ON RECURRENCE OVER SUBSETS AND WEAK MIXING

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ABSTRACT. The paper is a study of properties of weakly mixing sets (of order \( n \)) in relation to notions such as proximality, sensitivity, scrambled tuples, Xiong chaotic sets and independent sets. Our main emphasis is put on the structure of the set of transfer times \( N(U \cap A, V) \) between open sets \( U \) and \( V \), both intersecting a weakly mixing set \( A \). It leads to a few conditions on properties of the set \( A \) equivalent to weak mixing.

We also prove that on topological graphs weakly mixing sets of order 2 can be approximated arbitrarily close by a weakly mixing set of all orders. This property is known to hold on the unit interval and not true in general (there are systems with weakly mixing sets of order \( n \) but not \( n + 1 \)).

1. INTRODUCTION

This paper is a continuation of the previous papers on local aspects of topological weak mixing [OZ11, OZ12, OZ13, OZ14] in dynamical systems \((X, f)\), that is, continuous maps \( f: X \to X \) acting on compact metric spaces. When defining recurrent properties of dynamical systems, it is convenient to analyze properties of transfer times between sets, expressed in terms of the set \( N(U, V) = \{ n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset \} \), where \( U \) and \( V \) are nonempty open subsets of \( X \). For example, \((X, f)\) is transitive if \( N(U, V) \) is nonempty for any choice of two nonempty open sets. As mentioned before, the main concept in this paper is topological weak mixing (in fact, its local versions), which is usually defined as transitivity of \((X \times X, f \times f)\). In other words, \((X, f)\) is weakly mixing if \( N(U_1, V_1) \cap N(U_2, V_2) \neq \emptyset \) for any choice of four nonempty open sets \( U_1, U_2, V_1, V_2 \subset X \). It was shown by Furstenberg (see [Fur67]) that if \((X, f)\) is weakly mixing then for every \( n \geq 2 \) and every nonempty open sets \( U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n \subset X \) we have \( N(U_1, V_1) \cap N(U_2, V_2) \cap \cdots \cap N(U_n, V_n) \neq \emptyset \).

Note that weak mixing can be regarded as a ‘global’ property, while topological entropy is a ‘local’ one, since it can be supported on a small set in the space (e.g., a nowhere dense attractor). It is also not hard to see that in general there cannot be any implication between weak mixing and positive topological entropy. Then an appropriate ‘local’ version of weak mixing is needed. Such a concept was introduced by Blanchard and Huang in [BH08], where they defined a local concept of weak mixing. Strictly speaking, a non-trivial closed set \( A \subset X \) (i.e., not a singleton) is weakly mixing if for every \( n \geq 2 \) and every nonempty open sets \( U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n \subset X \) intersecting \( A \) (i.e., \( U_i \cap A \neq \emptyset \) and \( V_i \cap A \neq \emptyset \) for each \( i = 1, 2, \ldots, n \)) we have \( N(U_1 \cap A, V_1) \cap N(U_2 \cap A, V_2) \cap \cdots \cap N(U_n \cap A, \ldots, V_n) \neq \emptyset \).

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A, V_n) \neq \emptyset. As we can see, the above definition is consistent with the definition of weakly mixing map, and what’s more important, it is proved in [BH08] that every dynamical system with positive topological entropy contains many Cantor weakly mixing sets. 

Similarly, for a fixed integer \( n \geq 2 \), we say that a nontrivial closed subset \( A \) of \( X \) is \textit{weakly mixing of order} \( n \) if for every nonempty open subsets \( U_1, U_2, \ldots, U_n, V_1, V_2, \ldots, V_n \) of \( X \) intersecting \( A \), \( N(U_1 \cap A, V_1) \cap N(U_2 \cap A, V_2) \cap \cdots \cap N(U_n \cap A, V_n) \neq \emptyset. \) Unfortunately, the analog of Furstenberg’s theorem cannot be proved here. Namely, it is proved in [OZ11, OZ14] that for every \( n \geq 2 \) there exists a dynamical system which contains weakly mixing sets of order \( n \) but no weakly mixing sets of order \( n + 1 \).

Since Furstenberg’s theorem does not work for weakly mixing sets of order \( n \), it is natural to ask which criteria for weak mixing (i.e. equivalent conditions) can be used in the case of weakly mixing sets. It was proved in [Ban99] that most of the conditions that can be expressed in terms of intersections of sets \( N(U, V) \) lead to weak mixing. Of particular interest can be the condition, proved first by Petersen in [Pet70], which says that a dynamical system is weakly mixing if and only if \( N(U, V) \cap N(U, U) \neq \emptyset \) for any nonempty open sets \( U, V \subset X \). In the spirit of the above fact, we find the following criterion for weak mixing of order \( n \). It will be shown later, in Example 3.2, that we cannot use exactly the same condition as in [Pet70].

**Theorem 1.1.** Let \( (X, f) \) be a dynamical system, \( A \subset X \) a nontrivial closed subset and \( n \geq 2 \). Then \( A \) is a weakly mixing set of order \( n \) if and only if for any \( n + 1 \) open subsets \( U_1, V_1, V_2, \ldots, V_n \) of \( X \) intersecting \( A \),

\[
N(U_1 \cap A, V_1) \cap \bigcap_{i=2}^{n} N(V_i \cap A, V_i) \neq \emptyset.
\]

It is shown in [Li11, Theorem 3.2] that if a dynamical system \( (X, f) \) is weakly mixing, then there exists a residual subset \( K \) of \( X \) such that for every \( x \in K \) and every nonempty open subset \( U \) of \( X \), the set \( N(x, U) \) contains an IP-set. Later, it was generalized in [OZ13, Theorem 8], which states that if \( A \) is a weakly mixing set of order 2 and \( U \) is an open subset of \( X \) intersecting \( A \), then there is \( x \in U \cap A \) such that for every open subset \( V \) of \( X \) intersecting \( A \) the set \( N(x, V) \) contains an IP-set. Using the idea in the proof of [Li11, Theorem 3.2], we could extend the above mentioned fact from [OZ13] a little further.

**Theorem 1.2.** Let \( (X, f) \) be a dynamical system and \( A \subset X \) a weakly mixing set of order \( n \) with \( n \geq 2 \). Then there exists a residual subset \( K \) of \( A \) such that for any \( x \in K \) and any choice of \( n - 1 \) open subsets \( U_1, \ldots, U_{n-1} \) of \( X \) intersecting \( A \) there exist points \( y_i \in U_i \cap A \), where \( i = 1, \ldots, n - 1 \), such that \( N(x, U_1) \cap \bigcap_{i=1}^{n-1} N(y_i, U_i) \) contains an IP-set.

A subset \( A \) of \( X \) is \textit{transitive} in \( (X, f) \) if for any open subsets \( U \) and \( V \) of \( X \) intersecting \( A \), the set \( N(U \cap A, V) \) is not empty; \textit{totally transitive} if \( A \) is transitive in \( (X, f^k) \) for every \( k \in \mathbb{N} \). Let \( n \geq 2 \) be an integer. It is clear that a nontrivial closed subset \( A \subset X \) is weakly mixing of order \( n \) if and only if \( A^n \) is a transitive set in \( (X^n, f^{(n)}) \). Using Theorem 1.2, we have the following result.

**Proposition 1.3.** Let \( (X, f) \) be a dynamical system and \( A \subset X \) a nontrivial closed subset.

(1.3.1) If \( A \) is a weakly mixing set of order 2, then \( A \) is totally transitive.

(1.3.2) If \( A \) is a weakly mixing set of order \( n \) for \( f \) with \( n \geq 3 \), then, for every \( k \in \mathbb{N} \), \( A \) is a weakly mixing set of order \( n - 1 \) for \( f^k \).
The authors in [HLY12] proved that a dynamical system is weakly mixing if and only if it has the IP-independent property (a formal definition of independence will be given later). We will obtain a similar result for the case of weakly mixing sets.

Inspired by the result of Xiong and Yang [XY91], the authors of [BH08] provided an alternative definition of a weakly mixing set. Strictly speaking, it was proved in [BH08] that a nontrivial closed set $A \subset X$ is a weakly mixing set if and only if there exists a union $B \subset A$ of countably many Cantor sets, dense in $A$ and such that for any $C \subset B$, any continuous map $g : C \to A$ there exists an increasing sequence of natural numbers $\{n_i\}_{i=1}^\infty$ such that $\lim_{i \to \infty} f^{n_i}(x) = g(x)$ for any $x \in C$.

Similarly, we can introduce Xiong chaotic sets of a finite order as follows. A subset $K$ of $X$ with at least $n$ points is called a Xiong chaotic set of order $n$ if for any subset $E$ of $K$ with cardinality $n$ and for any map $g : E \to K$ there is an increasing subsequence $\{q_i\}_{i=1}^\infty$ in $\mathbb{N}$ such that $\lim_{i \to \infty} f^{q_i}(x) = g(x)$ for every $x \in E$. Later, we will show that a result analogous of [BH08] holds, that is, any nontrivial closed subset $A$ of $X$ is a weakly mixing set of order $n$ if and only if there exists a dense Mycielski subset $S$ of $A$ which is Xiong chaotic of order $n$. A dynamical system has a weakly mixing set of order $n$ if and only if it has an uncountable Xiong chaotic set of order $n$. One of advantages of Xiong chaotic sets is that they are hereditary by subsets, while weakly mixing sets are not.

For a dynamical system $(X, f)$, the proximal relation is

$$\text{Prox}_2(f) = \{(x, y) \in X \times X : \liminf_{k \to \infty} d(f^k(x), f^k(y)) = 0\},$$

and the proximal cell of a point $x \in X$ is $\text{Prox}_2(f)(x) = \{y \in X : (x, y) \in \text{Prox}_2(f)\}$. It was shown in [AK03] that if $(X, f)$ is weakly mixing, then for every $x \in X$, the proximal cell $\text{Prox}_2(f)(x)$ of $x$ is residual in $X$. The authors in [HSY04] studied the structure of proximal cells of points in weakly mixing systems and showed that there is a Xiong chaotic set in those proximal cells. In [OZ13] it was proved that for every closed weakly mixing set $A$ and every $x \in A$, the set $\text{Prox}_2(f)(x) \cap A$ is residual in $A$. We will show that the same is true if we consider proximal tuples instead of pairs. For a dynamical system $(X, f)$ and a positive integer $n \geq 2$, the $n$-th proximal relation is

$$\text{Prox}_n(f) = \{(x_1, \ldots, x_n) \in X^n : \liminf_{k \to \infty} \max_{1 \leq i < j \leq n} d(f^k(x_i), f^k(x_j)) = 0\},$$

and the $n$-th proximal cell of a point $x \in X$ is $\text{Prox}_n(f)(x) = \{(x_1, \ldots, x_{n-1}) \in X^{n-1} : (x, x_1, \ldots, x_{n-1}) \in \text{Prox}_n(f)\}$.

**Theorem 1.4.** Let $(X, f)$ be a dynamical system and $A \subset X$ a weakly mixing set. Then for every $x \in A$ and $n \geq 2$, the set $\text{Prox}_n(f)(x) \cap A^{n-1}$ is residual in $A^{n-1}$.

