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Fusion-Riesz frame in Hilbert space

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Abstract. Fusion-Riesz frame (Riesz frame of subspace) whose all subsets are fusion frame sequences with the same bounds is a special fusion frame. It is also considered a generalization of Riesz frame since it shares some important properties of Riesz frame. In this paper, we show a part of these properties of fusion-Riesz frame and the new results about the stabilities of fusion-Riesz frames under operator perturbation (simple named operator perturbation of fusion-Riesz frames). Moreover, we also compare the operator perturbation of fusion-Riesz frame with that of fusion frame, fusion-Riesz basis (also called Riesz decomposition or Riesz fusion basis) and exact fusion frame.

§1 Introduction

Frame is very useful in filter bank theory [6], signal and image processing [9] etc. Many remarkable achievements have been obtained in frame theory and its applications, refer to [15, 17, 18]. However, along with the emergence of new applications and development of frame theory, one single frame system is insufficient to meet the actual requirements. This drove the scholars to put forward further some generalized frames. Specially, fusion frame is one of these generalized frames.

Fusion frame (frame of subspace) was first considered by Casazza and Kutyniok for the purpose of the construction of frames in [12]. And then, Casazza, Kutyniok and Li showed the applications of fusion frame in [14], the fusion frame accommodate well the situation of distributed processing and parallel processing of large frame systems. Right now, there are many remarkable results about fusion frame theorem [1, 2, 5, 8, 11, 12, 14, 19, 32] and its applications [7, 10, 14].

It is known that the frame and fusion frame can be used to process signals. In signal processing, a signal will be transformed into frame coefficients firstly and the frame coefficients

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will be processed, lastly, the original signal will be reconstructed with the processed frame coefficients. However, while we process frame coefficients, the frame coefficients are probably missed for some reasons (such as the data transmission may lead to missing). In this situation, the robustness and redundancy of the frames and fusion frames are significant for the perfect reconstruction of signals, see [20,22,23] for more details. Now, we consider processing the signals by the frames and fusion frames with robustness, assume that part of frame coefficients would be lost, a fact is that the remaining frame coefficients is also probably excess to reconstructing efficiently the perfect signals. So we hope we can take part of the remaining frame coefficients as little as possible to reconstruct the perfect signals for the purpose of reducing the calculated amount of reconstruction. Specially, we hope the remaining frame coefficients contain an exact frame coefficient subset (non-redundancy), this requires that the frame and fusion frame have the corresponding proposition. It is known that Riesz frame contains an exact frame (non-redundancy) and its subset is either a Riesz frame or not a frame. In fact, the fusion-Riesz frame has also the same proposition. In view of this, it is one good choice to process the signals by Riesz frame or fusion-Riesz frame.

The main part of this paper shows some properties of fusion-Riesz frame (see Property 3.1-Property 3.3) and discusses the operator perturbation of fusion-Riesz frames:

- Let $T \in L(H)$ be a surjection and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion-Riesz frame for H. Then under what conditions the family $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H?

We can think about it in this way:

- Find a surjection $T \in L(H)$ such that for any $I \subseteq Z^+$, if $\{(W_j, v_j)\}_{j\in I}$ is a fusion frame for $\overline{span}\{W_j\}_{j\in I}$ with bounds A and B, then the family $\{(\overline{TW_j}, v_j)\}_{j\in I}$ is also a fusion frame for $\overline{span}\{\overline{TW_j}\}_{j\in I}$ with bounds C and D.

In order to do with the problem above, we study the existing results (i.e., Corollary 3.4 and Corollary 3.5 in this paper) [25, 31] and the operator perturbation of fusion frames which was considered recently by Casazza, Kutyniok, Li, Asgari and Gavryta etc [2,3,12,14,19,24,28,30]:

- Find a surjection $T \in L(H)$ such that: if $\{(W_j, v_j)\}_{j\in Z^+}$ is a fusion frame for H then the family $\{(\overline{TW_j}, v_j)\}_{j\in Z^+}$ is a fusion frame for H.

Compare the above two questions, they are quite similar. In fact, they have also many similar results. For example, it can be proved that: Let $T \in L(H)$ be a surjection and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion-Riesz frame (fusion frame) for H. If $T^+TW_j \subset W_j, j \in Z^+$, $T^*TW_j \subset W_j, j \in Z^+$ or T is an invertible operator, then $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame (fusion frame) for H. However, the operator perturbation of fusion frames and operator perturbation of fusion-Riesz frames are not equal, Example 3.2 shows they are distinct.

In Section 3, we prove some propositions of fusion-Riesz frame (see Propositions 3.1-3.3), show the distinction between operator perturbation of fusion frames and operator perturbation of fusion-Riesz frames by Example 3.2, and obtain some further results about operator perturbation of fusion-Riesz frames (see Theorem 3.1 − Corollary 3.3 and Corollary 3.6). We also show the relation $T^*TW_j \subset W_j \Rightarrow T^+TW_j \subset W_j$ which is helpful to the operator perturbation of fusion-Riesz frames and fusion frame.

§2 Preliminaries

Before giving our results, we need the following definitions and lemmas.

Definition 2.1. [12] Let $\{W_j\}_{j=1}^{\infty}$ be a family of closed subspaces of H, $\{v_j\}_{j=1}^{\infty}$ be a family of positive weights, i.e., $v_j > 0$ for $j = 1, 2, \cdots$. If there exist constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j}(f)\|^2 \leq B\|f\|^2$$

for all $f \in H$, then we call $\{(W_j, v_j)\}_{j=1}^{\infty}$ a fusion frame for H with bounds A and B. Further, we call the fusion frame an exact fusion frame, if it ceases to be a fusion frame once one element is deleted. We call $\{(W_j, v_j)\}_{j\in I\subseteq Z^+}$ a fusion frame sequence for H with bounds A and B if $\{(W_j, v_j)\}_{j\in I}$ is a fusion frame for $\overline{span}\{W_j\}_{j\in I}$ with bounds A and B.

