'_3$ to be the first point in $\alpha'_2$ along the direction from $x'_1$ to $x'_3$ such that

$$\delta_{D'}(x'_1, y'_3) = 2H\delta_{D'}(x'_1, x'_3),$$

and if $\alpha'_2[x'_1, x'_3] \subset \mathbb{B}(x'_1, 2H\delta_{D'}(x'_1, x'_3))$, then let $y'_3 \in \alpha'_2$ be such that

$$\delta_{D'}(x'_1, y'_3) > H\delta_{D'}(x'_1, x'_3).$$

Necessarily, we see that $\delta_{D}(x_1, x_3) \leq \delta_{D}(x_1, y_3)$. Also, the chosen point $y'_3$ satisfies all requirements in Claim 7. Hence the claim is true.

Let us continue the proof of this theorem. Let $E'_1 = \alpha'_1[x'_2, x'_4]$ and $E'_2 = \alpha'_2[x'_1, y'_3]$ (see Fig. 8). We need lower bounds for the quantity $\min\{\text{diam}(E_1), \text{diam}(E_2)\}$ and for the length of every arc connecting $E'_1$ and $E'_2$ in $D'$, respectively. For this, it follows from Claim 6 that

$$\text{diam}(E_1) \geq \delta_{D}(x_4, x_2) \geq H_2\delta_{D}(x_1, x_2)$$

and from Claim 7 that

$$\text{diam}(E_2) \geq \delta_{D}(x_1, y_3) \geq \delta_{D}(x_1, x_3),$$

whence the assumption “$\delta_{D}(x_1, x_2) > \delta_{D}(x_1, x_3)$” implies

$$\min\{\text{diam}(E_1), \text{diam}(E_2)\} \geq \min\{H_2\delta_{D}(x_1, x_2), \delta_{D}(x_1, x_3)\} \geq H_2\delta_{D}(x_1, x_3). \quad (3.41)$$

Since Claim 7(3) implies $\delta_{D'}(\alpha'_2[x'_1, y'_3]) \leq 4H\delta_{D'}(x'_1, x'_3)$, it follows from the choice of $x'_4$ and the assumption “$\delta_{D'}(x'_1, x'_2) > H_1\delta_{D'}(x'_1, x'_3)$” that for each $u' \in E'_1$ and $v' \in E'_2$, we have

$$\delta_{D'}(u', v') \geq \delta_{D'}(x'_1, x'_2) - \delta_{D'}(x'_2, u') - \delta_{D'}(x'_1, v') \geq \left(\frac{7}{8} - \frac{4H}{H_1}\right)\delta_{D'}(x'_1, x'_2). \quad (3.42)$$

The needed lower bounds have been obtained. In order to apply the quasiconformal invariance property of the conformal modulus of a curve family together with Theorems C and D to get a homeomorphism, we still need to construct a family of curves in $D'$. Let $\Gamma'$ denote the curves joining $E'_1$ and $E'_2$ in $D'$. Then (see Fig. 9)

$$\Gamma' = \Gamma'_1 \cup \Gamma'_2,$$
Fig. 9. The arcs $\gamma' \in \Gamma_1'$ and $\gamma'_2 \in \Gamma_2'$ in $D'$.

where

$$\Gamma_1' = \{ \gamma' \in \Gamma' : \gamma' \subset B(x_1', (\delta_D' (x_1', x_2') \delta_D' (x_1', y_3'))^{\frac{1}{2}}) \}$$

and

$$\Gamma_2' = \{ \gamma' \in \Gamma' : \gamma' \cap S(x_1', (\delta_D' (x_1', x_2') \delta_D' (x_1', y_3'))^{\frac{1}{2}}) \neq \emptyset \}.$$

Hence we infer from (3.42), Theorems C and D that

$$
\text{Mod}(E_1', E_2'; D') = \text{Mod}(\Gamma') \leq \text{Mod}(\Gamma_1') + \text{Mod}(\Gamma_2') \\
\leq \omega_{n-1} \left( \frac{\omega_{n-1} (\delta_D' (x_1', x_2') \delta_D' (x_1', y_3'))^{\frac{1}{2}}}{((7/8 - 4H/H_1) \delta_D' (x_1', x_2'))^n} \right) + \omega_{n-1} \left( \frac{1}{2} \log \frac{\delta_D' (x_1', x_2')}{\delta_D' (x_1', y_3')} \right)^{1-n} \\
\leq 2^n \omega_{n-1} \left( \log \frac{\delta_D' (x_1', x_2')}{H^2 \delta_D' (x_1', x_3')} \right)^{1-n}, \quad (3.43)
$$

since the statement (3) in Claim 7 shows that for each $\gamma' \in \Gamma_2'$, $\gamma' \cap S(x_1', 2H \delta_D' (x_1', x_3')) \neq \emptyset$. Now, the quasiconformal invariance property of the conformal modulus of a curve family together with (3.41) guarantees

$$\text{Mod}(\Gamma') \geq \frac{1}{K} \text{Mod}(\Gamma) \geq \frac{1}{K} \varphi \left( \frac{\delta_D (E_1, E_2)}{\min \{ \text{diam} (E_1), \text{diam} (E_2) \} } \right) \geq \frac{1}{K} \varphi \left( \frac{\delta_D (x_1, x_2)}{H^2 \delta_D (x_1, x_3)} \right),$$

which, combining with (3.43), shows that

$$\frac{\delta_D' (x_1', x_2')}{\delta_D' (x_1', x_3')} \leq H^2 \exp \left( 2 \left( \frac{2K \omega_{n-1}}{\tau} \right)^{\frac{1}{n-1}} \right),$$

where

$$\tau = \varphi \left( \frac{\delta_D (x_1, x_2)}{H^2 \delta_D (x_1, x_3)} \right).$$
We let
\[ \eta_3(t) = H^2 \exp \left( 2 \left( \frac{2K\omega_{n-1}}{\varphi(t/H_2)} \right)^{\frac{1}{n-1}} \right) \]  
for \( t > 1 \).

Now, we are ready to conclude the existence of the homeomorphism \( \eta \). We see from (3.37), (3.38) and (3.44) that the homeomorphism
\[ \eta(t) = \begin{cases} H_1 \mu_{11} \eta_1(t) & \text{if } t \in (0, 1], \\ \max\{\mu_{11} \eta_2(t), 36 \psi(1) \eta_3(t)\} & \text{if } t \in (1, \infty), \end{cases} \]
is the desired one, where
\[ \mu_{11} = \exp \left( 2 \left( \frac{2K\omega_{n-1}}{\varphi(1/H_2)} \right)^{\frac{1}{n-1}} \right). \]
The proof of Theorem 1 is complete. \( \square \)

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