In fact, we prove even more as presented in the following theorem, where $LY_n^\delta(X, f)(x)$ is the $n$-scrambled cell of $x$ with modular $\delta > 0$. More precisely, $LY_n^\delta(X, f)(x)$ is the collection of points $(x_1, \ldots, x_{n-1})$ in $X^{n-1}$ such that $(x, x_1, \ldots, x_{n-1})$ is proximal and

$$\limsup_{k \to \infty} \min_{1 \leq i < j \leq n} d(f^k(x_i), f^k(x_j)) \geq \delta.$$

**Theorem 1.5.** Let $(X, f)$ be a dynamical system, $A \subset X$ a weakly mixing set and $n \geq 2$. Then there exists $\delta > 0$ such that for every $x \in A$, $LY_n^\delta(X, f)(x) \cap A^{n-1}$ is residual in $A^{n-1}$. 
The following result shows that, when we look only at separation of trajectories of tuples, weak mixing of order 2 is enough to obtain rich structure of such points (see subsection 2.2 for definitions of sensitivity).

**Theorem 2.2.** Let \((X, f)\) be a dynamical system and \(A \subset X\) a weakly mixing set of order 2. Then \(A\) is a sensitive set in \((\text{Orb}(A, f), f)\). In particular, the system \((\text{Orb}(A, f), f)\) is \(n\)-sensitive for every \(n \geq 2\).

In the final section, we prove that on topological graphs weakly mixing sets of order 2 can be approximated arbitrarily close (in the Hausdorff metric) by a weakly mixing set of all orders. This completes our previous research in [OZ11].

## 2. Preliminaries

In this section, we provide some basic notations, definitions and results which will be used later in this paper. Denote by \(\mathbb{N}\) (\(\mathbb{N}_0, \mathbb{Z}\) and \(\mathbb{R}\), respectively) the set of all positive integers (nonnegative integers, integers and real numbers, respectively). A subset \(A\) of \(\mathbb{N}\) is an IP-set if there exists a sequence \(\{p_j\}_{j=1}^\infty\) in \(\mathbb{N}\) such that \(A = \text{FS}(\{p_j\}_{j=1}^\infty)\), where \(\text{FS}(\{p_j\}_{j=1}^\infty)\), the finite sums of \(\{p_j\}_{j=1}^\infty\), is defined as

\[
\text{FS}(\{p_j\}_{j=1}^\infty) = \left\{ \sum_{j \in \alpha} p_j : \alpha \text{ is a nonempty finite subset of } \mathbb{N} \right\}.
\]

Let \(X\) be a compact metric space. A subset \(C\) of \(X\) is a Cantor set if it is homeomorphic with standard Cantor trinary set (equivalently, it is a perfect and totally disconnected compact metric space). We say that a subset \(K\) of \(X\) is a Mycielski set if it can be presented as a countable union of Cantor sets. The next two facts help to deal with residual relations. They are important tools with numerous applications. See [Aki04] for a comprehensive treatment of this topic.

**Lemma 2.1** (Ulam Lemma). Let \(X\) be a perfect compact metric space. If \(R\) is a dense \(G_\delta\) subset of \(X^n\), then there exists a dense \(G_\delta\) subset \(K\) of \(X\) such that for every \(x \in K\), the set \(R(x) = \{(x_1, \ldots, x_{n-1}) \in X^{n-1} : (x, x_1, \ldots, x_{n-1}) \in R\}\) is residual in \(X^{n-1}\).

**Theorem 2.2** (Mycielski Theorem [M64]). Let \(X\) be a perfect compact metric space. If \(R\) is a dense \(G_\delta\) subset of \(X^n\), then there exists a dense Mycielski subset \(K\) of \(X\) such that for every \(n\) distinct points \(x_1, \ldots, x_n \in K\), we have \((x_1, x_2, \ldots, x_n) \in R\).

### 2.1. Topological dynamics

By a (topological) dynamical system we mean a pair \((X, f)\) consisting of a compact metric space \((X, d)\) and a continuous map \(f : X \to X\). If \(X\) is a singleton, then we say that \((X, f)\) is trivial. If \(K \subset X\) is a nonempty closed subset satisfying \(f(K) \subset K\), then we say that \((K, f)\) is a subsystem of \((X, f)\).

Let \((X, f)\) be a dynamical system, \(\emptyset \neq A \subset X\) and \(x \in X\). The set \(\text{Orb}(A, f) = \bigcup_{n \in \mathbb{N}_0} f^n(A)\) is said to be the (positive) orbit of \(A\) under \(f\). Clearly, \((\text{Orb}(A, f), f)\) is a subsystem of \((X, f)\). We will write \(\text{Orb}(x, f) = \text{Orb}(\{x\}, f)\) for short.

We say that a point \(x \in X\) is a periodic point of \((X, f)\) if \(f^n(x) = x\) for some \(n \in \mathbb{N}\); a recurrent point of \((X, f)\) if there exists an increasing sequence \(\{k_n\}_{n=1}^\infty\) in \(\mathbb{N}\) such that \(\lim_{n \to \infty} f^{k_n}(x) = x\); a transitive point of \((X, f)\) if \(\text{Orb}(x, f)\) is dense in \(X\). Denote by \(\text{Per}(X, f)\) (\(\text{Rec}(X, f)\) and \(\text{Tran}(X, f)\), respectively) the set of all periodic points (recurrent points, respectively).
points and transitive points, respectively) of \((X, f)\). A dynamical system \((X, f)\) is minimal if \(\text{Tran}(X, f) = X\). A point \(x \in X\) is minimal if the subsystem \((\text{Orb}(x, f), f)\) is minimal.

Let \((X, f)\) be a dynamical system, \(A \subset X\) and \(n \geq 2\). Denote \(A^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in A\}\) and \(\Delta_n(A) = \{(x, x, \ldots, x) : x \in A\}\). The map \(f^{(n)}\) is induced on \(X^n\) by the formula \(f^{(n)}(x_1, x_2, \ldots, x_n) = (f(x_1), f(x_2), \ldots, f(x_n))\).

Let \((X, f)\) be a dynamical system, \(x \in X\) and \(A, B \subset X\). Denote \(N(x, A) = \{n \in \mathbb{N} : f^n(x) \in A\}\) and \(N(A, B) = \{n \in \mathbb{N} : f^n(A) \cap B \neq \emptyset\}\). When we want to emphasize the map \(f\), we use \(N_f(x, A)\) and \(N_f(A, B)\) instead of \(N(x, A)\) and \(N(A, B)\). A dynamical system \((X, f)\) is called transitive if for any nonempty open subsets \(U\) and \(V\) of \(X\), the set \(N(U, V)\) is not empty; totally transitive if \((X, f^k)\) is transitive for every \(k \in \mathbb{N}\); weakly mixing if \((X^2, f^{(2)})\) is transitive. It is well known that if \((X, f)\) is transitive, then \(\text{Tran}(X, f)\) is a dense \(G_\delta\) subset of \(X\).

2.2. **Proximal and scrambled tuples.** We say that an \(n\)-tuple \((x_1, \ldots, x_n) \in X^n\) (where \(n \geq 2\)) is proximal if
\[
\liminf_{k \to \infty} \max_{1 \leq i < j \leq n} d(f^k(x_i), f^k(x_j)) = 0.
\]

Let \(\text{Prox}_n(f)\) denote the collection of all proximal \(n\)-tuples in \((X, f)\). It is easy to verify that \(\text{Prox}_n(f)\) is a \(G_\delta\) subset of \(X^n\). For \(x \in X\), define the \(n\)-th proximal cell of \(x\) as
\[
\text{Prox}_n(f)(x) = \{(x_1, \ldots, x_{n-1}) \in X^{n-1} : (x, x_1, \ldots, x_{n-1})\}.
\]

An \(n\)-tuple \((x_1, \ldots, x_n) \in X^n\) (where \(n \geq 2\)) is called scrambled (with modular \(\delta > 0\)) if it is proximal and
\[
\limsup_{k \to \infty} \min_{1 \leq i < j \leq n} d(f^k(x_i), f^k(x_j)) \geq \delta.
\]
A subset \(S\) of \(X\) is called \(n\)-scrambled if any \(n\) distinct points in \(S\) form a scrambled \(n\)-tuple. The system \((X, f)\) is called Li-Yorke \(n\)-chaotic if there exists an uncountable \(n\)-scrambled subset \(S\) of \(X\).

In [Xio05], Xiong introduced the conception of \(n\)-sensitivity, that is, a dynamical system \((X, f)\) is called \(n\)-sensitive, where \(n \geq 2\), if there exists \(\delta > 0\) such that for every nonempty open set \(U \subset X\) there are distinct points \(x_1, x_2, \ldots, x_n \in U\) and some \(m \in \mathbb{N}\) with
\[
\min_{1 \leq i < j \leq n} d(f^m(x_i), f^m(x_j)) > \delta.
\]
This definition was further generalized in [YZ08] to sensitive sets. A subset \(A\) of \(X\) is sensitive if for any \(n \geq 2\), any \(n\) distinct points \(x_1, x_2, \ldots, x_n \in A\), any neighborhood \(U_i\) of \(x_i\), \(i = 1, 2, \ldots, n\), and any nonempty open set \(U \subset X\) there exists \(k \in \mathbb{N}\) and \(y_i \in U\) such that \(f^k(y_i) \in U_i\) for \(i = 1, 2, \ldots, n\). It is shown in [YZ08] that a transitive system is \(n\)-sensitive if and only if there exists a sensitive set with cardinality \(n\). Note that 2-scrambled set, Li-Yorke 2-chaos and 2-sensitivity are classical definitions.

2.3. **Transitive sets and weakly mixing sets.** Let \((X, f)\) be a dynamical system. A subset \(A\) of \(X\) is transitive in \((X, f)\) if for any open subsets \(U\) and \(V\) of \(X\) intersecting \(A\), the set \(N(U \cap A, V)\) is not empty; totally transitive if \(A\) is transitive in \((X, f^k)\) for every \(k \in \mathbb{N}\). Let \(n \geq 2\) be an integer. A nontrivial closed subset \(A \subset X\) is weakly mixing of order \(n\) provided that \(A^n\) is a transitive set in \((X^n, f^{(n)})\); weakly mixing of all orders or simply weakly mixing if \(A\) is weakly mixing of order \(k\) for all \(k = 2, 3, \ldots,\)
Remark 2.3. In the present paper we require a weakly mixing set (of order $n$) to be closed and nontrivial which is a little more restrictive than the original definition in [OZ11].

The following result is derived from the definition directly.

Lemma 2.4. Let $(X, f)$ be a dynamical system, $A \subset X$ a nontrivial closed subset and $n \geq 2$. Then $A$ is weakly mixing of order $n$ if and only if for any open subsets $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ of $X$ intersecting $A$,

$$\bigcap_{i=1}^n N(U_i \cap A, V_i) \neq \emptyset.$$

The following lemmas, while simple in proof, are very useful in practice. The proofs can be found in [OZ11] and [OZ14].

Lemma 2.5. Let $(X, f)$ be a dynamical system and $A \subset X$ a weakly mixing subset of order 2. Then $A$ is perfect.