Reference [29] points out that the fusion frame can be regarded as a special case of g-frames. So we consider fusion-Riesz basis as a special case of g-Riesz basis (see Definition 3.1 in [29]).

Definition 2.2. We call $\{(W_j, v_j)\}_{j=1}^{\infty}$ a fusion-Riesz basis for H with bounds A and B, if $\{W_j\}_{j=1}^{\infty}$ is complete (i.e., $\overline{span}\{W_j\}_{j=1}^{\infty} = H$) and for any finite $I \subset Z^+$, there exist constants $0 < A \leq B < +\infty$ such that

$$A\sum_{i\in I} \|f_i\|^2 \leq \left\|\sum_{i\in I} v_i f_i\right\|^2 \leq B\sum_{i\in I} \|f_i\|^2, \{f_i\}_{i\in I} \in l^2\Big(\{W_i\}_{i\in I}\Big), \tag{1}$$

where

$$l^2\Big(\{W_i\}_{i\in I}\Big) = \left\{\{f_i \in W_i\}_{i\in I} \;\middle|\; \sum_{i\in I} \|f_i\|^2 \text{ convergence}\right\}.$$

Remark 2.1. Regardless of the sizes of (positive) bounds, the fusion-Riesz basis is equal to the Riesz decomposition (see Definition 4.4, Lemma 4.5 and Theorem 4.6 in [12]) and Riesz fusion basis (see Definition 3.1 and Theorem 3.6 in [4]). Especially, while $W_i = span\{g_i\}$(i.e., $|W_i| = 1$) and $v_i = \|g_i\|$ for all $i \in Z^+$, we have that $\{(\overline{span}\{g_i\}, \|g_i\|)\}_{i=1}^{\infty}$ is a fusion-Riesz basis for H with bounds A and B if and only if $\{g_i\}_{i=1}^{\infty}$ is a Riesz basis for H with bounds A and B (see Proposition 3.1).

Definition 2.3. [12] We call $\{(W_j, v_j)\}_{j=1}^{\infty}$ a fusion-Riesz frame (Riesz frame of subspace) for H with bounds A and B, if there exist constants $0 < A \leq B < +\infty$ such that

$$A\|f\|^2 \leq \sum_{j\in I} v_j^2 \|\pi_{W_j}(f)\|^2 \leq B\|f\|^2$$

for all $I \subseteq Z^+$ and $f \in \overline{span}\{W_j\}_{j\in I}$.

Lemma 2.1. *([15, 24]) Let H, K be two Hilbert spaces, and $U \in L(H, K)$ be a closed range operator. Then there exists a unique operator $U^+ \in L(K, H)$ such that*

$$N_{U^+} = U(H)^\perp, N_U^\perp = U^+(K), UU^+(f) = f, f \in U(H).$$

We call U^+ the pseudo-inverse of U.

Lemma 2.2. *([2, 15, 21, 24, 27]) Let H, K be two Hilbert spaces, $T \in L(H, K)$ be a closed range operator. Then*

(1) $TT^+ = \pi_{T(H)}$, $T^+T = \pi_{T^*(K)}$.

(2) $||T^*f|| \geq \frac{||f||}{||T^+||}$, $f \in T(H)$.

Lemma 2.3. *([19]) Let H, K be two Hilbert spaces and $T \in L(H, K)$, and let W be a closed subspace of H. Then $\pi_W T^* \pi_{\overline{TW}} = \pi_W T^*$.*

Lemma 2.4. *Let λ be a positive number, $T \in L(H)$ be a surjection and $\{(W_j, v_j)\}_{j=1}^\infty$ be a fusion-Riesz frame for H. Then the following two propositions are equivalent:*

(1) $\sum\limits_{j\in I} v_j^2 \|\pi_{\overline{TW_j}} f\|^2 \leq \lambda \|f\|^2$ *for all* $f \in \overline{span}\{\overline{TW_j}\}_{j\in I}$ *and* $I \subseteq Z^+$.

(2) $\sum\limits_{j=1}^{\infty} v_j^2 \|\pi_{\overline{TW_j}} f\|^2 \leq \lambda \|f\|^2$ *for all* $f \in H$.

Proof. From T is a surjection and $\{(W_j, v_j)\}_{j=1}^\infty$ is a fusion-Riesz frame, we have $H = \overline{span}\{\overline{TW_j}\}_{j=1}^\infty$. Moreover, $\sum\limits_{j\in I} v_j^2 \|\pi_{\overline{TW_j}} f\|^2 \leq \sum\limits_{j=1}^{\infty} v_j^2 \|\pi_{\overline{TW_j}} f\|^2$ for all $f \in \overline{span}\{\overline{TW_j}\}_{j\in I}$ and $I \subseteq Z^+$. So it is obvious that Lemma 2.4 holds. □

Definition 2.4. ([21, 28]) Suppose that H, K are two Hilbert spaces, and $T : H \to K$ is a bounded linear operator. Let

$$\gamma(T) = \inf\{\|Tx\| : x \in N_T^\perp, \|x\| = 1\}.$$

Then we call $\gamma(T)$ the minimum modulus of operator T.

§3 The main results

Fusion-Riesz frame shares many of properties of Riesz frame. For example, from the definitions, every subset of Riesz frame is a frame sequence, every subset of fusion-Riesz frame is a fusion frame sequence; Riesz basis is a Riesz frame, fusion-Riesz basis is a fusion-Riesz frame; Riesz frame contains an exact frame [16], fusion-Riesz frame contains an exact fusion frame, see Proposition 3.2; Riesz frame can be written as a finite union of exact frame sequences (by Theorem 4.2 in [13]), fusion-Riesz frame can be written as a finite union of exact fusion frame sequences, see Proposition 3.3. However, there are also some properties of Riesz frame can not be shared by fusion-Riesz frame. For example, any Riesz frame contains a Riesz basis [16], but some fusion-Riesz frames do not contain any fusion-Riesz basis, see Example 3.1.

