Products and decompositions of adding machines

Jie-Hua Mai, Cui-Jun Li, Xin-He Liu

A Institute of Mathematics, Shantou University, Shantou, Guangdong 515063, PR China
b School of Science, Beijing Jiaotong University, Beijing 100044, PR China
c Institute of Mathematics, Guangxi University, Nanning, Guangxi 530004, PR China

ABSTRACT

In this paper we give several necessary and sufficient conditions for the product $f_K$ of a countable family of adding machines to be topologically conjugate to an adding machine $f_K$. We also study decompositions of adding machines $f_K$, of carry systems $K$ and of $K$-adic topological groups $\Sigma_K$, investigate the relation between these decompositions, and obtain some necessary and sufficient conditions for a $K$-adic topological group $\Sigma_K$ to be decomposable to the direct product of a family of its compact subgroups. In addition, equivalent conditions of an adding machine having a non-trivial periodic orbit factor are given, and the calculation of elements of finite orders in $K$-adic topological groups $\Sigma_K$ is discussed.

© 2011 Elsevier Inc. All rights reserved.

0. Introduction

Topological conjugacy is one of the most important subjects in the study of dynamical systems. Topologically conjugate maps have the same topological dynamical properties. If a newly encountered system is proved to be topologically conjugate to a system with clear structure, then the dynamical properties of this new system become clear. Thus, topological conjugacy is a powerful tool used to study unknown systems from some known systems.

Adding machines can be used to describe the dynamical properties of subsystems of many maps. Therefore, it draws a wide attention of the researchers in the recent 30 years (e.g. see [2–4,7–11,13–16]). Let $f$ be a continuous map of a metric space $X$ into itself. Mai and Ye [17] proved that $f$ is pointwise recurrent (resp. minimal) and has the pseudo-orbit tracing property if and only if $f$ is uniformly conjugate to a subsystem of an adding-machine-like map (resp. of an adding-machine map). Using regularly recurrent points, Block and Keesling [5] obtained necessary and sufficient conditions for a minimal map of a compact metric space to be topologically semi-conjugate (topologically conjugate) to an adding machine. For every sequence $K = (p_1, p_2, \ldots)$ of integers $p_i \geq 2$, Block, Keesling and Misiurewicz [6] showed that there is a dense subset $A \subset [\sqrt{2}, 2]$ such that the tent map $T_a$ ($a \in A$) restricted to the $\omega$-limit set of the critical point $c = \frac{1}{2}$ is topologically conjugate to the adding machine $f_K$.

The above researches are mainly to seek for subsystems of maps on various spaces which are topologically conjugate to adding machines, and then one can use adding machines to describe dynamical properties of these maps or their subsystems. There are also some researches which specialize in adding machines themselves. For example, Buescu and Stewart [12] gave a topologically conjugacy classification of adding machines. Block and Keesling [5] obtained the same classification of
adding machines in different way. Another classification was obtained by Banks in [1]. Recently, in [18], the authors considered a more general definition of adding machines, and presented various necessary and sufficient conditions for an adding machine to be topologically conjugate or topologically semi-conjugate to another one.

In this paper we will study products and decompositions of adding machines. The paper is organized as follows: In Section 1 we introduce some concepts, notations and lemmas. In Section 2 we give several necessary and sufficient conditions for the product \( f_K \) of a countable family of adding machines to be topologically conjugate to an adding machine \( f_K \).

In Section 3 we study decompositions of adding machines \( f_K \), of carry systems \( K \) and of \( K \)-adic topological groups \( \Sigma_K \), investigate the relation between these decompositions, and obtain some necessary and sufficient conditions for a \( K \)-adic topological group \( \Sigma_K \) to be decomposable to the direct product of a family of its compact subgroups. In Section 4, some equivalent conditions of an adding machine having a non-trivial periodic orbit factor are given. In Section 5, we discuss the calculation of elements of finite orders in \( K \)-adic topological groups \( \Sigma_K \).

1. Preliminaries

In this section we first introduce some concepts and notations, which also appear in [18]. Let \( \mathbb{Z}, \mathbb{N} \) and \( \mathbb{Z}_+ \) be the sets of all integers, positive integers and nonnegative integers, respectively. For any \( n \in \mathbb{N} \), write \( \mathbb{N}_n = \{1, 2, \ldots, n\} \), and \( \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \).

Denote by \( \mathbb{N}^\mathbb{N} \) the set of all infinite sequences of positive integers. For any sequence \( K = (k_1, k_2, \ldots) \in \mathbb{N}^\mathbb{N} \), let

\[
\Sigma_K = \{(i_1, i_2, \ldots) : \text{for each } n \in \mathbb{N}, \ i_n \in \mathbb{Z}_{k_n}\}.
\]

Define a metric \( d_K \) on \( \Sigma_K \) by, for any \((a_1, a_2, \ldots)\) and \((b_1, b_2, \ldots)\) \(\in \Sigma_K \),

\[
d_K((a_1, a_2, \ldots), (b_1, b_2, \ldots)) = \max\{\rho(a_n, b_n)/2^{n-1} : n \in \mathbb{N}\},
\]

where

\[
\rho(a_n, b_n) = \begin{cases} 
0, & \text{if } a_n = b_n; \\
1, & \text{if } a_n \neq b_n.
\end{cases}
\]

In the following we will see that the metric \( d_K \) has better properties than many other metrics on \( \Sigma_K \) inducing the same topology. Thus we choose \( d_K \) as the standard metric on \( \Sigma_K \). The metric space \((\Sigma_K, d_K)\) is called the \( K \)-adic symbolic space. Define an additive operation \( \oplus_K \) in \( \Sigma_K \) by, for any \((a_1, a_2, \ldots)\) and \((b_1, b_2, \ldots)\) \(\in \Sigma_K \),

\[
(a_1, a_2, \ldots) \oplus_K (b_1, b_2, \ldots) = (c_1, c_2, \ldots),
\]

where

\[
c_n = a_n + b_n + \lambda_{n-1} - \lambda_n k_n
\]

with \( \lambda_0 = 0 \), and, for \( n = 1, 2, \ldots \),

\[
\lambda_n = \begin{cases} 
0, & \text{if } a_n + b_n + \lambda_{n-1} < k_n; \\
1, & \text{if } a_n + b_n + \lambda_{n-1} \geq k_n.
\end{cases}
\]

Under these definitions, \((\Sigma_K, d_K, \oplus_K)\) is a commutative compact topological group, called the \( K \)-adic topological group. If there would be no confusion, both \((\Sigma_K, d_K)\) and \((\Sigma_K, d_K, \oplus_K)\) are abbreviated to \( \Sigma_K \).