Lemma 2.6. Let $(X, f)$ be a dynamical system and $A$ a closed subset of $X$. If $A$ is a transitive set, then:

1. $(\text{Orb}(A, f), f)$ is a transitive subsystem of $(X, f)$,
2. $A \cap \text{Tran}(\text{Orb}(A, f), f)$ is a dense $G_δ$ subset of $A$.

2.4. Symbolic dynamics. Let $A$ be a finite set (an alphabet) endowed with the discrete topology and let $A^{\mathbb{N}_0}$ denote the Cantor space with respect to the product topology. We write elements of $A^{\mathbb{N}_0}$ as $x = x_0x_1 \ldots$. The shift transformation $σ : A^{\mathbb{N}_0} \to A^{\mathbb{N}_0}$ is given by $σ(x)_i = x_{i+1}$ for $i \in \mathbb{N}_0$. The dynamical system $(A^{\mathbb{N}_0}, σ)$ is called the full shift over $A$.

By a word (over $A$), we mean any finite sequence $u = u_0 \ldots u_{n-1}$, $n \geq 1$ where $u_i \in A$. The length of $u$ is denoted by $|u| = n$ and the set of all words is denoted by $A^+$. If $x \in A^{\mathbb{N}_0}$ and $0 \leq i < j$, then by $x_{[i,j]}$ we mean the sequence $x_i x_{i+1} \ldots x_j$. For simplicity, we use the notation $x_{[i,j]} = x_{[i,j-1]}$. If $a_1 \ldots a_m \in A^+$, then we define the cylinder set

$$C[a_1 \ldots a_m] = \{x \in A^{\mathbb{N}_0} : x_{[0,m]} = a_1 \ldots a_m\}.$$

If $X$ is a subshift, we denote the cylinder set $C_X[a_1 \ldots a_m] = C[a_1 \ldots a_m] \cap X$.

2.5. Topological graphs. Roughly speaking, a topological graph is a continuum which is the union of finite number of intervals which can intersect only at endpoints and do not have self intersections. More formally, a topological graph is a compact connected metric space $G$ which is homeomorphic to a polyhedron (a geometric realization) of some finite one-dimensional complex. In particular, we can naturally endow $G$ with the metric $d$ given by the length of the shortest arc joining $x, y$ in $G$ (induced on $G$ from the polyhedron). An arc $I \subset G$ is a closed interval if there is a homeomorphism $ϕ : [0, 1] \to I$ such that the set $ϕ([0, 1))$ is open in $G$.

Let $(G, f)$ be a dynamical system and let $I, J \subset G$ be closed intervals. If there exists a closed interval $K \subset I$ such that $f(K) = J$, then we say that $I$ $f$-covers $J$ and denote this fact by $I \xrightarrow{f} J$. In this paper we will need the following standard properties of $f$-covering (see [AdRR03], Page 590):

Lemma 2.7. Let $I, J, K, L \subset G$ be closed intervals and $f, g : G \to G$ continuous.
(2.7.1) If $I \subset K$, $L \subset J$ and $I \xrightarrow{f} J$, then $K \xrightarrow{f} L$.
(2.7.2) If $I \xrightarrow{f} J$ and $J \xrightarrow{g} K$, then $I \xrightarrow{g \circ f} K$.
(2.7.3) If $J \subset f(I)$, and $K_1, K_2 \subset J$ are closed intervals such that $K_1 \cap K_2$ is at most one point, then $I \xrightarrow{f} K_1$, or $I \xrightarrow{f} K_2$.

3. WEAKLY MIXING SETS OF FINITE ORDER

In this section we study weakly mixing sets of finite order. It is clear that a dynamical system $(X, f)$ is weakly mixing if and only if for any four nonempty open subsets $U_1, V_1, U_2, V_2$ of $X$, $N(U_1, V_1) \cap N(U_2, V_2) \neq \emptyset$. It is shown in [Pet70] that we can reduce four open sets in the characterization of weak mixing to two open sets, that is, a dynamical system $(X, f)$ is weakly mixing if and only if for any two nonempty open subsets $U$ and $V$ of $X$, $N(U, V) \cap N(U, U) \neq \emptyset$ (this was later extended in [Ban99] to show that most of possible conditions of this kind are equivalent to weak mixing). Similar to the above mentioned condition, we can simplify the condition in Lemma 2.4 obtaining an alternative definition of weakly mixing set of order $n$. The advantage is that we have to verify conditions on transfer times for only $n + 1$ open sets instead of $2n$ sets in the original definition.

**Theorem 3.1.** Let $(X, f)$ be a dynamical system, $A \subset X$ a nontrivial closed subset and $n \geq 2$. Then $A$ is a weakly mixing set of order $n$ if and only if for any $n + 1$ open subsets $U_1, V_1, V_2, \ldots, V_n$ of $X$ intersecting $A$,

$$N(U_1 \cap A, V_1) \cap \bigcap_{i=2}^{n} N(V_i \cap A, V_i) \neq \emptyset.$$ 

**Proof.** The necessity follows from Lemma 2.4. Now we prove the sufficiency. Fix any $2n$ open subsets $U_1, V_1, U_2, V_2, \ldots, U_n, V_n$ of $X$ intersecting $A$. Assume that for some $1 \leq j < n$ we have

$$\bigcap_{i=1}^{j} N(U_i \cap A, V_i) \cap \bigcap_{l=j+1}^{n} N(V_l \cap A, V_l) \neq \emptyset.$$ 

Then there is $k > 0$ and open subsets $U'_1, \ldots, U'_j$, $V'_{j+1}, \ldots, V'_n$ of $X$ intersecting $A$ such that $U'_i \subset U_i$ and $f^k(U'_i) \subset V_i$ for each $i = 1, \ldots, j$, and $V'_l \subset V_l$ and $f^k(V'_l) \subset V_l$ for each $l = j+1, \ldots, n$. By the assumption we can choose

$$m \in N(U_{j+1} \cap A, V'_{j+1}) \cap \bigcap_{i=1}^{j} N(U'_i \cap A, U'_i) \cap \bigcap_{l=j+2}^{n} N(V'_l \cap A, V'_l),$$

then

$$m + k \in \bigcap_{i=1}^{j+1} N(U_i \cap A, V_i) \cap \bigcap_{l=j+2}^{n} N(V_l \cap A, V_l).$$

Hence by induction on $j$, we eventually obtain that

$$\bigcap_{i=1}^{n} N(U_i \cap A, V_i) \neq \emptyset,$$

which implies that $A$ is weakly mixing of order $n$. \qed
Unfortunately, the above technique is not sufficient if we want directly copy the condition from [Pet70]. Simply, this condition is not sufficient to induce even smallest degree of local weak mixing, as shown by the next example. The technique used here is a modification of Example 6.1 from [OZ11]. Since the construction is somewhat long and complicated, we move it to Appendix A.

**Example 3.2.** There are a dynamical system \((X, f)\) and a nontrivial closed subset \(A\) of \(X\) satisfying the following two conditions:

1. \(N(U \cap A, V) \cap N(U \cap A, U) \neq \emptyset\) and \(N(U \cap A, V) \cap N(V \cap A, V) \neq \emptyset\) for any open subsets \(U, V\) of \(X\) intersecting \(A\),

2. \(A\) is not weakly mixing of order 2.

It is shown in [L11, Theorem 3.2] that if a dynamical system \((X, f)\) is weakly mixing, then there exists a residual subset \(K\) of \(X\) such that for every \(x \in K\) and every nonempty open subset \(U\) of \(X\), the set \(N(x, U)\) contains an IP-set. Later, it was generalized in [OZ13, Theorem 8], which states that if \(A\) is a weakly mixing set of order 2 and \(U\) is an open set of \(X\) intersecting \(A\), then there is \(x \in U \cap A\) such that for every open set \(V\) of \(X\) intersecting \(A\) the set \(N(x, V)\) contains an IP-set. The following lemma is inspired by the proof of [L11, Theorem 3.2]. It allows us to extend the above mentioned fact from [OZ13] a little further.

**Lemma 3.3.** Let \((X, f)\) be a dynamical system and \(n \geq 2\). If there are \(n\) points \(x, y_1, y_2, \ldots, y_{n-1} \in X\) with \(x \neq y_1\) such that

\[(y_1, y_1, y_2, \ldots, y_{n-1}) \in \text{Orb}((x, y_1, \ldots, y_{n-1}), f^{(n)}),\]

then for every choice of open neighborhoods \(U_i\) of \(y_i\) with \(i = 1, 2, \ldots, n-1\), the set

\[N(x, U_1) \cap \bigcap_{i=1}^{n-1} N(y_i, U_i)\]

contains an IP-set.

**Proof.** For each \(i = 1, 2, \ldots, n-1\) fix an open neighborhood \(U_i\) of \(y_i\). Since \(x \neq y_1\), we may assume that \(x \notin U_1\). We are going to construct an IP-set in \(N(x, U_1) \cap \bigcap_{i=1}^{n-1} N(y_i, U_i)\).

We start our construction by putting \(U_1^{(1)} = U_1\) for \(i = 1, \ldots, n-1\).

By (1), there exists \(p_1 \in \mathbb{N}\) such that \(f^{p_1}(x) \in U_1^{(1)}\) and \(f^{p_i}(y_i) \in U_i^{(1)}\) for \(i = 1, \ldots, n-1\). Let \(U_i^{(2)} = U_i^{(1)} \cap f^{-p_i}(U_i^{(1)})\) for \(i = 1, \ldots, n-1\). Clearly, \(U_i^{(2)}\) is also an open neighborhood of \(y_i\) for \(i = 1, \ldots, n-1\). By (1) again there exists \(p_2 > 0\) such that \(f^{p_2}(x) \in U_1^{(2)}\) and \(f^{p_i}(y_i) \in U_i^{(2)}\) for \(i = 1, \ldots, n-1\). Then for every \(m \in FS(\{p_j\}_{j=1}^k)\), we have \(f^m(x) \in U_1\) and \(f^m(y_i) \in U_i\) for \(i = 1, \ldots, n-1\). We continue this construction inductively.

Assume that for some \(k \geq 2\) positive integers \(p_1, p_2, \ldots, p_k\) have been constructed in such a way that if \(m \in FS(\{p_j\}_{j=1}^k)\) then \(f^m(x) \in U_1\) and \(f^m(y_i) \in U_i\) for \(i = 1, \ldots, n-1\). For each \(i = 1, \ldots, n-1\) put

\[U_i^{(k+1)} = U_i \cap \bigcap_{m \in FS(\{p_j\}_{j=1}^k)} f^{-m}(U_i)\]

and observe that each \(U_i^{(k+1)}\) is also an open neighborhood of \(y_i\) for \(i = 1, \ldots, n-1\). Again by (1) there exists \(p_{k+1} > 0\) such that \(f^{p_{k+1}}(x) \in U_1^{(k+1)}\) and \(f^{p_{k+1}}(y_i) \in U_i^{(k+1)}\)
for \( i = 1, \ldots, n - 1 \). Then, completing induction, for every \( m \in FS(\{p_j\}_{j=1}^{k+1}) \), we have \( f^m(x) \in U_i \) and \( f^m(y_i) \in U_i \) for \( i = 1, \ldots, n - 1 \). Thus, we get a sequence \( \{p_j\}_{j=1}^{\infty} \) such that \( FS(\{p_j\}_{j=1}^{\infty}) \subset N(x, U_1) \cap \bigcap_{i=1}^{n-1} N(y_i, U_i) \). The proof is finished. \( \square \)

**Theorem 3.4.** Let \((X, f)\) be a dynamical system and \( A \subset X \) a weakly mixing set of order \( n \) with \( n \geq 2 \). Then there exists a residual subset \( K \) of \( A \) such that for any \( x \in K \) and any choice of \( n-1 \) open subsets \( U_1, \ldots, U_{n-1} \) of \( X \) intersecting \( A \) there exist points \( y_i \in U_i \cap A \), where \( i = 1, \ldots, n-1 \), such that \( N(x, U_1) \cap \bigcap_{i=1}^{n-1} N(y_i, U_i) \) contains an IP-set.