The following Proposition 3.1 shows that Riesz frame is a special fusion-Riesz frame.

Proposition 3.1. *$\{(\overline{span}\{f_j\}, \|f_j\|)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H with bounds A and B if and only if $\{f_j\}_{j=1}^{\infty}$ is a Riesz frame for H with bounds A and B.*

Proof. Let $W_j = \overline{span}\{f_j\}$ for all $j \in Z^+$. Then for all $I \subseteq Z^+$,

$$\sum_{j\in I} \|f_j\|^2 \|\pi_{W_j}(f)\|^2 = \sum_{j\in I} \|f_j\|^2 \|\langle f, \frac{f_j}{\|f_j\|}\rangle \frac{f_j}{\|f_j\|}\|^2 = \sum_{j\in I} |\langle f, f_j\rangle|^2, f \in H.$$

So $\{(\overline{span}\{f_j\}, \|f_j\|)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H with bounds A and B if and only if $\{f_j\}_{j=1}^{\infty}$ is a Riesz frame for H with bounds A and B. □

Proposition 3.2. *Every fusion-Riesz frame contains an exact fusion frame.*

Proof. Reference [26] shows that "g-Riesz frame contains an exact g-frame". So we have that Proposition 3.2 holds. Below is a brief proof of Proposition 3.2.

Let $\{(W_i, v_i)\}_{i=1}^{\infty}$ be a fusion-Riesz frame for H with bounds A and B. Define the nonempty poset

$$\Omega = \left\{ I \subseteq Z^+ \middle| A\|f\|^2 \le \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \le B\|f\|^2, f \in H \right\} \ni Z^+$$

with order "$\prec := \supseteq$". For any totally ordered family of subsets $\{I_k \subseteq \Omega\}_{k\in J\subseteq Z^+}$, by lemma 3.1 in [16], $\bigcap_{k\in J} I_k \in \Omega$, i.e., $\{I_k\}_{k\in J}$ has upper bound $\bigcap_{k\in J} I_k \in \Omega$. By Zorn lemma, Ω has a maximum I. This shows that: fusion frame $\{(W_i, v_i)\}_{i\in I}$ ceases to be a fusion frame for H once one element is deleted, i.e., $\{(W_i, v_i)\}_{i\in I}$ is exact fusion frame for H. □

Proposition 3.3. *Any fusion-Riesz frame can be written as a finite union of exact fusion frame sequences.*

Proof. Let $\{(W_i, v_i)\}_{i=1}^{\infty}$ be a fusion-Riesz frame for H with bounds A and B. Then for any $\emptyset \ne I \subseteq Z^+$, $\{(W_i, v_i)\}_{i\in I}$ is a fusion-Riesz frame for $\overline{span}\{W_i\}_{i\in I}$ with bounds A and B. By Proposition 3.2, there exist subsets $I_k \subseteq Z^+, k = 1, 2, \cdots$ such that

(1) All non-empty subsets are disjoint to one another, i.e., $I_k \bigcap I_m = \emptyset$ for $k, m \in Z^+$.

(2) For all $I_k \ne \emptyset$, $\{(W_i, v_i)\}_{i\in I_k}$ is an exact fusion frame for $\overline{span}\{W_i\}_{i\in I_k}$ with bounds A and B.

(3) $\overline{span}\{W_i\}_{i\in I_1} = H$, $\overline{span}\{W_i\}_{i\in I_k} = \overline{span}\{W_i\}_{i\in Z^+ - \bigcup_{m=1}^{k-1} I_m}$, where $k \ge 2$ and $\overline{span}\{W_i\}_{i\in\emptyset} = \emptyset$.

Then

(a) $H = \overline{span}\{W_i\}_{i\in I_1} \supseteq \overline{span}\{W_i\}_{i\in I_2} \supseteq \cdots$.

(b) If $I_k = \emptyset$, then $I_{k+m} = \emptyset$ for $m = 1, 2, \cdots$.

Take $I_k \neq \emptyset, 1 \leq k \leq n$, computing

$$nA\|f\|^2 \leq \sum_{k=1}^{n} \sum_{i \in I_k} v_i^2 \|\pi_{W_i}(f)\|^2 \leq B\|f\|^2, f \in \overline{span}\{W_i\}_{i \in I_n},$$

so $n \leq \frac{B}{A}$. This shows that there has only finite (n') non-empty subsets $I_k, k \leq n' < +\infty$. Hence,

$$\overline{span}\{W_i\}_{i \in Z^+ - \bigcup_{m=1}^{n'} I_m} = \overline{span}\{W_i\}_{i \in I_{n'+1}} = \emptyset,$$

i.e., $Z^+ = \bigcup_{m=1}^{n'} I_m$. It follows that the fusion-Riesz frame $\{(W_i, v_i)\}_{i=1}^{\infty}$ can be written as a finite union of exact fusion frame sequences. □

Example 3.1 shows an fusion-Riesz frame which does not contain any fusion-Riesz basis.

Example 3.1. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonornal basis for H, $W_1 = span\{e_1, e_2\}$, $W_2 = span\{e_2, e_3\}$ and $W_k = span\{e_{k+1}\}$ for all $k \geq 3$. Then $\{(W_j, 1)\}_{j=1}^{\infty}$ does not contain a fusion-Riesz basis.

Proof. For any $f \in \overline{span}\{W_j\}_{j \in I}$ and $I \subseteq Z^+$, $\|f\|^2 \leq \sum_{j \in I} \|\pi_{W_j} f\|^2 \leq 2\|f\|^2$. By Definition 2.3, $\{(W_j, 1)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H.