For given \( v \in \Sigma_K \), define a map \( f_{vK} : \Sigma_K \to \Sigma_K \) by

\[
f_{vK}(x) = x \oplus_K v,
\]

where \( f_{vK}(x) \) is called the translation on \( \Sigma_K \) by adding \( v \). Write \( 0_K = (0, 0, 0, \ldots) \). Then \( 0_K \) is the unit element of the additive group \( \Sigma_K \). For any \( n \in \mathbb{Z}_+ \), let \( nv \) be the sum \( v \oplus_K v \oplus_K \cdots \oplus_K v \). Then \( nv = f_{vK}^n(0_K) \). Let \( -nv \) be the inverse element of \( nv \). Then \( -nv = f_{vK}^{-n}(0_K) \).

\[
1_K = \begin{cases} 
(0, 0, 0, \ldots), & \text{if } K = (1, 1, 1, \ldots); \\
(1, 0, 0, \ldots), & \text{if } k_1 > 1; \\
\underbrace{0, 0, 0, \ldots}_n, & \text{if } k_1 = \cdots = k_n = 1 \text{ and } k_{n+1} > 1 \text{ for some } n \in \mathbb{N}.
\end{cases}
\]

Let \( f_K = f_{1K} \) be the translation by adding \( 1_K \). \( f_K \) is called the adding machine on the \( K \)-adic topological group \( \Sigma_K \), and the sequence \( K \) is called the carry system of \( f_K \) and of the additive operation \( \oplus_K \). For any \( v \in \Sigma_K \), write \( Orb(v, f_K) = \{f_K^n(v) : n \in \mathbb{N}\} \). \( Orb(v, f_K) \) is called the orbit of \( v \) under \( f_K \).

\[
i_K = f_K^1(0_K), \quad \text{and} \quad O_K = Orb(0_K, f_K) = \{i_K : i \in \mathbb{Z}_+\}.
\]

\( O_K \) is called the basic orbit of \( f_K \).
Let \( K = (k_1, k_2, \ldots) \in \mathbb{N}^\mathbb{N} \) be as above. If \( k_n = 1 \) for all \( n \in \mathbb{N} \), then \( K, \Sigma_K \) and \( f_K \) are all said to be trivial. If the set \( \{n \in \mathbb{N} : k_n > 1\} \) is finite (resp. infinite), then \( K \) is said to be essentially finite (resp. essentially infinite). It is easy to see that (i) if \( K \) is trivial, then \( \Sigma_K \) has only one element \((0, 0, 0, \ldots)\), which is a fixed point of \( f_K \); (ii) if \( K \) is essentially finite, then \( \Sigma_K \) is a finite set, and the whole space \( \Sigma_K \) is a periodic orbit of \( f_K \) with period \( \prod_{n=1}^{N} k_n \); (iii) if \( K \) is essentially infinite, then \( \Sigma_K \) is a space homeomorphic to the Cantor set, and the whole \( \Sigma_K \) is a minimal set of \( f_K \) (for the definition of minimal sets of a map, see [3]).

A map \( \psi \) from a metric space \( X \) to itself is called a periodic map if there is \( n \in \mathbb{N} \) such that \( \psi^n \) is the identity map of \( X \); the smallest such \( n \) is called the period of \( \psi \). A periodic map \( \psi \) is called a cyclic permutation if \( \psi \) has only a periodic orbit. In most researches on adding machines, only carry systems which consist of integers greater than 1 are considered. In [18] and in this paper we allow carry systems of adding machines to contain 1's, so cyclic permutations can be regarded as special adding machines, and this extends the family of adding machines.

Let \( K = (k_1, k_2, \ldots) \) and \( L = (l_1, l_2, \ldots) \in \mathbb{N}^\mathbb{N} \). Suppose that \( K \neq L \) and there exists a sequence \( \{n_1 < n_2 < n_3 < \cdots\} \subset \mathbb{N} \) such that \( l_{n_i} = k_1 \) for all \( i \in \mathbb{N} \) and \( l_j = 1 \) for all \( j \in \mathbb{N} - \{n_1, n_2, n_3, \ldots\} \). Then the adding machines \( f_K \) and \( f_L \) will have the same dynamical properties. However, the standard metrics \( d_K \) and \( d_L \) on the topological groups \( \Sigma_K \) and \( \Sigma_L \) are different. In this paper we study products of infinitely many topological groups and of infinitely many adding machines. The difference between metrics on topological groups will influence the metrics on the products of these topological groups, and then influence the properties of products of these adding machines. Therefore, not only for extending the family of adding machines such that cyclic permutations are admitted, but also for studying products and decompositions of adding machines, it is necessary to allow carry systems of adding machines to contain 1's.

In this paper we need the following lemma, of which the proof is simple.

**Lemma 1.1.** (See [18, Lemma 2.1.]) Let \( K = (k_1, k_2, \ldots) \in \mathbb{N}^\mathbb{N} \) and \( \{u, v, w\} \subset \Sigma_K \). Then:

1. \( d_K(u, w) \leq \max\{d_K(u, v), d_K(v, w)\} \).
2. The translation \( f_K \) is an isometric homeomorphism, and \( f_K f_L = f_L f_K \).
3. For any \( i, j, n \in \mathbb{Z}^+ \), \( d_K(f_K^i(v), f_L^j(v)) \leq 2^{-n} \) if and only if \( (i - j)/\prod_{m=1}^{n} k_m \in \mathbb{Z} \).
4. Every orbit of the adding machine \( f_K \) is dense in \( \Sigma_K \).

In order to study the relation between adding machines, it is necessary to compare their carry systems.

**Definition 1.2.** (See [18, Definition 2.3.]) For any \( K = (k_1, k_2, \ldots) \) and \( L = (l_1, l_2, \ldots) \in \mathbb{N}^\mathbb{N} \), define the cumulative products

\[
\pi_0 K = 1, \quad \pi_0 L = 1, \quad \pi_n K = \prod_{i=1}^{n} k_i, \quad \text{and} \quad \pi_n L = \prod_{i=1}^{n} l_i, \quad \text{for all} \ n \in \mathbb{N}.
\]

(1.10)

We say that:

1. \( K \) dominates \( L \) (or \( K \) is richer than \( L \)) and write \( K \succ L \) or \( L \prec K \) if for any \( n \in \mathbb{N} \) there exists \( q_n \in \mathbb{N} \) such that \( \pi_n K / \pi_n L = q_n \in \mathbb{N} \).
2. \( K \) dominates \( L \) in product and write \( K \succpi L \) or \( L \precpi K \) if \( \pi_n K / \pi_n L \in \mathbb{N} \) for all \( n \in \mathbb{N} \).
3. \( K \) termwise dominates \( L \) (or \( K \) is termwise stronger than \( L \)) and write \( K \succpi L \) or \( L \precpi K \) if \( k_n / l_n \in \mathbb{N} \) for all \( n \in \mathbb{N} \).
4. \( K \) is similar to \( L \) (or \( K \) is equivalent to \( L \)) and write \( K \sim L \) if \( K \succ L \) and \( L \succ K \).

Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces. A map \( \psi : X \to Y \) is said to be non-expansive if \( d_Y(\psi(x), \psi(y)) \leq d_X(x, y) \) for any \( x, y \in X \). Let \( f : X \to X \) and \( g : Y \to Y \) be continuous maps. \( f \) is said to be topologically semi-conjugate (resp. non-expansively semi-conjugate, topologically conjugate, isometrically conjugate) to \( g \) if there exists a continuous surjection (resp. non-expansive surjection, homeomorphism, isometric homeomorphism) \( h : X \to Y \) such that \( hf = gh \). Such an \( h \) is called a topological semi-conjugacy (resp. non-expansive semi-conjugacy, topological conjugacy, isometric conjugacy) from \( f \) to \( g \). The following lemma is also useful in this paper.

**Lemma 1.3.** (See [18, Proposition 2.4.]) Suppose \( K \succ L \). Then there exists a unique continuous homomorphism \( \Psi_{KL} \) from the \( K \)-adic topological group \( \Sigma_K \) onto \( \Sigma_L \) such that \( \Psi_{KL}(k_i) = 1_i \) for all \( i \in \mathbb{Z}^+ \).

This homomorphism \( \Psi_{KL} \) is also a topological semi-conjugacy from \( f_K \) to \( f_L \). Furthermore, if \( K \succpi L \), then \( \Psi_{KL} \) is a non-expansive semi-conjugacy.

The continuous homomorphism \( \Psi_{KL} \) in Lemma 1.3 is called the natural homomorphism from \( \Sigma_K \) onto \( \Sigma_L \).
2. Products of adding machines

In this section we study products of infinitely many adding machines, and give several necessary and sufficient conditions for the products of infinitely many adding machines to be topologically conjugate to adding machines.