**Proof.** Since \( A^n \) is a transitive set in \((X^n, f^{(n)})\), by Lemma 2.6 the relation \( R = A^n \cap \text{Tran}(\text{Orb}(A^n, f^{(n)}), f^{(n)}) \) is a dense \( G_\delta \) subset of \( A^n \). By the Ulam Lemma, there exists a dense \( G_\delta \) subset \( K \) of \( A \) such that for every \( x \in K \) the section of \( R \) at \( x \), that is the set \( R(x) = \{(y_1, \ldots, y_{n-1}) \in A^{n-1} : (x, y_1, \ldots, y_{n-1}) \in R\} \), is residual in \( A^{n-1} \). It remains to show that \( K \) satisfies our requirement.

Fix \( x \in K \) and \( n-1 \) open subsets \( U_1, U_2, \ldots, U_{n-1} \) of \( X \) intersecting \( A \). Since \( R(x) \) is residual, we can select points \( y_i \in U_i \cap A \), where \( i = 1, \ldots, n-1 \), such that \( (x, y_1, \ldots, y_{n-1}) \in R \) and \( x \neq y_1 \) (recall that \( A \) is perfect by Lemma 2.5). By the definition of \( R \) we obtain \( (y_1, y_1, y_2, \ldots, y_{n-1}) \in \text{Orb}(\{x, y_1, \ldots, y_{n-1}\}, f^{(n)}) \).

Now the result follows by Lemma 3.3. \( \square \)

As a direct corollary, we have:

**Corollary 3.5.** Let \((X, f)\) be a dynamical system and \( A \subset X \) a weakly mixing set of order \( n \) with \( n \geq 2 \). Then for any \( n \) open subsets \( U_1, V_1, V_2, \ldots, V_{n-1} \) of \( X \) intersecting \( A \),

\[
N(U_1 \cap A, V_1) \cap \bigcap_{i=1}^{n-1} N(V_i \cap A, V_i)
\]

contains an IP-set.

It is shown in [OZ12, Theorem 6] that a weakly mixing set of order 2 is totally transitive. Now, with help of Corollary 3.5, we can extend it as follows.

**Proposition 3.6.** Let \((X, f)\) be a dynamical system and \( A \subset X \) a nontrivial closed subset.

(3.6.1) If \( A \) is a weakly mixing set of order 2, then \( A \) is totally transitive.

(3.6.2) If \( A \) is a weakly mixing set of order \( n \) for \( f \) with \( n \geq 3 \), then for every \( k \in \mathbb{N} \), \( A \) is a weakly mixing set of order \( n-1 \) for \( f^k \).

**Proof.** If \( F \) contains an IP-set, then \( F \cap n\mathbb{N} \neq \emptyset \) for every \( n \in \mathbb{N} \). Now the result follows by Theorem 3.1 and Corollary 3.5. \( \square \)

The above fact motivates us to state the following question for further investigation.

**Question 1.** Let \((X, f)\) be a dynamical system and \( k \in \mathbb{N}, n \geq 2 \). If a subset \( A \subset X \) is weakly mixing of order \( n \) for \( f \), is it weakly mixing of order \( n \) for \( f^k \)?

**Theorem 3.7.** Let \((X, f)\) be a dynamical system and \( A \subset X \) a weakly mixing set of order 2. Then \( A \) is a sensitive set in \((\text{Orb}(A, f), f)\).
Proof. Without loss of generality, we assume that \(\text{Orb}(A,f) = X\). First note that both \(A\) and \(X\) must be perfect. Now fix \(n \geq 2\) and \(n\) distinct points \(x_1, x_2, \ldots, x_n\) in \(A\). Let \(U\) be a nonempty open subset of \(X\) and \(U_i\) an open neighborhood of \(x_i\) for \(i = 1, 2, \ldots, n\). There is some \(k \geq 0\) such that \(f^k(A) \cap U \neq \emptyset\) and then there is an open subset \(V\) of \(X\) intersecting \(A\) such that \(f^k(V) \subset U\). Since \(A\) is a weakly mixing set of order \(2\), there exists \(m_2 \in \mathbb{N}\) such that \(U_1 \cap A \cap f^{-m_2}(U_2) \neq \emptyset\) and \(V \cap A \cap f^{-m_2}(V) \neq \emptyset\). By induction, there exist \(m_3, \ldots, m_n \in \mathbb{N}\) such that

\[
U_1 \cap A \cap \bigcap_{i=2}^n f^{-m_i}(U_i) \neq \emptyset \quad \text{and} \quad V \cap A \cap \bigcap_{i=2}^n f^{-m_i}(V) \neq \emptyset.
\]

And so there is a point \(y \in A\) such that \(\{y, f^{m_2}(y), \ldots, f^{m_n}(y)\} \subset V\). By Lemma 2.6, \(\text{Tran}(X,f) \cap A\) is a dense \(G_\delta\) subset of \(A\), then we can choose \(x \in \text{Tran}(X,f) \cap U_1 \cap A \cap \bigcap_{i=2}^n f^{-m_i}(U_i)\), that is \(x \in U_1\) and \(f^{m_i}(x) \in U_i\) for \(i = 2, \ldots, n\). As \(x\) is a transitive point in \((X,f)\) and observe that the space \(X\) is perfect, there exists \(p \in \mathbb{N}_0\) such that \(\{f^p(x), f^{p+m_2}(x), \ldots, f^{p+m_n}(x)\} \subset V\) and \(q > p + k\) such that \(f^q(x) \in U_1\) and \(f^{q+m_i}(x) \in U_i\) for \(i = 2, \ldots, n\). Let \(r = q - p - k\), \(y_1 = f^{p+k}(x)\) and \(y_i = f^{p+k+m_i}(x)\) for \(i = 2, \ldots, n\). Then \(y_i \in U\) and \(f^r(y_i) \in U_i\) for \(i = 1, 2, \ldots, n\), which implies that \(A\) is a sensitive set. \(\square\)

Combining Theorem 3.7 with Lemma 2.6, we obtain:

**Corollary 3.8.** Let \((X, f)\) be a dynamical system and \(A \subset X\) a weakly mixing set of order 2. Then \((\text{Orb}(A,f), f)\) is \(n\)-sensitive for every \(n \geq 2\).

4. **Xiong chaotic set of finite order**

In this section, we study Xiong chaotic sets of finite order and its connection with weakly mixing sets of finite order.

**Definition 4.1.** Let \((X, f)\) be a dynamical system and \(n \geq 2\). A subset \(K\) of \(X\) with at least \(n\) points is called a Xiong chaotic set of order \(n\) if for any subset \(E\) of \(K\) of cardinality \(n\) and for any map \(g : E \rightarrow \overline{K}\) there is an increasing subsequence \(\{q_i\}_{i=1}^\infty\) in \(\mathbb{N}\) such that \(\lim_{i \to \infty} f^{q_i}(x) = g(x)\) for every \(x \in E\).

The following result is straightforward by the definition.

**Proposition 4.2.** If \(K\) is a Xiong chaotic set of order \(n\), then there exists \(\delta > 0\) such that for every \(n\) distinct points \(x_1, x_2, \ldots, x_n\) in \(K\),

\[
\lim_{k \to \infty} \liminf_{1 \leq i < j \leq n} \frac{d(f^{k}(x_i), f^{k}(x_j))}{d(f^{k}(x_i), f^{k}(x_j))} = 0,
\]

\[
\lim_{k \to \infty} \limsup_{1 \leq i < j \leq n} \frac{d(f^{k}(x_i), f^{k}(x_j))}{d(f^{k}(x_i), f^{k}(x_j))} > \delta,
\]

and

\[
\lim_{k \to \infty} \liminf_{1 \leq i \leq n} d(f^{k}(x_i), x_i) = 0.
\]

In particular, \(K\) is \(n\)-scrambled with modular \(\delta\).

**Theorem 4.3.** Let \((X, f)\) be a dynamical system, \(A\) a perfect subset of \(X\) and \(n \geq 2\). Then the following conditions are equivalent:

(4.3.1) \(A\) is a weakly mixing set of order \(n\);

(4.3.2) there exists a dense Mycielski subset \(S\) of \(A\) which is Xiong chaotic of order \(n\);
(4.3.3) there exists a dense subset \( S \) of \( A \) which is Xiong chaotic of order \( n \).

Proof. \((4.3.1) \Rightarrow (4.3.2)\) First note that \( A \) is perfect. Since \( A^n \) is a transitive set in \( (X^n, f^{(n)}) \), by Lemma 2.6 the relation \( R = A^n \cap \text{Tran} (\text{Orb}(A^n, f^{(n)})) \) is a dense \( G_\delta \) subset of \( A^n \). By the Mycielski Theorem, there exists a dense Mycielski subset \( S \) of \( A \) such that for every \( n \) distinct points \( x_1, x_2, \ldots, x_n \in S \), we have \( (x_1, x_2, \ldots, x_n) \in R \). Fix a subset \( E \) of \( S \) with cardinality \( n \) and a map \( g : E \to A \). Enumerate \( E \) as \( \{x_1, x_2, \ldots, x_n\} \) and let \( y_i = g(x_i) \) for \( i = 1, 2, \ldots, n \). As \( (x_1, x_2, \ldots, x_n) \) is a transitive point in \( (\text{Orb}(A^n, f^{(n)}), f^{(n)}) \) and observe \( (y_1, y_2, \ldots, y_n) \in A^n \), there is an increasing subsequence \( \{q_k\}_{k=1}^\infty \in \mathbb{N} \) such that \( \lim_{k \to \infty} f^{q_k}(x_i) = g(x_i) \) for \( i = 1, 2, \ldots, n \), thus \( S \) is a Xiong chaotic set of order \( n \).

(4.3.2) \( \Rightarrow (4.3.3) \) The implication is trivial.