Assume $\{(W_j, 1)\}_{j=1}^{\infty}$ contains a fusion-Riesz basis $\{(W_j, 1)\}_{j \in I}$. Then $\overline{span}\{W_j\}_{j \in I} = H$. This implies that $I = Z^+$. So $\{(W_j, 1)\}_{j=1}^{\infty}$ must be a fusion-Riesz basis. Take $e_2 \in W_1$ and $-e_2 \in W_2$, then $\|e_2 + (-e_2)\|^2 = 0 \cdot (\|e_2\|^2 + \|-e_2\|^2)$. By Definition 2.2, $\{(W_j, 1)\}_{j=1}^{\infty}$ is not a fusion-Riesz basis, i.e., assumption is invalid.

Next, we consider the operator perturbation of fusion-Riesz frames. At first, we want to know whether the operator perturbation of fusion-Riesz frame is equal to the operator perturbation of fusion frame. If they are equal, then it is not necessary to study the operator perturbation of fusion-Riesz frame specially. But the following example shows that they are distinct.

Example 3.2. Let $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H and $W_k = span\{e_k\}$ for each $k \in Z^+$. Define a surjection $T \in L(H)$ as follows:

$$Tf = \sum_{k=1}^{\infty} \langle f, e_{2k-1} \rangle e_k + \sum_{k=1}^{\infty} \langle f, e_{4k-2} \rangle e_{4k-2} + \sum_{k=1}^{\infty} \frac{\langle f, e_{4k} \rangle k}{\sqrt{1+k^2}} e_{4k-2} + \sum_{k=1}^{\infty} \frac{\langle f, e_{4k} \rangle}{\sqrt{1+k^2}} e_{4k}, f \in H.$$

Then we can show that

(1) $\{(\overline{TW_j}, 1)\}_{j=1}^{\infty}$ is a fusion frame for H.

(2) $\{(\overline{TW_j}, 1)\}_{j=1}^{\infty}$ is not a fusion-Riesz frame for H.

Proof. In fact, $f = T\left(\sum_{k=1}^{\infty} \langle f, e_k \rangle e_{2k-1}\right)$ for all $f \in H$, i.e., T is a surjection. We compute

$$TW_{4k} = span\left\{\frac{k}{\sqrt{1+k^2}} e_{4k-2} + \frac{1}{\sqrt{1+k^2}} e_{4k}\right\}, TW_{4k-2} = W_{4k-2}, TW_{2k-1} = W_k, k \in Z^+.$$

It is easy to obtain that

$$\|f\|^2 = \sum_{k=1}^{\infty} \|\pi_{W_k} f\|^2 \leq \sum_{k=1}^{\infty} \|\pi_{TW_k} f\|^2 \leq 3\|f\|^2, f \in H.$$

Hence $\{(\overline{TW_j},1)\}_{j=1}^{\infty}$ is a fusion frame for H with bounds 1 and 3. By Lemma 2.4,

$$\sum_{k\in I}\|\pi_{TW_k}f\|^2\le 3\|f\|^2, f\in \overline{span}\{TW_k\}_{k\in I} \textit{ and } I\subset Z^+.$$

But,

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\|\pi_{TW_{2k}}e_{4n}\|^2=\lim_{n\to\infty}\frac{1}{1+n^2}=0,$$

where $e_{4n}\in\overline{span}\{TW_{2k}\}_{k=1}^{\infty}$. Hence $\{(\overline{TW_j},1)\}_{j=1}^{\infty}$ is not a fusion-Riesz frame for H. □

Now, we give a sufficient condition for the operator perturbation of fusion-Riesz frames.

Theorem 3.1. *Let $T\in L(H)$ be a closed range operator and $\{(W_j,v_j)\}_{j=1}^{\infty}$ be a fusion-Riesz frame for H with bounds A and B. If there exist two positive numbers $\alpha\le\beta$ and a family*

$$\{T_I\}_{I\subseteq Z^+} = \left\{ T_I\in L(H) \;\middle|\; N_{T_I}^{\perp}\subseteq \mathcal{W}_I, \overline{T_I(H)}\supseteq \overline{span}\{\overline{TW_j}\}_{j\in I} \right. \left. \textit{and } \; 0<\inf_{I\subseteq Z^+}\gamma(T_I)\le\sup_{I\subseteq Z^+}\|T_I\|<\infty \right\}$$

such that

$$\alpha\sum_{j\in I}v_j^2\|\pi_{W_j}T_I^*(f)\|^2\le\sum_{j\in I}v_j^2\|\pi_{\overline{TW_j}}(f)\|^2\le\beta\sum_{j\in I}v_j^2\|\pi_{W_j}T_I^*(f)\|^2, f\in\overline{span}\{\overline{TW_j}\}_{j\in I}$$

or

$$\alpha\sum_{j\in I}v_j^2\|\pi_{W_j}(f)\|^2\le\sum_{j\in I}v_j^2\|\pi_{\overline{TW_j}}T_I(f)\|^2\le\beta\sum_{j\in I}v_j^2\|\pi_{W_j}(f)\|^2, f\in\mathcal{W}_I=\overline{span}\{W_j\}_{j\in I}$$

for $I\subseteq Z^+$. Then $\{(\overline{TW_j},v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for $T(H)$.