Let \( K_i \in \mathbb{N}^N \) be given, for every \( i \in \mathbb{N} \). Let \( K = (K_1, K_2, \ldots) \in (\mathbb{N}^N)^\mathbb{N} \) be the sequence of sequences \( K_1, K_2, \ldots \)

For \( n \geq 2 \), let \( K_n = (K_1, \ldots, K_n) \in (\mathbb{N}^N)^n \) be the sequence of sequences \( K_1, \ldots, K_n \). Let

\[
\Sigma_K = \prod_{i=1}^\infty \Sigma_{K_i} \quad \text{and} \quad \Sigma_{K_n} = \prod_{i=1}^n \Sigma_{K_i}
\]

be the Cartesian products of sets \( \Sigma_{K_1}, \Sigma_{K_2}, \ldots \) and of \( \Sigma_{K_1}, \ldots, \Sigma_{K_n} \), respectively. Define the

standard metrics \( d_K \) on \( \Sigma_K \) and \( d_{K_n} = d_{K_1 \cup \cdots \cup K_n} \) on \( \Sigma_{K_n} \) by, for any \((x_1, x_2, \ldots, x_n)\) and \((y_1, y_2, \ldots, y_n)\) in \( \Sigma_K \),

\[
d_K((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \max \{ d_{K_i}(x_i, y_i) : i \in \mathbb{N} \},
\]

\[
d_{K_n}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max \{ d_{K_i}(x_i, y_i) : i \in \mathbb{N}_n \}.
\]

The metric spaces \((\Sigma_K, d_K)\) and \((\Sigma_{K_n}, d_{K_n})\) are both called compound symbolic spaces. Define additive operations \( \oplus \) in \( \Sigma_K \) and \( \oplus_{K_n} = \oplus_{K_1 \cup \cdots \cup K_n} \) in \( \Sigma_{K_n} \) by

\[
(x_1, x_2, \ldots) \oplus (y_1, y_2, \ldots) = (x_1 \oplus K_1 y_1, x_2 \oplus K_2 y_2, \ldots),
\]

\[
(x_1, \ldots, x_n) \oplus_{K_n} (y_1, \ldots, y_n) = (x_1 \oplus K_1 y_1, \ldots, x_n \oplus K_n y_n),
\]

for any \((x_1, x_2, \ldots, x_n, \ldots)\) and \((y_1, y_2, \ldots, y_n, \ldots)\) in \( \Sigma_K \). Then both \((\Sigma_K, d_K, \oplus_K)\) and \((\Sigma_{K_n}, d_{K_n}, \oplus_{K_n})\) are commutative topological groups, which are called the direct product groups of \( \Sigma_{K_1}, \Sigma_{K_2}, \ldots \) and of \( \Sigma_{K_1}, \ldots, \Sigma_{K_n} \), respectively. If no confusion arises, both \((\Sigma_K, d_K)\) and \((\Sigma_{K_n}, d_{K_n})\) are abbreviated to \( \Sigma_K \), and both \((\Sigma_K, d_K)\) and \((\Sigma_{K_n}, d_{K_n})\) are abbreviated to \( \Sigma_{K_n} \).

Define the products, of adding machines,

\[
f_K = \prod_{i=1}^\infty f_{K_i} : \Sigma_K \to \Sigma_K \quad \text{and} \quad f_{K_n} = f_{K_1 \cup \cdots \cup K_n} = \prod_{i=1}^n f_{K_i} : \Sigma_{K_n} \to \Sigma_{K_n}
\]

by, for any \((x_1, x_2, \ldots, x_n, \ldots)\) in \( \Sigma_K \),

\[
f_K(x_1, x_2, \ldots) = (f_{K_1}(x_1), f_{K_2}(x_2), \ldots).
\]

\[
f_{K_n}(x_1, \ldots, x_n) = (f_{K_1}(x_1), \ldots, f_{K_n}(x_n)).
\]

Then both \( f_K \) and \( f_{K_n} \) are isometric homeomorphisms, and both are called compound adding machines.

Suppose that \( K_i = (k_{i1}, k_{i2}, k_{i3}, \ldots) \), for every \( i \in \mathbb{N} \). Write

\[
k_{ij} = \sum_{i=1}^\infty k_{ij}
\]

and

\[
k_{1n} = \sum_{i=1}^\infty k_{i1} \quad \text{for all} \quad j \in \mathbb{N}.
\]

\[
K_N = \bigotimes_{i=1}^\infty K_i = (k_1^*, k_2^*, k_3^*, \ldots),
\]

and

\[
K_{1n} = \bigotimes_{i=1}^n K_i = K_1 \otimes \cdots \otimes K_n = (k_1^{(n)}, k_2^{(n)}, k_3^{(n)}, \ldots).
\]

\( K_N \) (resp. \( K_{1n} \)) is called the termwise product of \( K_1, K_2, \ldots \) (resp. of \( K_1, \ldots, K_n \)). Note that \( K_{1n} \in \mathbb{N}^N \). But it is possible that \( k_{ij} = \infty \) for some \( j \in \mathbb{N} \). Thus we only have \( K_N \in (\mathbb{N} \cup \{\infty\})^\mathbb{N} \).

Denotes by \( \text{diam}(X) \) the diameter of a metric space \( X \). Obviously, we have \text{diam}(X) → 0 as \( i \to \infty \).

**Lemma 2.1.** The following four conditions are equivalent:

(a) \( K_N = \bigotimes_{i=1}^\infty K_i \in \mathbb{N}^N \) (that is, \( k_{ij}^* = \prod_{i=1}^\infty k_{ij} < \infty \) for all \( j \in \mathbb{N} \)).

(b) For any \( j \in \mathbb{N} \), \( \{i \in \mathbb{N} : k_{ij} > 1\} \) is a finite set.

(c) \((\Sigma_K, d_K)\) is a compact metric space.

(d) \( \text{diam}(\Sigma_{K_n}) \to 0 \) as \( i \to \infty \).
In the following we always assume that the condition (a) in Lemma 2.1 holds. Then, for any integers 1 \leq i \leq n, we have \( K_i \rightarrow K_{n_i} \rightarrow K_i \), and it follows from Lemma 1.3 that there exist non-expansive natural homomorphisms
\[
\psi_{K_{n_i}K_i} : \Sigma_{K_{n_i}} \rightarrow \Sigma_{K_i} \quad \text{and} \quad \psi_{K_{n_i}n_iK_i} : \Sigma_{K_{n_i}n_i} \rightarrow \Sigma_{K_i},
\]
which satisfy
\[
\psi_{K_{n_i}K_i}(j_{K_i}) = j_{K_i} \quad \text{and} \quad \psi_{K_{n_i}n_iK_i}(j_{K_{n_i}n_i}) = j_{K_i} \quad \text{for all } j \in \mathbb{Z}_+.
\]
Define the compound natural homomorphisms
\[
\psi_{\Sigma} = \psi_{K_{n_i}K_i} = (\psi_{K_{n_i}K_i1}, \psi_{K_{n_i}K_i2}, \ldots) : \Sigma_{K_{n_i}} \rightarrow \Sigma_{\Sigma} \left( = \prod_{i=1}^{\infty} \Sigma_{K_i} \right)
\]
and
\[
\psi_{K_{n_i}} = \psi_{K_{n_i}n_i} : \Sigma_{K_{n_i}n_i} \rightarrow \Sigma_{\Sigma} \left( = \prod_{i=1}^{n} \Sigma_{K_i} \right)
\]
by
\[
\psi_{\Sigma}(x) = (\psi_{K_{n_i}K_i1}(x), \psi_{K_{n_i}K_i2}(x), \ldots), \quad \text{for any } x \in \Sigma_{K_{n_i}}
\]
and
\[
\psi_{K_{n_i}}(x) = (\psi_{K_{n_i}K_i1}(x), \ldots, \psi_{K_{n_i}n_iK_i}(x)), \quad \text{for any } x \in \Sigma_{K_{n_i}n_i}.
\]
From Lemma 1.3 and the definition of metric \( d_{\Sigma} \) on \( \Sigma_{\Sigma} \) we obtain immediately