(4.3.3) \( \Rightarrow (4.3.1) \) Fix any open subsets \( U_1, V_1, U_2, V_2, \ldots, U_n, V_n \) of \( X \) intersecting \( A \). Choose \( n \) distinct points \( x_i \in U_i \cap S \) and \( n \) points \( y_i \in V_i \cap A \) for \( i = 1, 2, \ldots, n \). Define a map \( g : \{x_1, x_2, \ldots, x_n\} \to A \) as \( g(x_i) = y_i \) for \( i = 1, 2, \ldots, n \). Then there exists \( k \geq 1 \) such that \( f^k(x_i) \in V_i \) for \( i = 1, 2, \ldots, n \). In particular the set \( \bigcap_{i=1}^n N(U_i \cap A, V_i) \) is not empty, which completes the proof.

\[\square\]

Corollary 4.4. Let \((X, f)\) be a dynamical system and \( n \geq 2 \). Then \((X, f)\) has a weakly mixing set of order \( n \) if and only if it has an uncountable Xiong chaotic set of order \( n \).

Proof. The necessity follows by Theorem 4.3 since a Mycielski set is uncountable. Now we prove the sufficiency. Let \( S \) be an uncountable Xiong chaotic set of order \( n \). By compactness of \( X \), we can divide the closure \( \bar{S} \) of \( S \) into \( K_1 \cup K_2 \), where \( K_1 \) is perfect and \( K_2 \) is at most countable. It is easy to see that \( K_1 \cap S \) is a Xiong chaotic set which is dense in \( K_1 \). By Theorem 4.3, \( K_1 \) is a weakly mixing set of order \( n \).

\[\square\]

Remark 4.5. It should be noticed that weakly mixing sets (of finite order \( n \)) are perfect. And hence it is more restrained than Xiong chaotic sets, because any infinite subsets of Xiong chaotic sets (of finite order \( n \)) are also Xiong chaotic sets.

Let \((X, f)\) be a dynamical system with \( x_0 \in X \), \( n \geq 2 \) and \( \delta > 0 \). Denote
\[D_n^\delta(X, f) = \left\{ (x_1, x_2, \ldots, x_n) \in X^n : \limsup_{k \to \infty} \min_{1 \leq i < j \leq n} d(f^k(x_i), f^k(x_j)) \geq \delta \right\},\]
and
\[D_n^\delta(X, f)(x_0) = \left\{ (x_1, \ldots, x_{n-1}) \in X^{n-1} : (x_0, x_1, \ldots, x_{n-1}) \in D_n^\delta(X, f) \right\}.

Proposition 4.6. Let \((X, f)\) be a dynamical system and \( A \subset X \) a weakly mixing set of order \( 2(n-1) \) with \( n \geq 2 \). Then there exists \( \delta > 0 \) such that for every \( x_0 \in A \), \( D_n^\delta(X, f)(x_0) \cap A^{n-1} \) is residual in \( A^{n-1} \).

Proof. Since \( A \) is perfect, pick \( 2(n-1) \) distinct points \( u_{1,1}, u_{1,2}, \ldots, u_{n-1,1}, u_{n-1,2} \in A \) and \( \delta > 0 \) such that \( d(u_{i_1, j_2}, u_{j_1, j_2}) > 4\delta \) for \((i_1,j_2) \neq (j_1,j_2)\). Fix \( x_0 \in A \). For every \( \varepsilon > 0 \), put
\[D_\varepsilon = \left\{ (x_1, \ldots, x_{n-1}) \in X^{n-1} : \min_{0 \leq i < j \leq n-1} d(f^k(x_i), f^k(x_j)) > \delta - \varepsilon \text{ for some } k > \frac{1}{\varepsilon} \right\}.

It is easy to verify that \( D_\varepsilon \) is an open subset of \( X^{n-1} \) and \( D_n^\delta(X, f)(x_0) = \bigcap_{m=1}^\infty D_\frac{1}{m} \).
Then it is sufficient to prove $D_\varepsilon \cap A^{n-1}$ is dense in $A^{n-1}$ for every $\varepsilon > 0$.

Fix $\varepsilon > 0$ and $n - 1$ open subsets $U_1, \ldots, U_{n-1}$ of $X$ intersecting $A$. By Theorem 4.3, there is a Xiong chaotic set $S$ of order $(n - 1)$ which is dense in $A$. Observing that $A$ is perfect, for each $i = 1, \ldots, n - 1$, choose $y_{i,1}, y_{i,2} \in U_i \cap S$ with $y_{i,1} \neq y_{i,2}$. Define a map $g : \{y_{1,1}, y_{1,2}, \ldots, y_{n-1,1}, y_{n-1,2}\} \to A$ as $g(y_{i,j}) = u_{i,j}$ for $i = 1, \ldots, n - 1$ and $j = 1, 2$. Then there is an increasing subsequence $\{q_l\}_{l=1}^\infty$ in $\mathbb{N}$ such that $\lim_{l \to \infty} f^{q_l}(y_{i,j}) = g(y_{i,j}) = u_{i,j}$ for $i = 1, \ldots, n - 1$ and $j = 1, 2$. Pick $k > \frac{1}{\varepsilon}$ such that $d(f^j(y_{i,j}), u_{i,j}) < \delta$ for $i = 1, \ldots, n - 1$ and $j = 1, 2$. There is at most one pair $(i_0, j_0)$ such that $d(f^k(x_0), u_{i_0,j_0}) < 2\delta$. For each $i = 1, \ldots, n - 1$, if $i \neq i_0$, let $x_i = y_{i,1}$; if $i = i_0$, let $x_i = y_{i_0,j_0}$, where $j_0 \in \{1,2\}$ and $j_0 \neq j_0$. Then $x_i \in U_i$ for $i = 1, \ldots, n - 1$ and

$$\min_{0 \leq i < j \leq n - 1} d(f^j(x_i), f^j(x_j)) > \delta,$$

which implies that $D_\varepsilon \cap A^{n-1}$ is dense in $A^{n-1}$. \hfill \Box

5. Weakly mixing sets

5.1. Weakly mixing sets. By Proposition 3.6, we have the following result.

**Lemma 5.1.** Let $(X, f)$ be a dynamical system, $A \subset X$ a nontrivial closed subset and $n \geq 2$. Then $A$ is weakly mixing for $f$ if and only if it is weakly mixing for $f^n$.

A dynamical system $(X, f)$ is called an $F$-system if it is totally transitive and has a dense set of periodic points. It is shown in [Fur67] that an $F$-system is disjoint from any minimal system. It is not hard to see that every $F$-system is weakly mixing (see [Ban97] Theorem 1.1)). We say a dynamical system $(X, f)$ has dense small periodic sets if for any nonempty open subset $U$ of $X$ there exists a nonempty closed subset $K$ of $U$ and $k \in \mathbb{N}$ such that $f^k(K) \subset K$. A dynamical system $(X, f)$ is called an $HY$-system if it is totally transitive and has dense small periodic sets. It is shown in [HY05] that an $HY$-system is weakly mixing and disjoint from any minimal system.

It is interesting that when a totally transitive set or a weakly mixing set of finite order is also a weak mixing set. Recall that a point $x \in X$ is distal provided that if $(x, y)$ is proximal and $y \in \text{Orb}(x, f)$ then $x = y$. The following fact is [OZ13] Corollary 11.

**Theorem 5.2.** Let $(X, f)$ be a dynamical system and $A \subset X$ a weakly mixing set of order 2. If the set of all distal points in $A$ is dense in $A$, then $A$ is weakly mixing of all orders.

We show that Theorem 5.2 can be generalized in the following way.

**Theorem 5.3.** Let $(X, f)$ be a dynamical system and $A \subset X$ a weakly mixing set of order 2. If for every open subset $U$ of $X$ intersecting $A$, there is a dynamical system $(Y, g)$ with a distal point $y \in Y$ and an open neighborhood $V \subset Y$ of $y$ and a point $x \in A \cap U$ such that $N(y, V) \subset N(x, U)$, then $A$ is weakly mixing of all orders.

**Proof.** By Theorem 3.1, it is sufficient to show that for any $n \geq 2$ and any $n + 1$ open subsets $U_1, V_1, V_2, \ldots, V_n$ of $X$ intersecting $A$,

$$N(U_1 \cap A, V_1) \cap \bigcap_{i=2}^n N(V_i \cap A, V_i) \neq \emptyset.$$
By assumptions, for $i = 2, \ldots, n$ there are points $x_i \in V_i \cap A$ and distal points $y_i$ (in some dynamical systems) and their open neighborhoods $W_i$ such that $N(y_i, W_i) \subset N(x_i, V_i)$ and hence also $\bigcap_{i=2}^{n} N(y_i, W_i) \subset \bigcap_{i=2}^{n} N(x_i, V_i)$. But the product of distal points is also distal, thus by [Fur81] the following set intersects every IP-set:

$$N((y_2, \ldots, y_n), W_2 \times \ldots \times W_n) \subset \bigcap_{i=2}^{n} N(y_i, W_i) \subset \bigcap_{i=2}^{n} N(x_i, V_i).$$

But by Corollary 3.5 the set $N(U_1 \cap A, V_1)$ contains an IP-set. The proof is finished. □

We say that a subset $A$ of $X$ has dense small periodic sets if for any open subset $U$ of $X$ intersecting $A$, there exists a closed subset $K$ of $U$ intersecting $A$ and $k \in \mathbb{N}$ such that $f^k(K) \subset K$. Then we have (observe that weak mixing of order 2 implies total transitivity):

**Theorem 5.4.** Let $(X, f)$ be a dynamical system and $A \subset X$ a nontrivial closed subset. If $A$ is totally transitive and has dense small periodic sets, then $A$ is weakly mixing.

**Proof.** First we show that $A$ is weakly mixing of order 2. Let $U_1, V_1, V_2$ be open subsets of $X$ intersecting $A$. Since $A$ has dense small periodic sets, there exists a closed subset $K$ of $V_2$ intersecting $A$ and $k \in \mathbb{N}$ such that $f^k(K) \subset K$. Since $A$ is transitive for $f^k$, there is $m \in \mathbb{N}$ such that $m \in N_{f^k}(U_1 \cap A, V_1)$. Then $km \in N(U_1 \cap A, V_1) \cap N(V_2 \cap A, V_2)$, which implies that $A$ is weakly mixing of order 2 by Theorem 3.1.

Now we show that $A$ satisfies the requirement of Theorem 5.3. Fix an open subset $U$ of $X$ intersecting $A$. There exists a closed subset $S$ of $U$ intersecting $A$ and $k \in \mathbb{N}$ such that $f^k(S) \subset S$. Pick a point $x \in S \cap A$. Then $k\mathbb{N} \subset N(x, U)$. Let $Y = \{0, 1, \ldots, k-1\}$ and $g : Y \to Y, g(i) = i+1 \pmod{k}$. Let $y = 0$ and $V = \{0\}$. Then $y$ is a distal point in $(Y, g)$ and $N(y, V) = k\mathbb{N} \subset N(x, U)$. Hence $A$ is weakly mixing of all orders by Theorem 5.3. □

5.2. **Proximal relations.** It is shown in [AK03] that if $(X, f)$ is weakly mixing, then for every $x \in X$, the set $\text{Prox}_2(f)(x)$ is residual in $X$. In [OZ13] it was proved that for every weakly mixing set $A$ and every $x \in A$, the set $\text{Prox}_2(f)(x) \cap A$ is residual in $A$. We will show that the same is true if we consider proximal tuples instead of pairs. First, we use a method of construction from [OZ13] Lemma 16) to prove the following result.