Proof. Let $C=\inf_{I\subseteq Z^+}\gamma(T_I)$ and $D=\sup_{I\subseteq Z^+}\|T_I\|$. We computing

$$\begin{aligned}\sum_{j\in I}v_j^2\|\pi_{\overline{TW_j}}(f)\|^2 &\le \beta\sum_{j\in I}v_j^2\|\pi_{W_j}T_I^*(f)\|^2\le\beta B\|T_I^*(f)\|^2\le\beta B\|T_I\|^2\|f\|^2\le\beta BD^2\|f\|^2,\\ \sum_{j\in I}v_j^2\|\pi_{\overline{TW_j}}(f)\|^2 &\ge \alpha\sum_{j\in I}v_j^2\|\pi_{W_j}T_I^*(f)\|^2\ge\alpha A\|T_I^*(f)\|^2\ge\alpha A\gamma(T_I)^2\|f\|^2\ge\alpha AC^2\|f\|^2\end{aligned}$$

for all $f\in\overline{span}\{\overline{TW_j}\}_{j\in I}$. Noticing that $\overline{span}\{\overline{TW_j}\}_{j\in Z^+}=T(H)$ if $\{(W_j,v_j)\}_{j=1}^{\infty}$ is a fusion frame for H and T is a closed range operator. Hence $\{(\overline{TW_j},v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for $T(H)$ with bounds αAC^2 and βBD^2 by Definition 2.3.

On the other hand, by computing

$$\begin{aligned}\frac{\alpha A}{\|T_I\|^2}\|T_I(f)\|^2\le\alpha A\|f\|^2 &\le \alpha\sum_{j\in I}v_j^2\|\pi_{W_j}(f)\|^2\le\sum_{j\in I}v_j^2\|\pi_{\overline{TW_j}}T_I(f)\|^2\\ &\le \beta\sum_{j\in I}v_j^2\|\pi_{W_j}(f)\|^2\le\beta B\|f\|^2\le\frac{\beta B}{\gamma(T_I)^2}\|T_I(f)\|^2,\ f\in\mathcal{W}_I.\end{aligned}$$

Then for all $g\in\overline{span}\{\overline{TW_j}\}_{j\in I}\subseteq\overline{T_I(H)}=T_I(N_{T_I}^{\perp})=T_I(\mathcal{W}_I)$,

$$\frac{\alpha A}{D^2}\|g\|^2\le\frac{\alpha A}{\|T_I\|^2}\|g\|^2\le\sum_{j\in I}v_j^2\|\pi_{\overline{TW_j}}g\|^2\le\frac{\beta B}{\gamma(T_I)^2}\|g\|^2\le\frac{\beta B}{C^2}\|g\|^2,\ g\in\overline{span}\{\overline{TW_j}\}_{j\in I}.$$

Hence $\{(\overline{TW_j},v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for $T(H)$ with bounds $\frac{\alpha A}{D^2}$ and $\frac{\beta B}{C^2}$ by Definition 2.3. □

Especially, if "T is a closed range operator" is replaced by "$T \in L(H)$ is a surjection" in Theorem 3.1, then $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H.

Corollary 3.1. *Let $T \in L(H)$ be a surjection, and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion-Riesz frame for H with bounds A and B. Then $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H, if $\eta = \inf\limits_{I\subseteq Z^+} \gamma(\pi_{\mathcal{W}_I} T^*) > 0$.*

Proof. For all $j \in I$, from $W_j \subset \mathcal{W}_I$, we have that $\pi_{W_j}\pi_{\mathcal{W}_I} = \pi_{W_j}$. By Lemma 2.3 and Definition 2.4,

$$\begin{aligned}
\sum_{j\in I} v_j^2 \|\pi_{W_j}\pi_{\mathcal{W}_I} T^*(f)\|^2 &= \sum_{j\in I} v_j^2 \|\pi_{W_j} T^*(f)\|^2 = \sum_{j\in I} v_j^2 \|\pi_{W_j} T^* \pi_{\overline{TW_j}}(f)\|^2 \\
&\le \|T\|^2 \sum_{j\in I} v_j^2 \|\pi_{\overline{TW_j}}(f)\|^2, \\
\sum_{j\in I} v_j^2 \|\pi_{W_j}\pi_{\mathcal{W}_I} T^*(f)\|^2 &= \sum_{j\in I} v_j^2 \|\pi_{W_j} T^* \pi_{\overline{TW_j}}(f)\|^2 \ge \sum_{j\in I} v_j^2 \gamma(\pi_{\mathcal{W}_j} T^*)^2 \|\pi_{\overline{TW_j}}(f)\|^2 \\
&\ge \eta^2 \sum_{j\in I} v_j^2 \|\pi_{\overline{TW_j}}(f)\|^2,
\end{aligned}$$

for all $f \in \overline{span}\{\overline{TW_j}\}_{j\in I}$. Take $\{T_I\}_{I\subseteq Z^+} = \{T\pi_{\mathcal{W}_I}\}_{I\subseteq Z^+}$. From Theorem 3.1, we have $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H with bounds $A\|T\|^{-2}\eta^2$ and $B\eta^{-2}\|T\|^2$. □

Combined with the following Theorem 3.2, some further results can be obtained.

Theorem 3.2. *Let T, $U \in L(H)$ be closed range operators, and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion-Riesz frame for H with bounds A and B. If $\ker T = \ker U$, then the following two conditions are equivalent:*

(1) *$\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for $T(H)$.*

(2) *$\{(\overline{UW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for $U(H)$.*

Proof. (1) ⇒ (2) Let $\mathcal{K}_I = \overline{span}\{\overline{TW_j}\}_{j\in I}$ for all $I \subset Z^+$. Define $U_0 \in L\big(H, U(H)\big)$ as follows: $U_0 f = UT^+ f$, $f \in H$. Then U_0 is a surjection since $\ker T = \ker U$. By Corollary 3.1 and Lemma 2.2, $\{(\overline{UW_j}, v_j)\}_{j=1}^{\infty} = \{(\overline{UU^+UW_j}, v_j)\}_{j=1}^{\infty} = \{(\overline{UT^+TW_j}, v_j)\}_{j=1}^{\infty} = \{(\overline{U_0TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for $U(H)$, if $\inf\limits_{I\subseteq Z^+} \gamma(\pi_{\mathcal{K}_I} U_0^*) > 0$.