Lemma 2.2. Suppose that the condition (a) in Lemma 2.1 holds. Then \( \psi_{\Sigma} \) is a non-expansive homomorphism, and \( \psi_{\Sigma} \circ f_{K_{n_i}} = f_{\Sigma} \circ \psi_{\Sigma} \), that is, the following diagram is commutative.

\[
\begin{array}{ccc}
\Sigma_{K_{n_i}} & \xrightarrow{f_{K_{n_i}}} & \Sigma_{K_i} \\
\psi_{\Sigma} & & \psi_{\Sigma} (= (\psi_{K_{n_i}K_i1}, \psi_{K_{n_i}K_i2}, \ldots)) \\
\Sigma_{\Sigma} & \xrightarrow{f_{\Sigma} (= \prod_{i=1}^{\infty} \Sigma_{K_i})} & \Sigma_{\Sigma} (= \prod_{i=1}^{\infty} \Sigma_{K_i})
\end{array}
\]

In general, the compound natural homomorphisms \( \psi_{\Sigma} \) may not be a surjection. Thus the commutativity of the above diagram does not guarantee that \( \psi_{\Sigma} \) is a topological semi-conjugacy from \( f_{K_{n_i}} \) to \( f_{\Sigma} \).

Let \( K = (k_1, k_2, \ldots) \) and \( L = (l_1, l_2, \ldots) \in \mathbb{N}^{\mathbb{N}} \). The sequences \( K \) and \( L \) are said to be relatively prime if \( k_i \) and \( l_j \) are relatively prime for all \( i, j \in \mathbb{N} \).

The following theorem is the main result of this section, which gives various necessary and sufficient conditions for the compound adding machine \( f_{\Sigma} \) to be topologically conjugate to an adding machine.

Theorem 2.3. Let \( K = (K_1, K_2, \ldots) \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \). Suppose that the condition (a) in Lemma 2.1 holds. Let \( K_{n_i} = \bigotimes_{i=1}^{\infty} K_i \) be the termwise product with the corresponding topological group \( \Sigma_{K_{n_i}} \) and adding machine \( f_{K_{n_i}} : \Sigma_{K_{n_i}} \rightarrow \Sigma_{K_{n_i}} \), let \( \Sigma_{\Sigma} = \prod_{i=1}^{\infty} \Sigma_{K_i} \) be the direct product group with the corresponding compound adding machine \( f_{\Sigma} = \prod_{i=1}^{\infty} f_{K_i} : \Sigma \rightarrow \Sigma \), and let \( \psi_{\Sigma} = \psi_{K_{n_i}K_i} : \Sigma_{K_{n_i}} \rightarrow \Sigma_{\Sigma} \) be the compound natural homomorphism, defined as above. Then the following seven conditions are equivalent:

1. \( f_{\Sigma} \) is topologically conjugate to an adding machine.
2. \( f_{\Sigma} \) is topologically conjugate to \( f_{K_{n_i}} \).
3. \( \psi_{\Sigma} \) is an isometric conjugacy from \( f_{K_{n_i}} \) to \( f_{\Sigma} \).
4. \( \psi_{\Sigma} \) is an isometric isomorphism from \( \Sigma_{K_{n_i}} \) to \( \Sigma_{\Sigma} \).
5. \( \psi_{\Sigma} \) is an isometric injection.
6. \( \psi_{\Sigma} \) is a surjection.
7. For any integers \( m > i \geq 1 \), \( K_i \) and \( K_m \) are relatively prime.

Remark 2.4. If all of \( K_{n+1}, K_{n+2}, \ldots \) are trivial sequences, for some \( n > 1 \), then \( \Sigma_{\Sigma} \) and \( \Sigma_{K_{n+1}} \) are isometrically isomorphic, \( f_{\Sigma} \) and \( f_{\Sigma_{K_{n+1}}} \) are isometrically conjugate, \( \psi_{\Sigma} \) and \( \psi_{\Sigma_{K_{n+1}}} \) are isometrically conjugate, and \( K_{n+1} = K_{n+1} \). Thus Lemma 2.2 and Theorem 2.3 still hold if \( f_{\Sigma}, f_{K_{n+1}}, \psi_{\Sigma}, \Sigma_{K_{n}} \) and \( \Sigma_{\Sigma} \) are replaced by \( f_{\Sigma_{K_{n+1}}}, f_{K_{n+1}}, \psi_{\Sigma_{K_{n+1}}}, \Sigma_{K_{n+1}} \) and \( \Sigma_{\Sigma_{K_{n+1}}} \), respectively.
In order to prove Theorem 2.3, we need the following three lemmas.

**Lemma 2.5.** Let $K$ and $L \in \mathbb{N}^n$. If $K$ and $L$ are not relatively prime, then no orbit of the product $f_K \times f_L$ of adding machines $f_K$ and $f_L$ is dense in $\Sigma_K \times \Sigma_L$.

**Proof.** For any $v \in \Sigma_K$ and any $w \in \Sigma_L$, the product $f_K \times f_L : \Sigma_K \times \Sigma_L \to \Sigma_K \times \Sigma_L$ of translations $f_K$ and $f_L$ is a homeomorphism which sends the basic orbit $\text{Orb}(0_K, 0_L, f_K \times f_L)$ onto the orbit $\text{Orb}(v, w, f_K \times f_L)$. Thus it suffices to show that $\text{Orb}(0_K, 0_L, f_K \times f_L)$ is not dense in $\Sigma_K \times \Sigma_L$. Since $K$ and $L$ are not relatively prime, there exist $n \in \mathbb{N}$ such that the greatest common divisor $\gcd(\pi_{nK}, \pi_{nL}) = m > 1$ (where $\pi_{nK}$ and $\pi_{nL}$ are defined as in (1.10)). Let

$$K' = (m, \pi_{nK}/m, k_{n+1}, k_{n+2}, \ldots), \quad L' = (m, \pi_{nL}/m, l_{n+1}, l_{n+2}, \ldots).$$

Then $K' \sim K$, and $L' \sim L$. By Lemma 1.3, there exist natural homomorphisms

$$\psi_{KK'} : K \to K', \quad \psi_{KK'} : K' \to K, \quad \psi_{LL'} : L \to L', \quad \psi_{LL'} : L' \to L,$$

and $\psi_{KK'} = \psi_{KK'}^{-1}$ (resp. $\psi_{LL'} = \psi_{LL'}^{-1}$) is a topological conjugacy from $f_K$ to $f_{K'}$ (resp. from $f_L$ to $f_{L'}$). Hence the product $\psi_{KK'} \times \psi_{LL'} : \Sigma_K \times \Sigma_L \to \Sigma_{K'} \times \Sigma_{L'}$ is also a topological conjugacy from $f_K \times f_L$ to $f_{K'} \times f_{L'}$. For any $a \in \mathbb{Z}_m$, write

$$V_a = \{(i_1, i_2, \ldots) \in \Sigma_K : i_1 = a\}, \quad W_a = \{(i_1, i_2, \ldots) \in \Sigma_L : i_1 = a\}.$$

Then $V_a$ and $W_a$ are both closed and open subsets of $\Sigma_K$ and $\Sigma_L$, respectively. Since the orbit $\text{Orb}((0_{K'}, 0_L), f_{K'} \times f_{L'}) = \{(j_{K'}, j_{L'}) : j^{(i_{K'}, i_{L'})} \in \mathbb{Z}_m\}$ is dense in $\Sigma_{K'} \times \Sigma_L$, Lemma 2.5 is proven. □

**Lemma 2.6.** Suppose $(J, K, L) \in \mathbb{N}^2$ with $J > K$ and $J > L$. Define the compound natural homomorphism $\psi_{J,K,L} = (\psi_{J,K}, \psi_{J,L}) : \Sigma_J \to \Sigma_K \times \Sigma_L$ by

$$\psi_{J,K,L}(x) = (\psi_{J,K}(x), \psi_{J,L}(x)) \quad \text{for any } x \in \Sigma_J. \tag{2.12}$$

Then $\psi_{J,K,L}$ is a surjection if and only if $K$ and $L$ are relatively prime.