**Lemma 5.5.** Let $(X, f)$ be a dynamical system and $A \subset X$ a weakly mixing set with $x \in A$. Then for every $n \geq 2$, every open subsets $U_1, U_2, \ldots, U_n$ of $X$ intersecting $A$ and each $\varepsilon > 0$, there are $y_i \in U_i \cap A$ for $i = 1, 2, \ldots, n$ and $m \in \mathbb{N}$ such that (we put $y_0 = x$)

$$\max_{0 \leq i < j \leq n} d(f^m(y_i), f^m(y_j)) \leq \varepsilon.$$

**Proof.** Fix $n \geq 2$, open subsets $U_1, U_2, \ldots, U_n$ of $X$ intersecting $A$ and $\varepsilon > 0$. Let $\{V_1, \ldots, V_k\}$ be a cover of $X$ consisting of open sets with diameters less than $\varepsilon/2$.

There exists $x_1 \in \{1, 2, \ldots, k\}$ with $x \in V_{x_1}$. By weak mixing of $A$, there exist $m_1 > 0$ and an open set $U_i^{(1,1)} \subset U_i$ intersecting $A$ such that $f^{m_1}(U_i^{(1,1)}) \subset V_{x_1}$ for $i = 1, 2, \ldots, n$.

Assume for some $q \geq 1$ we have constructed open sets $U_i^{(q,1)}, U_i^{(q,2)}, \ldots, U_i^{(q,q)} \subset U_i$ intersecting $A$ for $i = 1, 2, \ldots, n$, an integer $m_q > 0$ and pairwise distinct integers $s_1, s_2, \ldots, s_q \subset \{1, 2, \ldots, k\}$ such that

$$f^{m_q}(U_i^{(q,r)}) \subset V_{s_r} \text{ for } r = 1, 2, \ldots, q, i = 1, 2, \ldots, n.$$
If \( f^{m_q}(x) \not\in \bigcup_{r=1}^{q} V_r \), then we can choose \( s_{q+1} \in \{1, 2, \ldots, k\} \setminus \{s_1, s_2, \ldots, s_q\} \) and an open set \( U^{(q,q+1)} \) containing \( x \) (and so intersecting \( A \)) such that \( f^{m_q}(U^{(q,q+1)}) \subset V_{q+1} \). By weak mixing of \( A \), there are open sets \( U_i^{(q+1,1)}, U_i^{(q+1,2)}, \ldots, U_i^{(q+1,q+1)} \subset U_i \) intersecting \( A \) and \( p > 0 \) such that \( f^p(U_i^{(q+1,r)}) \subset U_i^{(q,r)} \) for \( r = 1, 2, \ldots, q + 1 \), \( i = 1, 2, \ldots, n \), where \( U_i^{(q,q+1)} = U(q,q+1) \) for \( i = 1, 2, \ldots, n \). Now if we put \( m_{q+1} = m_q + p \), then for \( r = 1, 2, \ldots, q + 1 \) and \( i = 1, 2, \ldots, n \)
\[
f^{m+1}(U_i^{(q+1,r)}) = f^{m_q}(f^p(U_i^{(q+1,r)})) \subset f^{m_q}(U_i^{(q,r)}) \subset V_s.
\]
Obviously at most step \( q \leq k \), we cannot extend the sequence \( s_1, s_2, \ldots, s_q \) any further by the above procedure. Hence, we have that \( f^{m_q}(x) \in \bigcup_{s=1}^{q} V_s \), in particular \( f^{m_q}(x) \in V_s \) for some \( s \in \{s_1, \ldots, s_q\} \). But then by the construction, for \( i = 1, 2, \ldots, n \), \( f^{m_q}(U_i^{(q,\ell)}) \subset V_s \), and so if we fix any \( y_i \in U_i^{(q,\ell)} \cap A \subset U_i \cap A \) then \( f^{m_q}(y_i) \in V_s \), finishing the proof.

Then we obtain:

**Theorem 5.6.** Let \((X, f)\) be a dynamical system and \( A \subset X \) a weakly mixing set. Then for every \( x_0 \in A \) and \( n \geq 2 \), the set \( \text{Prox}_n(f)(x_0) \cap A^{n-1} \) is residual in \( A^{n-1} \).

**Proof.** Fix any \( x_0 \in A \) and \( n \geq 2 \). For every \( \varepsilon > 0 \), put
\[
P_\varepsilon = \left\{ (x_1, \ldots, x_{n-1}) \in X^{n-1} : \max_{0 \leq i < j \leq n-1} d(f^k(x_i), f^k(x_j)) < \varepsilon \text{ for some } k \geq 0 \right\}.
\]
It is easy to verify that \( P_\varepsilon \) is an open subset of \( X^{n-1} \). By Lemma 5.5, \( P_\varepsilon \cap A^{n-1} \) is dense in \( A^{n-1} \). This, by the fact
\[
\text{Prox}_n(f)(x_0) = \bigcap_{m=1}^{\infty} P_{\frac{1}{m}},
\]
proves that \( \text{Prox}_n(f)(x_0) \cap A^{n-1} \) is residual in \( A^{n-1} \). \( \square \)

Let \((X, f)\) be a dynamical system with \( x_0 \in X \), \( n \geq 2 \) and \( \delta > 0 \). Denote
\[
LY_n^\delta(X,f) = \left\{ (x_1, x_2, \ldots, x_n) \in X^n : (x_1, x_2, \ldots, x_n) \text{ is } n\text{-scrambled with modular } \delta \right\},
\]
and
\[
LY_n^\delta(X,f)(x_0) = \left\{ (x_1, \ldots, x_{n-1}) \in X^{n-1} : (x_0, x_1, \ldots, x_{n-1}) \in LY_n^\delta(X,f) \right\}.
\]
As a direct corollary of Proposition 4.6 and Theorem 5.6 we obtain the following fact.

**Theorem 5.7.** Let \((X, f)\) be a dynamical system and \( A \subset X \) a weakly mixing set. Then for every \( n \geq 2 \), there exists \( \delta > 0 \) such that for every \( x \in A \), \( LY_n^\delta(X,f)(x) \cap A^{n-1} \) is residual in \( A^{n-1} \).

### 5.3. Local independent sets.
Let \((X, f)\) be a dynamical system. Following [KL07], for a tuple \( A = (A_1, A_2, \ldots, A_k) \) of subsets of \( X \), we say that a nonempty subset \( F \subset \mathbb{N}_0 \) is an independence set for \( A \) if for any nonempty finite subset \( J \subset F \), we have
\[
\bigcap_{j \in J} f^{-i}(A_{s(j)}) \neq \emptyset
\]
for any \( s \in \{1, \ldots, k\}^J \). We shall denote the collection of all independence sets for \( A \) by \( \text{Ind}(A_1, A_2, \ldots, A_k) \) or \( \text{Ind}A \). According to the best knowledge of the authors, the above
mentioned notion of independence sets was first presented in [HY06] under the name interpolating set (see also [GW95]), and in [H06] when defining strong scrambled pairs. Later, the authors of [HLY12] systematically studied independence sets in topological and measurable dynamics. In particular, they proved the following result (see [HLY12] Theorem 5.1).

**Theorem 5.8.** For a dynamical system \((X,f)\), the following conditions are equivalent:

1. \((5.8.1)\) \((X,f)\) is weakly mixing;
2. \((5.8.2)\) for any two nonempty open subsets \(U_1,U_2\) of \(X\), \(\text{Ind}(U_1,U_2)\) contains an infinite set;
3. \((5.8.3)\) for any \(n \in \mathbb{N}\) and nonempty open subsets \(U_1,U_2,\ldots,U_n\) of \(X\), there is an IP set in \(\text{Ind}(U_1,U_2,\ldots,U_n)\).

In the spirit of [HLY12] we introduce a local definition of independence sets as follows.

**Definition 5.9.** Let \((X,f)\) be a dynamical system and \(\emptyset \neq A \subset X\). Let \(U_1,U_2,\ldots,U_n\) be open subsets of \(X\) intersecting \(A\). We say that a nonempty subset \(F\) of \(X\) is an independence set for \((U_1,U_2,\ldots,U_n)\) with respect to \(A\), if for every nonempty finite subset \(J \subset F\), and \(s \in \{1,2,\ldots,n\}^J\),

\[ \bigcap_{j \in J} f^{-j}(U_{s(j)}) \]

is a nonempty open subset of \(X\) intersecting \(A\).

Now we can employ introduced notion, to state a theorem analogous to Theorem 5.8.

**Theorem 5.10.** Let \((X,f)\) be a dynamical system and \(A \subset X\) a nontrivial closed set. Then the following conditions are equivalent:

1. \((5.10.1)\) \(A\) is a weakly mixing set;
2. \((5.10.2)\) for every \(n \geq 2\) and every open subsets \(U_1,U_2,\ldots,U_n\) of \(X\) intersecting \(A\), there exists \(t \in \mathbb{N}\) such that \(\{0,t\}\) is an independence set for \((U_1,U_2,\ldots,U_n)\) with respect to \(A\);
3. \((5.10.3)\) for every \(n \geq 2\) and every open subsets \(U_1,U_2,\ldots,U_n\) of \(X\) intersecting \(A\), there exists a sequence \(\{t_j\}_{j=1}^{\infty}\) in \(\mathbb{N}\) such that \(\{0\} \cup F S(\{t_j\}_{j=1}^{\infty})\) is an independence set for \((U_1,U_2,\ldots,U_n)\) with respect to \(A\).

**Proof.** \((5.10.2) \Rightarrow (5.10.1)\) Fix \(n \geq 2\) and open subsets \(U_1,U_2,\ldots,U_n, V_1,V_2,\ldots,V_n\) of \(X\) intersecting \(A\). By assumption there exists \(t \geq 1\) such that \(\{0,t\}\) is an independence set for \((U_1,U_2,\ldots,U_n, V_1,V_2,\ldots,V_n)\) with respect to \(A\). For \(i = 1,2,\ldots,n\), \(U_i \cap f^{-t}(V_i)\) is a nonempty open subset of \(X\) intersecting \(A\). Then \(t \in \bigcap_{i=1}^{n} N(U_i \cap A, V_i)\), which implies that \(A\) is weakly mixing of order \(n\).