From $\ker T = \ker U$, we have $N_{U_0\pi_{\mathcal{K}_I}}^{\perp} \subseteq \mathcal{K}_I \subseteq N_{U_0}^{\perp}$ for all $I \subset Z^+$. By Definition 2.4,

$$\gamma(\pi_{\mathcal{K}_I} U_0^*) = \gamma(U_0 \pi_{\mathcal{K}_I}) \ge \gamma(U_0) = \gamma(UT^+) \ge \gamma(U)\gamma(T^+) > 0,\ I \subseteq Z^+.$$

This implies that $\inf\limits_{I\subseteq Z^+} \gamma(\pi_{\mathcal{K}_I} U_0^*) \ge \gamma(U)\gamma(T^+) > 0$.

(2) ⇒ (1) It is similar to "(1) ⇒ (2)". □

Theorem 3.3. *Let $T \in L(H)$ be a closed range operator and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion-Riesz frame (fusion frame) for H with bounds A and B. Then $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame (fusion frame) for H, if there exists a self-adjoint closed range operator $S \in L(H)$ such that $\ker S = \ker T$ and $SW_j \subseteq W_j$ for all $j \in Z^+$.*

Proof. From $\ker S = \ker T$ and $S^* = S$,

$$\begin{aligned} A\gamma(S)^2\|f\|^2 &\leq A\|S^* f\|^2 \leq \sum_{j\in I} v_j^2 \|\pi_{W_j} S^*(f)\|^2 \\ &= \sum_{j\in I} v_j^2 \|\pi_{W_j} S^* \pi_{\overline{SW_j}}(f)\|^2 \leq \|S^*\|^2 \sum_{j\in I} v_j^2 \|\pi_{\overline{SW_j}}(f)\|^2 \\ &\leq \|S\|^2 \sum_{j\in I} v_j^2 \|\pi_{W_j}(f)\|^2 \leq B\|S\|^2\|f\|^2, \end{aligned} \tag{2}$$

where $I \subseteq Z^+$ and $f \in \overline{span}\{\overline{SW_j}\}_{j\in I}$. It follows that $\{(\overline{SW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H with bounds $A\gamma(S)^2$ and $B\|S\|^2$. By Theorem 3.2, $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame (fusion frame) for H. Especially, let $I = Z^+$ in inequality (2). By Theorem 3.1 in [24], we can obtain that $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion frame for H from that $\{(W_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame for H. □

Theorem 3.3 is a generalization of the results obtained by Casazza, Kutyniok, Li, Asgari and Gavryta [12,14,19,28,30]. It can be proved that T^+T and $(T^*T)^k$ are self-adjoint for all $k \in Z^+$.

Corollary 3.2. *Let $T \in L(H)$ be a surjection, and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion-Riesz frame for H. Then $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H, if one of the following conditions holds:*

(1) $\inf_{I\subset Z^+} \gamma(\pi_{\mathcal{W}_I} T^+) > 0$.

(2) $\inf_{I\subset Z^+} \gamma(\pi_{\mathcal{W}_I} T^+T) > 0$.

(3) $\inf_{I\subset Z^+} \gamma\left(\pi_{\mathcal{W}_I} (T^*T)^k\right) > 0$ *for some positive integer k.*

(4) $\inf_{I\subset Z^+} \gamma\left(\pi_{\mathcal{W}_I} (T^*T)^k T^*\right) > 0$ *for some positive integer k.*

(5) $\inf_{I\subset Z^+} \gamma\left(\pi_{\mathcal{W}_I} \left(T^+(T^+)^*\right)^k\right) > 0$ *for some positive integer k.*

(6) $\inf_{I\subset Z^+} \gamma\left(\pi_{\mathcal{W}_I} \left(T^+(T^+)^*\right)^k T^+\right) > 0$ *for some positive integer k.*

Proof. It follows immediately from Corollary 3.1 and Theorem 3.2. It is to note here that $\ker T = \ker(T^+)^* = \ker T^+T = \ker(T^*T)^k = \ker T(T^*T)^k = \ker(T^+)^*(T^+(T^+)^*)^k = \ker(T^+(T^+)^*)^k$. □

Corollary 3.3. *Let $T \in L(H)$ be a surjection, and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion-Riesz frame for H. Then $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame for H, if one of the following conditions holds:*

(1) $T^+TW_j \subset W_j$ *for all $j \in Z^+$.*

(2) *There exists a positive integer k such that $(T^*T)^k W_j \subset W_j$ for all $j \in Z^+$.*

(3) *There exists a positive integer k such that $(T^+(T^+)^*)^k W_j \subset W_j$ for all $j \in Z^+$.*

Proof. From $(T^*T)^kW_j \subset W_j$ for all $j \in Z^+$, we have $\left((T^*T)^k\right)^* \mathcal{W}_I = (T^*T)^k\mathcal{W}_I \subset \mathcal{W}_I$ for any $I \subset Z^+$. Further, $(T^*T)^k \left((T^*T)^k\right)^* \mathcal{W}_I \subset (T^*T)^k\mathcal{W}_I \subset \mathcal{W}_I$ for any $I \subset Z^+$. This implies that

$$\|\pi_{\mathcal{W}_I}(T^*T)^k f\| = \|(T^*T)^k f\| \geq \left(\gamma(T^*)\gamma(T)\right)^k \|f\|,$$

for all $f \in \left((T^*T)^k\right)^* \mathcal{W}_I$ and $I \subset Z^+$. By Definition 2.4,

$$\inf_{I \subset Z^+} \gamma\left(\pi_{\mathcal{W}_I}(T^*T)^k\right) \geq \left(\gamma(T^*)\gamma(T)\right)^k = \gamma(T)^{2k} > 0.$$

From Corollary 3.2, $\{(\overline{TW_j}, v_j)\}_{j=1}^\infty$ is a fusion-Riesz frame for H. Similarly, from condition (1) or (3), we have $\{(\overline{TW_j}, v_j)\}_{j=1}^\infty$ is a fusion-Riesz frame for H. □

The following Corollary 3.4 is the result in [31] and Corollary 3.5 is the result in [25].