**Proof.** If $K$ and $L$ are not relatively prime, then, by Lemma 2.5,

$$\psi_{J,K,L}(\text{Orb}(0_J, f_J)) = \{(i_k, j_l) : i \in \mathbb{Z}_m\} = \text{Orb}((0_K, 0_L), f_K \times f_L)$$

is not dense in $\Sigma_K \times \Sigma_L$. Since the orbit $\text{Orb}(0_J, f_J)$ is dense in $\Sigma_J$ and $\psi_{J,K,L}$ is continuous, $\psi_{J,K,L}$ is not surjective.

Conversely, if $K$ and $L$ are relatively prime, then for any given $n \in \mathbb{N}$, $\pi_{nK}$ and $\pi_{nL}$ are relatively prime. Thus, for any given $(i, j) \in \mathbb{Z}_m$, there exists $b \in \mathbb{Z}_{\pi_{nK}}$ such that $i + b \pi_{nK} \equiv j \pmod{\pi_{nL}}$. Take $m = i + b \pi_{nK}$. Then $m \equiv i \pmod{\pi_{nK}}$ and $m \equiv j \pmod{\pi_{nL}}$. By (3) of Lemma 1.1 and the definition of the standard metric $d_{KL}$ on $\Sigma_K \times \Sigma_L$, we have

$$d_{KL}(\psi_{J,K,L}(m_{J}), (i_k, j_l)) = \max\{d_K(m_k, i_k), d_L(m_l, j_l)\} \leq 2^{-n}. \tag{2.13}$$

By (4) of Lemma 1.1, the product $O_K \times O_L$ of basic orbits $O_K$ and $O_L$ is dense in $\Sigma_K \times \Sigma_L$. Hence, it follows from (2.13) that $\psi_{J,K,L}(\Sigma_J)$ is dense in $\Sigma_K \times \Sigma_L$, and therefore, since $\Sigma_J$ is compact and $\psi_{J,K,L}$ is continuous, $\psi_{J,K,L}$ is a surjection. □

**Lemma 2.7.** Let $K, L \in \mathbb{N}^n$, and let the compound natural homomorphism $\psi = \psi_{K \otimes L, K \otimes L} : \Sigma_K \otimes \Sigma_L \to \Sigma_K \times \Sigma_L$ be defined as in (2.12). If $K$ and $L$ are relatively prime, then $\psi$ is an isometric isomorphism.

**Proof.** For any given $(i, j) \in \mathbb{Z}_{\mathbb{N}}^+$ with $j \otimes K \neq i \otimes L$, there exists $n \in \mathbb{N}$ such that $d_{K \otimes L}(j \otimes K, i \otimes L) = 2^{-n+1}$. By (3) of Lemma 1.1, we have

$$(j - i)/\pi_{n,K \otimes L} \notin \mathbb{Z}. \tag{2.14}$$

Since $K$ and $L$ are relatively prime, $\pi_{nK}$ and $\pi_{nL}$ are relatively prime. Since $\pi_{n,K \otimes L} = \pi_{nK} \cdot \pi_{nL}$, it follows from (2.14) that $(j - i)/\pi_{nK}, (j - i)/\pi_{nL} \notin \mathbb{Z}$. Thus, by (3) of Lemma 1.1 we have $\max\{d_K(j_k, i_k), d_L(j_l, i_l)\} > 2^{-n}$. On the other hand, by Lemma 2.2 and Remark 2.4, we have

$$d_{K \otimes L}(j \otimes K, i \otimes L) = 2^{-n+1} \geq d_{KL}(\psi(j \otimes K), \psi(i \otimes L)) \quad d_{KL}(j_k, i_k) = \max\{d_K(j_k, i_k), d_L(j_l, i_l)\}.$$
Thus \( d_{KL}(\Psi(jK \otimes L), \Psi(iK \otimes L)) = \max\{d_K(jK, iK), d_L(jL, iL)\} = d_{KL}(jK \otimes L, iK \otimes L) = 2^{-n+1} \). This means that \( \Psi |_{O_{K \otimes L}} \) is isometric since the orbit \( O_{K \otimes L} \) is dense in \( \Sigma_{K \otimes L} \). \( \Psi \) itself is also isometric. In addition, by Lemma 2.6, \( \Psi \) is surjective; thus \( \Psi \) is an isometric isomorphism. Lemma 2.7 is proven. \( \square \)

We now give

**Proof of Theorem 2.3.** It is clear that (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1), (4) \( \Rightarrow \) (5) and (4) \( \Rightarrow \) (6).

By Lemma 2.2, \( \Psi_{K_i} \) is a homomorphism. Thus (3) \( \Rightarrow \) (4).

By Lemma 2.2, we have \( \Psi_{K_i} \circ f_{K_i} = f_K \circ \Psi_{K_i} \). Thus (4) \( \Rightarrow \) (3).

We now need only to show that (1) \( \Rightarrow \) (7), (6) \( \Rightarrow \) (7), (5) \( \Rightarrow \) (7) and (7) \( \Rightarrow \) (4).

(1) \( \Rightarrow \) (7). If there exist integers \( m > i \geq 1 \) such that \( K_i \) and \( K_m \) are not relatively prime, then by Lemma 2.5, no orbit of the compound adding machine \( f_{K_i} \) is dense in \( \Sigma_{K_i} \times \Sigma_{K_m} \), and hence, no orbit of the compound adding machine \( f_{K} = f_{K_1} \times \cdots \times f_{K_i} \times \cdots \times f_{K_m} \times \cdots \) is dense in the compound symbolic space \( \Sigma_K = \Sigma_{K_1} \times \cdots \times \Sigma_{K_i} \times \cdots \times \Sigma_{K_m} \times \cdots \). This implies that \( f_{K} \) cannot be topologically conjugate to an adding machine. Thus (1) \( \Rightarrow \) (7).

(6) \( \Rightarrow \) (7). If there exist integers \( m > i \geq 1 \) such that \( K_i \) and \( K_m \) are not relatively prime, then by Lemma 2.6, the compound natural homomorphism \( \Psi_{K_i,K_m} : \Sigma_{K_i} \times \Sigma_{K_m} \rightarrow \Sigma_{K_i} \times \Sigma_{K_m} \) is not surjective, which implies that the compound natural homomorphism \( \Psi_{K_i} = \Psi_{K_i,K_i} : \Sigma_{K_i} \rightarrow \Sigma_{K_i} \) is not surjective. Thus (6) \( \Rightarrow \) (7).

(5) \( \Rightarrow \) (7). If there exist integers \( m > i \geq 1 \) such that \( K_i \) and \( K_m \) are not relatively prime, then there exist \( n \) and \( a \in \mathbb{N} \) such that the greatest common divisor \( \gcd(nK_i, nK_m) = a > 1 \). By (1.10) and (2.8), we have \( \pi_{nK_i} = \prod_{j=1}^{\infty} \pi_{nK_j} \). Thus, for any \( j \in \mathbb{N} \), we have \( b/\pi_{nK_j} = (\prod_{j=1}^{\infty} \pi_{nK_j})/(a\pi_{nK_j}) \in \mathbb{N} \), but \( b/\pi_{nK_i} = 1/a \not\in \mathbb{N} \). Therefore, by (2.10), (2.2) and (3) of Lemma 1.1, we have

\[
d_{K_j}(\Psi_{K}(b_{K_j}), \Psi_{K}(0_{K_j})) = \max\{d_{K_j}(b_{K_j}, 0_{K_j}) : j \in \mathbb{N} \} \leq 2^{-n} < d_{K_i}(b_{K_i}, 0_{K_i}).
\]

This implies that \( \Psi_{K} \) is not isometric. Thus (5) \( \Rightarrow \) (7).