\((5.10.1) \Rightarrow (5.10.3)\) Let \(U_1,U_2,\ldots,U_n\) be open subsets of \(X\) intersecting \(A\). First, there exists \(t_1 \in \mathbb{N}\) such that

\[ t_1 \in \bigcap_{i_1,i_2 \in \{1,2,\ldots,n\}} N(U_{i_1} \cap A, U_{i_2}). \]

That is for every \(i_1,i_2 \in \{1,2,\ldots,n\}\), \(U_{i_1} \cap f^{-t_1}(U_{i_2})\) is a nonempty open set intersecting \(A\). And then, there exists \(t_2 \in \mathbb{N}\) such that

\[ t_2 \in \bigcap_{i_1,i_2,i_3,i_4 \in \{1,2,\ldots,n\}} N(U_{i_1} \cap f^{-t_1}(U_{i_2}) \cap A, U_{i_3} \cap f^{-t_1}(U_{i_4})). \]
That is, for every \( i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, n\} \),
\[
U_{i_1} \cap f^{-t_1}(U_{i_2}) \cap f^{-t_2}U_{i_3} \cap f^{-(t_1+t_2)}(U_{i_4})
\]
is a nonempty open set of \( X \) intersecting \( A \). Then \( \{0, t_1, t_2, t_1 + t_2\} \) is an independent set of \((U_1, U_2, \ldots, U_n)\) with respect to \( A \) and the result follows by induction. 

\[ 5.10.3 \Rightarrow 5.10.2 \] The implication is trivial. \( \square \)

**Remark 5.11.** Let \( A = (A_1, \ldots, A_k) \) be a tuple of subsets of \( X \). If \( F \) is an independence set for \( A \), then for every \( m \in \mathbb{N}_0 \), \( F - m \), the subset \( \{n - m : n \geq m \text{ and } n \in F\} \), is also an independence set for \( A \). So we may also assume that an independence set of \( A \) containing \( 0 \). But in Theorem 5.10 we cannot replace \( \{0\} \cup FS(\{t_j\}^\infty_{j=1}) \) by \( FS(\{t_j\}^\infty_{j=1}) \) as shown by the following example.

**Example 5.12.** Consider \( X = \Sigma_3^+ = \{0, 1, 2\}^{\mathbb{N}_0} \) and denote
\[
A = \{1, 2\}^{\mathbb{N}_0} \cup C_X[00].
\]
For every open subsets \( U_1, \ldots, U_n \) of \( X \) intersecting \( A \), we can easily define words \( w^{(1)}, \ldots, w^{(n)} \) of the same length \( M \geq 2 \) (with symbols in the alphabet \( \{0, 1, 2\} \) such that \( C[w^{(0)}] \subset A \) for all \( i = 1, \ldots, n \). This easily implies that the set \( J = \{kM : k \in \mathbb{N}\} \) is an independence set for \((U_1, U_2, \ldots, U_n)\) with respect to \( A \). But \( A \) is not weakly mixing of order 2, because for example \( N(A \cap C_X[1], C_X[0]) = \emptyset \).

**Question 2.** Let \((X, f)\) be a dynamical system and \( A \subset X \) a weakly mixing set of order 2. Is it true that for every two open subsets \( U_1, U_2 \) of \( X \) intersecting \( A \), there exists \( t \in \mathbb{N} \) such that \( \{0, t\} \) is an independence set for \((U_1, U_2)\) with respect to \( A \)?

6. **Topological graphs**

For any integer \( n \geq 2 \), it is known that a weakly mixing set of order \( n \) does not have to be weakly mixing of order \( n+1 \), or even worse, it may happen that there is no a weakly mixing set of order \( n+1 \) in the system with weakly mixing sets of order \( n \) \([OZ14]\). Note that examples in \([OZ14]\) are subshifts, and for every dynamical system on the unit interval \(([0, 1], f)\) with positive topological entropy there is an \( m > 0 \) and a closed set \( A \) invariant for \( f^m \) such that \((A, f^m)\) is conjugated with the full shift on two symbols. In particular, in every interval map with weakly mixing sets we can find sets which are weakly mixing of order \( n \) but not \( n+1 \). However, in \([OZ11]\) the authors proved that on the unit interval, every weakly mixing set of order 2 is arbitrarily close (in the Hausdorff metric) to a weakly mixing set of all orders. So even if these sets are not the same, they are present arbitrarily close to each other. Theorem 6.1 completes our research on weakly mixing sets in dimension one, showing that the above fact also holds for all topological graphs.

**Theorem 6.1.** Let \((G, f)\) be a dynamical system acting on the topological graph \( G \) and let \( A \subset G \) be a weakly mixing subset of order 2. Then for every \( \varepsilon > 0 \) there is a weakly mixing subset \( D \subset G \) such that \( \mathcal{H}_d(A, D) \leq \varepsilon \), where \( \mathcal{H}_d(A, D) \) denotes the Hausdorff distance between \( A \) and \( D \).

**Proof.** Let \( \varepsilon > 0 \). Pick nonempty open subsets \( U_1, \ldots, U_s \) of \( G \) with diameters at most \( \varepsilon \) such that \( A \subset \bigcup_{i=1}^s U_i \) and \( A \cap U_i \neq \emptyset \) for \( i = 1, \ldots, s \). By Lemma 2.5 the set \( A \) is perfect, therefore for every \( i = 1, \ldots, s \) it is possible to select an open set \( V_i \subset U_i \) contained in the interior of an edge of \( G \) such that \( V_i \cap A \neq \emptyset \) and \( V_1, \ldots, V_s \) are pairwise disjoint.
Claim 1. For every \( i = 1, \ldots, s \) there is an interval \( I_i \subset V_i \), its disjoint closed subintervals \( K_{2i}, K_{2i+1} \) and an integer \( n_i > 0 \) such that

\[
(6.1.1) \quad K_{2i}, K_{2i+1} \text{ form a strong 2-horseshoe for } f^{n_i}, \text{ that is, } K_p \xrightarrow{f^{n_i}} K_q \text{ for all } p, q \in \{2i, 2i+1\}, \quad \text{and}
\]

\[
(6.1.2) \quad \text{both sets } \text{int}(K_{2i}), \text{int}(K_{2i+1}) \text{ as well as every connected component of the set } I_i \setminus (K_{2i} \cup K_{2i+1}) \text{ intersect } A.
\]

Proof of Claim. Let \( I_i \) be any closed interval contained in \( V_i \) such that \( \text{int}(I_i) \cap A = \emptyset \). Let us identify \( I_i \) with \([0, 1]\). Observe that \( A \) is a weakly mixing set of order 2, and so it contains no isolated points. Thus there are points \( 0 = a_0 < a_1 < \cdots < a_6 < a_7 = 1 \) in \( I_i \) such that \((a_j, a_{j+1}) \cap A = \emptyset \) for all \( j = 0, \ldots, 6 \). Denote \( I_{i,j} = [a_{2j}, a_{2j+1}) \) for \( j = 0, 1, 2, 3 \). Since \( A \) is weakly mixing of order 2, there are \( k > 0, r > 0 \) such that

\[
f^r(I_{i,1}) \cap (a_1, a_2) \neq \emptyset, \quad f^k(I_{i,2}) \cap (a_1, a_2) \neq \emptyset,
\]

\[
f^r(I_{i,1}) \cap (a_5, a_6) \neq \emptyset, \quad f^k(I_{i,2}) \cap (a_5, a_6) \neq \emptyset.
\]

If \( I_{i,1} \xrightarrow{f^r} I_{i,1}, I_{i,1} \xrightarrow{f^r} I_{i,2} \) and \( I_{i,2} \xrightarrow{f^k} I_{i,1}, I_{i,2} \xrightarrow{f^k} I_{i,2} \) then by Lemma 2.7 intervals \( I_{i,1} \) and \( I_{i,2} \) form a 2-horseshoe for \( f^{k+r} \).

Otherwise there are \( p \in \{1, 2\} \) and \( j \in \{k, r\} \) such that \( I_{i,0} \xrightarrow{f^j} I_{i,0} \) and \( I_{i,p} \xrightarrow{f^j} I_{i,3} \). Next, if we consider \( I_{i,0} \), then there is \( l > 0 \) such that \( I_{i,0} \xrightarrow{f^l} I_{i,0} \) and \( I_{i,0} \xrightarrow{f^l} I_{i,3} \), or we have the second possibility that \( I_{i,0} \xrightarrow{f^l} I_{i,1} \) and \( I_{i,0} \xrightarrow{f^l} I_{i,2} \) which implies that \( I_{i,0} \xrightarrow{f^{l+j}} I_{i,0} \) and \( I_{i,0} \xrightarrow{f^{l+j}} I_{i,3} \) again by applying Lemma 2.7. We can repeat the same arguments for \( I_{i,3} \), and with the help of Lemma 2.7 finally obtain that \( I_{i,0}, I_{i,3} \) form a 2-horseshoe for some iterate of \( f \). Indeed the claim holds. \( \square \)

Now for each \( i = 1, \ldots, s \) let sets \( K_{2i}, K_{2i+1} \) be provided for \( V_i \) by Claim 1, and let \( J_i \) be the connected component of \( I_i \setminus (K_{2i} \cup K_{2i+1}) \) such that \( K_{2i} \) and \( K_{2i+1} \) are contained in different connected components of \( I_i \setminus J_i \). We prove by induction that for every \( m = 1, \ldots, s \) intervals \( K_2, \ldots, K_{2m+1} \) form a horseshoe for some iterate \( f^h, h \in \mathbb{N} \), that is, \( K_p \xrightarrow{f^h} K_q \) for all \( p, q \in \{2, \ldots, 2m+1\} \).

By the construction we have proved the above statement for \( m = 1 \), so we may assume that it holds for some \( 1 \leq m < s \). There is \( t > 0 \) such that \( K_p \xrightarrow{f^t} K_q \) for \( 1 < p, q \leq 2m+1 \) and we may also assume (replacing \( t \) and \( n_i \) by their multiply) that \( K_{2m+2}, K_{2m+3} \) form a 2-horseshoe for \( f^t \). Since \( A \) is weakly mixing of order 2, there are \( k, r > 0 \) such that

\[
f^r(K_2) \cap J_1 \neq \emptyset, \quad f^r(K_2) \cap J_{m+1} \neq \emptyset,
\]

\[
f^k(K_{2m+3}) \cap J_1 \neq \emptyset, \quad f^k(K_{2m+3}) \cap J_{m+1} \neq \emptyset.
\]

From the construction and the first two conditions we see that \( K_2 \xrightarrow{f^r} K_2 \) or \( K_2 \xrightarrow{f^r} K_3 \) and at the same time \( K_2 \xrightarrow{f^r} K_{2m+2} \) or \( K_2 \xrightarrow{f^r} K_{2m+3} \), which implies that \( K_2 \xrightarrow{f^{r+t}} K_q \) for every \( q = 2, \ldots, 2m+3 \) by Lemma 2.7. By a symmetric argument, we see that \( K_{2m+3} \xrightarrow{f^{r+t}} K_q \) for every \( q = 2, \ldots, 2m+3 \). Now applying Lemma 2.7 it is easy to verify that \( K_p \xrightarrow{f^{r+t+k}} K_q \) for every \( 2 \leq p, q \leq 2m+3 \). This completes the induction.
Since $K_i$, $i = 2, \ldots, 2s + 1$ form a horseshoe, rewriting arguments in the proof of [M11 Theorem 9] (stated there for horseshoes in interval maps) we obtain that there is $n > 0$, an $f^n$-invariant closed subset $\Gamma \subset \bigcup_{i=1}^5 V_i$ and a topological conjugacy $\pi: (\Gamma, f^n) \to (\Sigma_\epsilon, \sigma)$ between dynamical systems, such that $\Gamma \cap V_i \neq \emptyset$ for each $i = 1, \ldots, s$ (and hence $\mathcal{H}_d(\Lambda, \Gamma) \leq \epsilon$), where $(\Sigma_\epsilon, \sigma)$ is the full shift over the alphabet $\{1, \ldots, s\}$. In particular $f^n$ is mixing on $\Gamma$, so indeed $\Gamma$ is a weakly mixing subset, which completes the proof. \qed

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Appendix A.