Corollary 3.4. *[31] Let $T \in L(H)$ be an invertible operator, and $\{(W_j, v_j)\}_{j=1}^\infty$ be a fusion-Riesz frame for H. Then $\{(\overline{TW_j}, v_j)\}_{j=1}^\infty$ is a fusion-Riesz frame for H.*

Proof. From $T \in L(H)$ is an invertible operator, $W_j = T^+TW_j \subset W_j$ for all $j \in Z^+$. By Corollary 3.3, Corollary 3.4 holds. □

Corollary 3.5. *[25] Let $T \in L(H)$ be a surjection and $\{W_j\}_{j=1}^\infty$ be a fusion-Riesz frame for H. If*

$$c = \sup_{I \subset Z^+} \delta(\overline{span}\{\pi_{N_T^\perp} W_j\}_{j\in I}, \overline{span}\{W_j\}_{j\in I}) < 1$$

and $\sup_{j\in Z^+} ||(\pi_{W_j}T^+T)^+|| < +\infty$, *then* $\{(\overline{TW_j}, v_j)\}_{j=1}^\infty$ *is a fusion-Riesz frame for* H, *where* $\delta(\cdot,\cdot)$ *is spaces gap.*

Proof. From $T^+T\overline{span}\{T^+TW_j\}_{j\in I} = \overline{span}\{\pi_{N_T^\perp} W_j\}_{j\in I}$ and Lemma 2.5 in [24], we have that

$$\|\pi_{\mathcal{W}_I}T^+T(f)\| \geq \sqrt{1-c^2}\|f\|, f \in \overline{span}\{T^+TW_j\}_{j\in I}.$$

Computing

$N^\perp_{\pi_{\mathcal{W}_I}T^+T} = T^+T\mathcal{W}_I(H) = T^+T\overline{span}\{W_j\}_{j\in I} \subseteq \overline{T^+T span\{W_j\}_{j\in I}} = \overline{span}\{T^+TW_j\}_{j\in I}$,

so $\gamma(\pi_{\mathcal{W}_I}T^+T) \geq \sqrt{1-c^2} > 0$. By Corollary 3.2, $\{(\overline{TW_j}, v_j)\}_{j=1}^\infty$ is a fusion-Riesz frame for H. □

At last part of Section 3, we consider the relation between $T^+TW_j \subset W_j$ and $T^*TW_j \subset W_j$.

Lemma 3.1. *Suppose $T \in L(H)$ has closed range and W_j is a closed subspace of H. If $T^*T(W_j) \subset W_j$, then $T^+T(W_j) \subset W_j$.*

Proof. Let $V = N_T^\perp$. Then $\pi_{W_j}(T^*T\pi_{W_j}f) = T^*T\pi_{W_j}f, f \in H$ from $T^*T\pi_{W_j}f \in T^*T(W_j) \subset W_j, f \in H$. It follows that $\pi_{W_j}T^*T\pi_{W_j} = T^*T\pi_{W_j}$.

Since $T^*T(W_j) \subset T^*T(H) \subset T^*(H) \subset V$ and $T^*T(W_j) \subset W_j$, we have $T^*T(W_j) \subset W_j \cap V$, i.e., $T^*T\pi_{W_j}f \in W_j \cap V$ for all $f \in H$. Then $\pi_{W_j\cap V}(T^*T\pi_{W_j}f) = T^*T\pi_{W_j}f$ for all $f \in H$, so $\pi_{W_j\cap V}T^*T\pi_{W_j} = T^*T\pi_{W_j}$. Hence we have

$$T^*T\pi_{W_j\cap V} = \pi_{W_j}T^*T\pi_{W_j\cap V} = (\pi_{W_j\cap V}T^*T\pi_{W_j})^* = (T^*T\pi_{W_j})^* = \pi_{W_j}T^*T.$$

it follows that $T^*T(W_j \cap V) =$

$$T^*T\pi_{W_j \cap V}(W_j) = \pi_{W_j}T^*T(W_j) = \pi_{W_j}T^*T\pi_{W_j}(H) = T^*T\pi_{W_j}(H) = T^*T(W_j). \quad (3)$$

On the other hand, from Lemma 2.2 ,

$$T^+(T^+)^*T^*T = T^+(TT^+)^*T = T^+\pi^*_{T(H)}T = T^+\pi_{T(H)}T = T^+T. \quad (4)$$

Combining with equation 3 and equation 4 we obtain

$$\begin{aligned} T^+TW_j &= T^+(T^+)^*T^*TW_j = T^+(T^+)^*T^*T(W_j \cap V) = T^+T(W_j \cap V) \\ &= \pi_{T^*(H)}(W_j \cap V) = \pi_V(W_j \cap V)W_j \cap V \\ &\subset W_j. \end{aligned}$$

This completes the proof. □

Corollary 3.6. *Let $T \in L(H)$ be a surjection, and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion-Riesz frame (fusion-Riesz basis, fusion frame) for H. Then $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame (fusion-Riesz basis, fusion frame) for H, if for all $j \in Z^+$, one of the following conditions holds:*

(1) $T^+TW_j \subset W_j$.

(2) $(T^*T)^{k_j} W_j \subset W_j$ *for some* $k_j \in Z^+$.

(3) $(T^+(T^+)^*)^{k_j} W_j \subset W_j$ *for some* $k_j \in Z^+$.