(7) \( \Rightarrow \) (4) Suppose that (7) holds. For any \( n \in \mathbb{N} \), let

\[
\Sigma_{K_n} = \Sigma_{K_1} \times \Sigma_{K_2} \times \cdots \times \Sigma_{K_n} \times [0_{K_{n+1}}] \times [0_{K_{n+2}}] \times \cdots.
\]

Then \( \Sigma_{K_n} \) is a subgroup of \( \Sigma_{K_i} \), and \( \Sigma_{K_n} \) is isometrically isomorphic to \( \Sigma_{K_n} = \Sigma_{K_1} \times \Sigma_{K_2} \times \cdots \times \Sigma_{K_n} \). Define compound natural homomorphisms \( \psi_n : \Sigma_{K_1} \times \cdots \times \Sigma_{K_n} \rightarrow \Sigma_{K_i} \) and \( \psi_n : \Sigma_{K_n} \rightarrow \Sigma_{K_i} \) by \( \psi_n((x_1, \ldots, x_n)) = (y_1, \ldots, y_n) \), where \( x_1 = \psi_{K_1,K_2}(x_1), y_1 = \psi_{K_i,K_j}(y_1), \ldots, x_n = \psi_{K_{n-1},K_n}(x_n), y_n = \psi_{K_i}(y_n) \). Note that \( K_1, \ldots, K_n, L \) are pairwise relatively prime. By induction, it is easy to show from (2.3) and Lemma 2.7 that

\[
d_{K_i}(x, y) = \max\{d_{K_1}(x_1, y_1), \ldots, d_{K_n}(x_n, y_n), d_{L}(v_n, w_n)\}. \tag{2.15}
\]

By Lemma 2.1, we have \( \lim_{n \to \infty} \text{diam}(\Sigma_{L_n}) = 0 \), which with (2.15) leads to

\[
d_{K_i}(x, y) = \lim_{n \to \infty} \max\{d_{K_i}(x_1, y_1), \ldots, d_{K_n}(x_n, y_n)\} = \max\{d_{K_i}(x_1, y_1) : i \in \mathbb{N} \}
\]

\[
= d_{K_i}((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = d_{K_i}(\psi_{K_i}(x), \psi_{K_i}(y)). \tag{2.16}
\]

Thus, \( \psi_{K_i} \) is an isometric surjection, which with Lemma 2.2 implies that \( \psi_{K_i} \) is an isometric isomorphism from \( \Sigma_{K_i} \) onto \( \Sigma_{K_i} \), and we get (7) \( \Rightarrow \) (4). Theorem 2.3 is proven. \( \square \)

A sequence \( K \in \mathbb{N}^\mathbb{N} \) is called a pure sequence and the corresponding adding machine \( f_K \) is called a pure adding machine if there exists a prime number \( p \) such that \( \log_p k_n \in \mathbb{Z}_+ \) for all \( n \in \mathbb{N} \). Obviously, by the equivalence between the conditions (7) and (3) in Theorem 2.3, we have

**Corollary 2.8.** For any \( K \in \mathbb{N}^\mathbb{N} \), there exist pairwise relatively prime pure sequences \( K_1, K_2, \ldots \in \mathbb{N}^\mathbb{N} \) such that \( K = K_N \equiv \bigotimes_{j=1}^{\infty} K_j \), and hence, the adding machine \( f_K \) is isometrically conjugate to the product \( f_K \) of the pure adding machines \( f_{K_1}, f_{K_2}, \ldots \).
standard metrics $d_n$ on $G_n$, $d_N$ on $G_N$ and additive operations $\oplus_n$ in $G_n$, $\oplus_N$ in $G_N$ by, for any $(x_1, x_2, \ldots, x_n, \ldots)$ and $(y_1, y_2, \ldots, y_n, \ldots) \in G_N$,

\[
d_n((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{d_K(x_i, y_i) : i \in \mathbb{N}_n\}, \tag{3.1}
\]

\[
d_N((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \max\{d_K(x_i, y_i) : i \in \mathbb{N}\}, \tag{3.2}
\]

\[
(x_1, \ldots, x_n) \oplus_n (y_1, \ldots, y_n) = (x_1 \oplus y_1, \ldots, x_n \oplus y_n). \tag{3.3}
\]

\[
(x_1, x_2, \ldots) \oplus_N (y_1, y_2, \ldots) = (x_1 \oplus y_1, x_2 \oplus y_2, \ldots). \tag{3.4}
\]

Then $(G_n, d_n, \oplus_n)$ and $(G_N, d_N, \oplus_N)$ are topological groups, which are called the direct products of $G_1, \ldots, G_n$ and of $G_1, G_2, \ldots$ respectively (since each $G_i$ is commutative, $(G_n, d_n, \oplus_n)$ is also called the direct sum of $G_1, \ldots, G_n$). If there would be no confusion, we abbreviate them to $G_n$ and $G_N$. Evidently, we have

**Lemma 3.1.** Let $K \in \mathbb{N}^N$, and let $G_1, G_2, \ldots$ be compact subgroups of the topological group $\Sigma_K$. Then the following three conditions are equivalent:

1. For any $(x_1, x_2, \ldots) \in G_N$, $x_1 \oplus \cdots \oplus x_n \rightarrow \infty$ as $n \rightarrow \infty$.
2. The direct product group $G_N = \prod_{n=1}^\infty G_n$ under the metric $d_N$ is compact.
3. $\lim_{n \rightarrow \infty} \text{diam}(G_n) = 0$.

In the following we always assume that the condition (3) in Lemma 3.1 holds. Define the natural addition homomorphisms $\Phi_n = \Phi_{G_n} : G_n \rightarrow \Sigma_K$ and $\Phi_N = \Phi_{G_N} : G_N \rightarrow \Sigma_K$ by, for any $(x_1, x_2, \ldots, x_n, \ldots) \in G_N$,

\[
\Phi_n(x_1, \ldots, x_n) = x_1 \oplus \cdots \oplus x_n, \tag{3.5}
\]

\[
\Phi_N(x_1, x_2, \ldots) = x_1 \oplus x_2 \oplus \cdots \equiv \lim_{n \rightarrow \infty} (x_1 \oplus \cdots \oplus x_n). \tag{3.6}
\]

**Proposition 3.2.** The natural addition homomorphisms $\Phi_n$ and $\Phi_N$ are non-expansive.

**Proof.** For any $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots) \in G_N$, write $v_n = (x_1, \ldots, x_n)$ and $w_n = (y_1, \ldots, y_n)$. By (2) and (1) of Lemma 1.1 we have

\[
d_K(\Phi_n(v_n), \Phi_n(w_n)) = d_K(\Phi_n(v_n) \oplus \Phi_n(w_n), 0_K) = d_K((x_1 \oplus y_1) \oplus \cdots \oplus (x_n \oplus y_n), 0_K) \leq \max\{d_K(x_i \oplus y_i, 0_K) : i \in \mathbb{N}_n\} = d_n(v_n, w_n).
\]

Thus $\Phi_n$ is non-expansive. Noting that $\lim_{n \rightarrow \infty} \text{diam}(G_n) = 0$, by (3.5) and (3.6) we have

\[
d_K(\Phi_N(x), \Phi_N(y)) = \lim_{n \rightarrow \infty} d_K(\Phi_n(v_n), \Phi_n(w_n)) \leq \lim_{n \rightarrow \infty} d_n(v_n, w_n) = d_n(x, y).
\]

Hence $\Phi_N$ is also non-expansive. $\square$

**Lemma 3.3.** (See [18, Corollary 3.5].) For any $K, L \in \mathbb{N}^N$, the adding machines $f_K$ and $f_L$ are topologically (resp. isometrically) conjugate if and only if the topological groups $\Sigma_K$ and $\Sigma_L$ are topologically (resp. isometrically) isomorphic.