Proof of Example 3.2. Let us endow $[0, 1]$ with the Euclidean metric. Take any increasing sequences $\{a_i\}_{i \in \mathbb{Z}}$, $\{b_i\}_{i \in \mathbb{Z}} \subset \mathbb{R} \setminus \mathbb{Q}$ such that $1/2 < a_{-1} < a_0 < b_0 < b_1 < a_1$ and $\lim_{i \to \infty} a_{-i} = \lim_{i \to \infty} b_{-i} = 0$ and $\lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i = 1$. Furthermore, we assume that every interval $(a_i, a_{i+1})$ contains at most one element of the set $\{2^{-k} : k \in \mathbb{N}\}$.

We will define homeomorphisms $F_i: [0, 1] \to [0, 1]$ for $i = 0, \ldots, 8$. Let $F_0 = \text{id}$ and put $F_i(0) = 0$ and $F_i(1) = 1$ for every $i$. For each $i \in \mathbb{Z}$ we put $F_1(a_i) = a_{i+1}$ and $F_3(b_i) = b_{i+1}$. On each interval $[b_i, b_{i+1}]$ we define $F_3$ as a linear map, which completes the definition of $F_3$, since values of $F_3$ at endpoints of every such interval have already been set. For $i \geq 0$ we define $F_1$ on $[a_i, a_{i+1}]$ as a linear map. Now, fix any sequence of distinct points $(c_k)_{k=1}^\infty \subset (a_0, a_1)$ in such a way that $\{c_k : k \in \mathbb{N}\} = [a_0, a_1]$. We are ready to define $F_1$ on intervals $[a_i, a_{i+1}]$ for $i < 0$. Suppose that $F_1|_{[a_{-n}, 1]}$ is already defined for some $n \geq 0$. If $\{2^{-k} : k \in \mathbb{N}\} \cap [a_{-n-1}, a_{-n}] = 0$ then we define $F_1$ as a linear map on $[a_{-n-1}, a_{-n}]$, and as a result $F_1$ is well defined on the interval $[a_{-n-1}, 1]$. Otherwise there is $k > 0$ such that $2^{-k} \in (a_{-n-1}, a_{-n})$ (hence $n \geq 1$). Denote

$$G: [a_{-n}, a_{-n+1}] \ni x \mapsto F_1|_{[a_{-n}, a_0]} \circ \cdots \circ F_1|_{[a_{-n}, a_{-n+1}]}(x) \in [a_0, a_1]$$

and observe that there is $q \in [a_{-n}, a_{-n+1}]$ such that $G(q) = c_k$. Now, we put $F_1(2^{-k}) = q$ and define $F_1$ to be linear on each of the intervals $[a_{-n-1}, 2^{-k}]$ and $[2^{-k}, a_{-n}]$. Then in this case $F_1$ is also well defined on $[a_{-n-1}, 1]$. Induction completes the construction. Denote inverses $F_2 = F_1^{-1}$ and $F_4 = F_3^{-1}$. Then for every $k$ there is $n > 0$ such that $F_2^n(c_k) = F_1^{-n}(c_k) = 2^{-k}$.

We define $F_3(x) = 1/2 + 1/2(2x-1)^3$ and $F_6 = F_5^{-1}$ as the inverse. Finally $F_7(2^{-k-1}) = 2^{-k}$ for $k \in \mathbb{N}$, $F_7(1/2) = a_1$ and $F_7(a_i) = a_{i+1}$ for $i \in \mathbb{N}$. Between any two consecutive points in the set $\bigcup_{k \in \mathbb{N}} \{2^{-k}, a_k\}$ the map $F_7$ is linear, which again gives a well-defined homeomorphism. As the last map we take $F_8 = F_7^{-1}$. Observe that for any $1/2 > \delta, \epsilon > 0$ there is $n > 0$ such that $F_2^n([1/2 - \delta, 1/2 + \delta]) \supset (\epsilon, 1 - \epsilon)$.

Let $X = \Sigma_+^+ \times [0, 1]$ (endowed with the product metric given by the maximum of distance on each coordinate) where $\Sigma_+^+ = \{0, 1, \ldots, 8\}^{\mathbb{N}_0}$ and let $T: X \to X$ be defined by

$$T(\omega, x) = (\sigma(\omega), F_0(\omega, x))$$
with $\sigma$ the standard shift transformation on $\Sigma_0^+$. Thus $X$ is compact and $T$ is continuous.

For any symbol $a \in \{0, 1, \ldots, 8\}$ let $\tilde{a}$ be replacement of $a$ by the second element of respective pair $(0, 0), (1, 2), (3, 4), (5, 6), (7, 8)$, for example $\tilde{8} = 7$. We extend this definition to words, putting $\tilde{w}_0 \cdots \tilde{w}_n = \tilde{w}_n \cdots \tilde{w}_0$. If for a finite sequence $w$ of symbols in $\{0, 1, \ldots, 8\}$ denote by $F_w$ the composition $F_{w} = F_{w_{|w|-1}} \circ \cdots \circ F_{w_{j+1}} \circ F_{w_{j}}$ then $F_{\tilde{w}_{|\tilde{w}|}} = F_{\tilde{w}} \circ F_{w} = \text{id}$.

Before proceeding with the construction of a set $A$, let us make a few observations on these maps $F_i$. Fix any nonempty open sets $U, V$ with $\mathcal{U}, \mathcal{V} \subset (0, 1)$. First of all $U \cap (b_i, b_{i+1}) \neq \emptyset$ for some $i \in \mathbb{Z}$, and hence there is a word $u$ consisting only of symbols 0, 3 or 4 and such that

$$F_{u}(U) \cap (a_0, a_1) \supset F_{u}(U) \cap (b_0, b_1) \neq \emptyset.$$  

In particular, there is $k \in \mathbb{N}$ such that $c_k \in F_{u}(U)$. But then there is also an $s > 0$ such that if we put $v = 2^s 7^{k-1}$ (i.e. $v$ is a concatenation of $s$ repetitions of symbol 2 and $k - 1$ repetitions of symbol 7) then $F_{v}(c_k) = F_{7}^{s-1}(F_{7}^{k-1}(c_k)) = F_{7}^{k-1}(2^{-k}) = 1/2$. In particular, there is $\delta > 0$ such that $[1/2 - \delta, 1/2 + \delta] \subset F_{u}(U)$, where as usual $uv = u_0u_1 \ldots u_{|u|-1}v_0v_1 \ldots v_{|v|-1}$. But then there are $\varepsilon > 0$ and $m > 0$ such that

$$F_{w}([1/2 - \delta, 1/2 + \delta]) \supset (\varepsilon, 1 - \varepsilon) \supset U \cup V$$

if we put $w = 6^m$. We have just shown that for any nonempty open sets $U, V$ with $\mathcal{U}, \mathcal{V} \subset (0, 1)$ there are words $u, v, w$ such that

$$(2) \quad F_{uvw}(U) \supset U \cup V.$$

Now we are ready for appropriate constructing set $A$. Let $\{\omega_i\}_{i=1}^{\infty}$ be any sequence containing all possible words (finite sequences) over symbols 0, 1, ..., 8. Let

$$\xi = \omega_1 \tilde{\omega}_1 \omega_2 \tilde{\omega}_2 \ldots \omega_t \tilde{\omega}_t \ldots \in \Sigma_0^+ \quad \text{and} \quad A = \{\xi\} \times [0, 1].$$

Take any nonempty open set $\tilde{U}, \tilde{V}$ intersecting $A$. Then there are $i > 0$ and open intervals $U, V$ such that $\mathcal{U}, \mathcal{V} \subset (0, 1)$ and

$$C[\omega_1 \tilde{\omega}_1 \ldots \omega_t \tilde{\omega}_t] \times U \subset \tilde{U} \quad \text{and} \quad C[\omega_1 \tilde{\omega}_1 \ldots \omega_t \tilde{\omega}_t] \times V \subset \tilde{V}.$$

Let words $u, v, w$ be provided for $U$ and $V$ by (2). By the definition, there is $j > 1$ such that $\omega_j = uvw \omega_1 \tilde{\omega}_1 \ldots \omega_t \tilde{\omega}_t$. Denote $t = \sum_{r=1}^{j-1} 2|\omega_r|$ and $p = t + |uvw|$. Note that

$$T^j(\{\xi\} \times U) = \{\sigma^j(\xi)\} \times F_{\omega_1 \tilde{\omega}_1 \ldots \omega_{j-1} \tilde{\omega}_{j-1}}(U) = \{\sigma^j(\xi)\} \times U$$

$$= \{\omega_j \tilde{\omega}_j \ldots \} \times U = \{uvw \omega_1 \tilde{\omega}_1 \omega_2 \tilde{\omega}_2 \ldots \omega_t \tilde{\omega}_t \ldots\} \times U$$

and therefore

$$T^p(\tilde{U} \cap A) \supset T^p(\{\xi\} \times U) \supset \{\omega_1 \tilde{\omega}_1 \ldots \omega_t \tilde{\omega}_t \ldots\} \times F_{uvw}(U)$$

$$\supset \{\omega_1 \tilde{\omega}_1 \ldots \omega_t \tilde{\omega}_t \ldots\} \times (U \cup V).$$

We have just shown that $p \in N(\tilde{U} \cap A, \tilde{U}) \cap N(\tilde{V} \cap A, \tilde{V})$.

Similarly, if in the above calculations $j$ was such that $\omega_j = \tilde{w}_1 \tilde{w}_2 \tilde{w}_3 \ldots \tilde{w}_n \tilde{w}_{n+1}$ then, since $F_{uvw}(U \cup V) = F_{uvw}^{-1}(U \cup V) \subset U$ by (2), we obtain that $p \in N(\tilde{U} \cap A, \tilde{U}) \cap N(\tilde{V} \cap A, \tilde{V})$, as

$$T^p(\tilde{U} \cap A) \supset \{\omega_1 \tilde{\omega}_1 \ldots \omega_t \tilde{\omega}_t \ldots\} \times F_{uvw}(U) \subset \tilde{U},$$

$$T^p(\tilde{V} \cap A) \supset \{\omega_1 \tilde{\omega}_1 \ldots \omega_t \tilde{\omega}_t \ldots\} \times F_{uvw}(V) \subset \tilde{V}.$$

Finally, observe that each map $F_i$ preserves the ordering of $[0, 1]$. Put $U = (a, b)$ and $V = (c, d)$ where $b < c$. If for some word $w$ we have $F_w(U) \cap U \neq \emptyset$, then $F_w(U) \subset [0, b)$,
in particular $F_w(U) \cap V = \emptyset$. Therefore, if we put $ar{U} = \Sigma_0^+ \times U$ and $ar{V} = \Sigma_0^+ \times V$, both intersecting $A$, then $N(\bar{U} \cap A, \bar{V}) \cap N(\bar{V} \cap A, \bar{U}) = \emptyset$. This shows that $A$ is not a weakly mixing set of order 2, completing the proof. 

\section*{REFERENCES}


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