Proof. Suppose (2) or (3) holds. From $T^*(H)$ is a closed subspace of H and

$$(T^*T)^{k_j}(H) = T^*(H), (T^+(T^+)^*)^{k_j}(H) = T^+(H) = T^*(H),$$

we know that $(T^*T)^{k_j}(H)$ and $(T^+(T^+)^*)^{k_j}(H)$ are closed subspaces of H. So the pseudo-inverse of $(T^*T)^{k_j}$ and $(T^+(T^+)^*)^{k_j}$ exist. By Lemma 2.2,

$$((T^*T)^{k_j})^+(T^*T)^{k_j} = \pi_{((T^*T)^{k_j})^*(H)} = \pi_{(T^*T)^{k_j}(H)} = \pi_{T^*(H)} = T^+T,$$

and

$$\left((T^+(T^+)^*)^{k_j}\right)^+(T^+(T^+)^*)^{k_j} = \pi_{((T^+(T^+)^*)^{k_j})^*(H)} = \pi_{(T^+(T^+)^*)^{k_j}(H)} = T^+T.$$

By Lemma 3.1, $T^+TW_j \subset W_j$, i.e., (1) holds.

Suppose (1) holds, i.e., $T^+TW_j \subset W_j$ for all $j \in Z^+$. While $\{(W_j, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame (fusion frame) for H, from Corollary 3.3 and Theorem 1 in [31] (or Theorem 3.5 in [24]), we have $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz frame (fusion frame) for H. On the other hand, while $\{(W_j, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz basis, by Theorem 4.6 in [12], we have $\{W_j\}_{j=1}^{\infty}$ is minimal, i.e., $W_j \bigcap W_i = \{0\}$, $j \neq i \in Z^+$. From $T^+TW_j \subset W_j$, $\{T^+TW_j\}_{j=1}^{\infty}$ is minimal. This shows that $\{\overline{TW_j}\}_{j=1}^{\infty} = \{\overline{T(T^+TW_j)}\}_{j=1}^{\infty}$ is minimal. By Theorem 4.6 in [12], $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is a fusion-Riesz basis for H. □

For the sake of contrast, there does not exist result similar to Corollary 3.6 in the operator perturbation of exact fusion frame by Exemple 3.3. Especially, while $\{(W_j, v_j)\}_{j=1}^{\infty}$ is an exact fusion frame for H, we can't obtain $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ is an exact fusion frame for H from $T^+TW_j \subset W_j, j \in Z^+$.

Example 3.3. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H and

$$W_1 = \overline{span}\{e_k\}_{k\neq 2,3}, \overline{W_2} = span\{e_k\}_{k\neq 1,3}, \overline{W_3} = span\{e_k\}_{k\neq 1,2}.$$

Define a surjection $T \in L(H)$ as follows

$$T(f) = \sum_{k=1}^{\infty} \langle f, e_{k+3}\rangle e_k, \ \ f \in H.$$

Then we have

(1) $\{(\overline{W_j}, 1)\}_{j=1}^{3}$ is an exact fusion frame for H. For all $j = 1, 2, 3$, the following conditions hold: $T^+TW_j \subset W_j$, $(T^*T)^{k_j} W_j \subset W_j$ for some $k_j \in Z^+$ and $(T^+(T^+)^*)^{k_j} W_j \subset W_j$ for some $k_j \in Z^+$.

(2) $\{(\overline{TW_j}, 1)\}_{j=1}^{3}$ is not an exact fusion frame for H.

Proof. By computing

$$2\|f\|^2 \leq \sum_{j=1}^{3} \|\pi_{W_j} f\|^2 \leq 3\|f\|^2, \qquad f \in H$$

and

$$\overline{span}\{W_1, W_2\} \neq H, \ \ \overline{span}\{W_1, W_3\} \neq H, \ \ \overline{span}\{W_2, W_3\} \neq H.$$

So $\{(\overline{W_j}, 1)\}_{j=1}^{3}$ is an exact fusion frame for H with bounds 2 and 3.

For any $f, g \in H$,

$$\langle f, T^*(g)\rangle = \langle T(f), g\rangle = \left\langle \sum_{k=1}^{\infty} \langle f, e_{k+3}\rangle e_k, g \right\rangle = \left\langle f, \sum_{k=1}^{\infty} \langle g, e_k\rangle e_{k+3} \right\rangle.$$

Hence

$$T^*(g) = \sum_{k=1}^{\infty} \langle g, e_k\rangle e_{k+3}, g \in H.$$

By Lemma 2.1, we can verify $T^+ = T^*$.
Further,

$$\left(T^+(T^+)^*\right)^{k_j} W_j = (T^*T)^{k_j} W_j = T^+TW_j = \overline{span}\{e_k\}_{k=4}^{\infty} \subset W_j$$

holds for any $j = 1, 2, 3$.

From $TW_1 = TW_2 = TW_3 = H$, $\{(\overline{TW_j}, 1)\}_{j=1}^{3}$ is not an exact fusion frame for H. □

Theorem 3.4. *Let* $\ker T = \ker U, T \in L(H), U \in L(H, K)$ *and* $\{(W_j, v_j)\}_{j=1}^{\infty}$ *is an exact fusion frame for* H. *Then* $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ *is an exact fusion frame for* H *if and only if* $\{(\overline{UW_j}, v_j)\}_{j=1}^{\infty}$ *is an exact fusion frame for* K.

Proof. It follows immediately from the Theorem 3.1 in [24]. □

Corollary 3.7. *Let* $T \in L(H)$ *be an invertible operator. Then* $\{(\overline{TW_j}, v_j)\}_{j=1}^{\infty}$ *is an exact fusion frame for* H *if and only if* $\{(W_j, v_j)\}_{j=1}^{\infty}$ *is an exact fusion frame for* H.

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