**Lemma 3.4.** (See [18, Theorem 2.9].)

1. $f_K$ is topologically conjugate to $f_L$ if and only if $K \sim L$.
2. $f_K$ is isometrically conjugate to $f_L$ if and only if $K = L$.

A map $\varphi$ from a metric space $(X, d_X)$ to a metric space $(Y, d_Y)$ is said to be non-contractive if $d_Y(\varphi(x), \varphi(v)) \geq d_X(x, v)$ for any $x, v \in X$.

**Theorem 3.5.** Let $K \in \mathbb{N}^N$, let $Q_1, Q_2, \ldots$ be compact subgroups of $\Sigma_K$, and let $Q_N = \prod_{n=1}^\infty Q_n$ be the direct product group. If $\lim_{n \rightarrow \infty} \text{diam}(Q_n) = 0$, and there exists a continuous isomorphism $\varphi : Q_N \rightarrow \Sigma_K$, then all of the following seven statements hold:

1. There exist compact subgroups $G_1, G_2, \ldots$ of $\Sigma_K$ such that $\lim_{n \rightarrow \infty} \text{diam}(G_n) = 0$, and for each $n \in \mathbb{N}$ there exists a continuous isomorphism $h_n : Q_n \rightarrow G_n$.
2. Let $G_N = \prod_{n=1}^\infty G_n$ be the direct product group, let $H_N = \prod_{n=1}^\infty h_n : Q_N \rightarrow G_N$ be the continuous isomorphism defined by $H_N(x) = (h_1(x_1), h_2(x_2), \ldots)$ for any $x = (x_1, x_2, \ldots) \in Q_N$, and let $\Phi_N = \Phi_{G_N} : G_N \rightarrow \Sigma_K$ be the natural addition homomorphism. Then $\Phi_N H_N = \varphi$. 


(3) For each $n \in \mathbb{N}$ there exist a sequence $K_n \in \mathbb{N}^n$ and an isometric isomorphism $\Psi_n : G_n \to \Sigma_n$ such that $K_n \approx \Sigma$, and $K_1, K_2, \ldots$ are pairwise relatively prime.

(4) Let $K_\infty = \bigotimes_{n=1}^{\infty} K_n$ be the termwise product of $K_1, K_2, \ldots$ Then $\Sigma_\infty = K_\infty$.

(5) Let $K = (K_1, K_2, \ldots) \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$, let $\Sigma_K = \prod_{n=1}^{\infty} \Sigma_n$ be the direct product group with the metric $d_K$ defined by (2.2), let $\Psi_n : G_n \to \Sigma_n$ be the maps of products $\Psi_n : G_n \to \Sigma_n$ and let $\Psi_K = \Psi_K : \Sigma_K \to \Sigma_K$ be the compound natural homomorphism defined by $\Psi_K(v) = (\Psi_{K_1}(v), \Psi_{K_2}(v), \ldots)$ for any $v \in \Sigma_K$. Then $\Psi_n \Sigma_n = \Psi_K \Phi_N$.

(6) $\psi$ is a non-contractive isomorphism.

(7) $\Phi_n \Sigma_n \Phi_K$ and $\Psi_K$ are isometric isomorphisms.

**Remark.** (2) and (5) of Theorem 3.5 claim that $\Phi_n H_N = \varphi$ and $\Psi_n \Sigma_n = \psi \Phi_n$. This means that the following diagram is commutative.

$$
\begin{array}{ccc}
\prod_{n=1}^{\infty} Q_n = Q_
 & \xrightarrow{\psi} & \Sigma_K \\
\prod_{n=1}^{\infty} h_n = H_n & \xrightarrow{\Phi_n} & \Psi_K \\
\prod_{n=1}^{\infty} G_n = G_n & \xrightarrow{\Psi_n \Sigma_n} & \Sigma_K = \prod_{n=1}^{\infty} \Sigma_n
\end{array}
$$

**Proof of Theorem 3.5.** Note that the unit element $0_K$ of additive group $\Sigma_K$ is also the unit element of subgroups $Q_1, Q_2, \ldots$ of $\Sigma_K$. For any $n \in \mathbb{N}$, let

$$Q_n' = \{ (x_1, x_2, \ldots) \in Q_n : x_i = Q_n, \text{ and } x_i = 0_K \text{ for all } i \in \mathbb{N} - \{n\} \},$$

and let $G_n = \varphi(Q_n')$. Then $Q_n'$ and $G_n$ are compact subgroups of $Q_n$ and $\Sigma_K$, respectively. Since $\varphi$ is uniformly continuous, it follows from $\lim_{n \to n} \text{diam}(Q_n) = 0$ that $\lim_{n \to n} \text{diam}(G_n) = 0$. Define the map $h_n : Q_n \to G_n$ by

$$h_n(x_n) = \varphi(0_K, \ldots, 0_K, x_n, 0_K, \ldots) \quad \text{for any } x_n \in Q_n.$$

It is easy to see that $h_n$ is a continuous isomorphism. Let $G_n = \prod_{n=1}^{\infty} G_n$. Then the map $H_n = \prod_{n=1}^{\infty} h_n : Q_n \to G_n$ is also a continuous isomorphism. Let $\Phi_n = \Phi_{G_n} : G_n \to \Sigma_K$ be the natural addition homomorphism. Noting that

$$\varphi(x) = \lim_{n \to \text{diam}} \varphi_n \left( \bigoplus_{i=1}^{n} \left( \bigoplus_{j=1}^{\infty} 0_K, \ldots, 0_K, x_i, 0_K, \ldots \right) \right) = \lim_{n \to \infty} \bigoplus_{i=1}^{n} h_i(x_i),$$

for any $x = (x_1, x_2, \ldots) \in Q_n$, we obtain $\Phi_n H_n = \varphi$. Hence $\Phi_n = \psi H_n^{-1}$ is a continuous isomorphism. Since $Q_n, G_n$ and $\Sigma_K$ are compact metric spaces, all of the continuous isomorphisms $\varphi, H_n$ and $\Phi_n$ are homeomorphisms.

For any $n \in \mathbb{N}$, let $p_n : G_n \to G_n$ be the projection defined by $p_n(x) = x_n$ for all $x = (x_1, x_2, \ldots) \in G_n$, and let $\Psi_{G_n} = p_n \Phi_n^{-1} : K_n \to G_n$. Define a map

$$\Psi_n = \Psi_{G_n} = (\Psi_{G_1}, \Psi_{G_2}, \ldots) : \Sigma_K \to G_n$$

by $\Psi_n(v) = (\Psi(v_1), \Psi(v_2), \ldots)$, for all $v \in \Sigma_K$. Then $\Psi_n = \Phi_n^{-1}$. Write $v_n = \Psi_{G_n}(1_k)$. Let $f(v_n) : G_n \to G_n$ be the translation on $G_n$ by adding $v_n$. Note that the unit element $0_K$ of $\Sigma_K$ is also the unit element of $G_n$. Since the basic orbit $O_K$ of adding machine $f_K$ is dense in $\Sigma_K$, the orbit $\text{Orb}(0_K, f(v_n)) = \Psi_{G_n}(0_K)$ is dense in $G_n$. Therefore, by Theorem 3.1 of [18], $\Psi_{G_n}$ is non-expansive, and there exist a sequence $K_n \in \mathbb{N}^n$ and an isometric isomorphism $\Psi_{G_n} : G_n \to \Sigma_n$ such that $K_n \approx \Sigma_k$, $\Psi_{G_n}(v_n) = 1_{k_n}$ and $\Psi_{G_n} \circ \psi_{G_n} = \psi_{G_n}$. These imply that $\Psi_n = \Psi_{G_n} = (\Psi_{G_1}, \Psi_{G_2}, \ldots)$ is a non-expansive isomorphism, $\Psi_{G_1} \Sigma_K = \prod_{n=1}^{\infty} \Psi_{G_n} \Sigma_n : G_n \to \Sigma_K$ is an isometric isomorphism, and the compound natural homomorphism $\Psi_K : (\Psi_{G_1}, \Psi_{G_2}, \ldots) = (\prod_{n=1}^{\infty} \Psi_{G_n} \Sigma_n) \circ \Psi_n = \Psi_{G_n} \Sigma_K \circ \Phi_n^{-1}$.

Since $\Psi_{G_n}$ is non-expansive, $\Psi_n = \Phi_n^{-1}$ is non-continuous. This with Proposition 3.2 implies that $\Phi_n$ is an isometric isomorphism, and hence, $\Psi_n = \Psi_{G_n} \Sigma_K \circ \Phi_n^{-1}$ is also an isometric isomorphism.

Note that, in the proof of $\Phi_n$ being non-continuous, it suffices that $\Phi_n$ is a continuous isomorphism. Hence we can similarly prove that $\varphi$ is non-continuous.

For any $i, m \in \mathbb{N}$ with $i < m$, define the compound natural homomorphism $\Psi_{K_i K_m} = (\Psi_{K_i}, \Psi_{K_m}) : \Sigma_K \to \Sigma_{K_i} \times \Sigma_{K_m}$ by

$$\Psi_{K_i K_m}(v) = (\Psi_{K_i}(v), \Psi_{K_m}(v)) \quad \text{for any } v \in \Sigma_K.$$
$K_n \in \mathbb{N}^N$. By Theorem 2.3, $\Sigma_{K_n}$ is isometrically isomorphic to $\Sigma_K = \prod_{n=1}^{\infty} \Sigma_{K_n}$, and hence, $\Sigma_{K_n}$ is isometrically isomorphic to $\Sigma_K$. Therefore, by Lemmas 3.3 and 3.4, we have $K = K_n = \bigotimes_{n=1}^{\infty} K_n$. Theorem 3.5 is proven. □

Let $\Phi_{Q_n, K} : Q_n \rightarrow \Sigma_K$ be the natural addition homomorphism. From the proof of Theorem 3.5 we see that, in Theorem 3.5, if $\varphi = \Phi_{Q_n, K}$, that is, if the natural addition homomorphism $\Phi_{Q_n, K}$ itself is an isomorphism, then we can take $G_n = Q_n$ (i.e., $G_n = Q_n$ for each $n \in \mathbb{N}$), and take $h_n$ to be the identity map of $Q_n$ (i.e., for each $n \in \mathbb{N}$, take $h_n$ to be the identity map of $Q_n$). Thus, from Theorem 3.5 we get

**Corollary 3.6.** Let $K \in \mathbb{N}^N$, let $G_1, G_2, \ldots$ be compact subgroups of $\Sigma_K$, and let $G_n = \prod_{i=1}^{\infty} G_i$. If $\lim_{n \to \infty} \text{diam}(G_n) = 0$, and the natural addition homomorphism $\Phi_n = \Phi_{Q_n, K} : G_n \rightarrow \Sigma_K$ is an isomorphism, then $\Phi_n$ is an isometric isomorphism.

**Remark.** A subgroup $G$ of $\Sigma_K$ is said to be trivial if $G$ contains only the unit element $0_K$. In Theorem 3.5, if there exists $m > 1$ such that all of $Q_{m+1}, Q_{m+2}, \ldots$ are trivial subgroups of $\Sigma_K$, then $G_{m+1}, G_{m+2}, \ldots$ are trivial subgroups, $K_{m+1}, K_{m+2}, \ldots$ are trivial sequences, and $G_n$ is isometrically isomorphic to $G_m$. Let $\Phi_n$ be isometrically conjugate to $\Phi_m$. Hence Theorem 3.5 and Corollary 3.6 still hold if we replace $Q_n, G_n, H_n, \ldots, \Phi_n$ by $Q_m, G_m, H_m, \ldots, \Psi_m$, respectively.

In order to study the structures of $K$, $\Sigma_K$, and $f_K$, we raise the following definition.

**Definition 3.7.** Let $K \in \mathbb{N}^N$, and let $G$ be a compact subgroups of $\Sigma_K$.

1. A sequence $K = (K_1, K_2, \ldots) \in (\mathbb{N}^N)^{\mathbb{N}}$ is called a decomposition (resp. normal decomposition) of $K$ if $K \sim \bigotimes_{n=1}^{\infty} K_n$ (resp. $K = \bigotimes_{n=1}^{\infty} K_n$), and $K_1, K_2, \ldots$ are pairwise relatively prime in this case, every $K_n$ is called a factor (resp. normal factor) of $K$.

2. A sequence $(Q_1, Q_2, \ldots)$ of compact subgroups of $G$ is called a decomposition (resp. normal decomposition) of $G$ if $G$ is continuously isomorphic (resp. isometrically isomorphic) to the direct product group $G_n = \prod_{i=1}^{\infty} G_i$. In this case, every $Q_n$ is a factor (resp. normal factor) of $G$.

3. Let $K = (K_1, K_2, \ldots) \in (\mathbb{N}^N)^{\mathbb{N}}$. The corresponding sequence $(f_{K_1}, f_{K_2}, \ldots)$ of adding machines is called a decomposition (resp. normal decomposition) of the adding machine $f_K : \Sigma_K \rightarrow \Sigma_K$ if $f_K$ is topologically conjugate (resp. isometrically conjugate) to the compound adding machine $f_K = \prod_{n=1}^{\infty} f_{K_n} : \Sigma_K \rightarrow \Sigma_K$ (resp. $\bigotimes_{n=1}^{\infty} \Sigma_{K_n}$). In this case, every $f_{K_n}$ is called a factor (resp. normal factor) of $f_K$.

4. A decomposition $(K_1, K_2, \ldots)$ (resp. $(Q_1, Q_2, \ldots)$, resp. $(f_{K_1}, f_{K_2}, \ldots)$ of $K$ (resp. $G$, resp. of $f_K$) is said to be trivial if there exists $m \in \mathbb{N}$ such that, for any $n \in \mathbb{N} - \{m\}$, $K_n$ (resp. $Q_n$, resp. $f_{K_n}$) is trivial.

5. A (or $G$, or $f_K$) is said to be decomposable (resp. normally decomposable) if it has a non-trivial decomposition (resp. non-trivial normal decomposition).

6. A decomposition $(K_1, K_2, \ldots)$ (resp. $(Q_1, Q_2, \ldots)$, resp. $(f_{K_1}, f_{K_2}, \ldots)$ of $K$ (resp. $G$, resp. of $f_K$) is said to be complete if, for any $n \in \mathbb{N}$, $K_n$ (resp. $Q_n$, resp. $f_{K_n}$) is indecomposable.

Between decomposabilities of $K$, $\Sigma_K$, and $f_K$, we have the following theorem.

**Theorem 3.8.** For any sequence $K \in \mathbb{N}^N$, the following seven conditions are equivalent:

A) $K$ is not a pure sequence.

B) $K$ is decomposable.

C) $K$ is normally decomposable.

D) The topological group $\Sigma_K$ is decomposable.

E) The topological group $\Sigma_K$ is normally decomposable.

F) The adding machine $f_K : \Sigma_K \rightarrow \Sigma_K$ is decomposable.

G) The adding machine $f_K : \Sigma_K \rightarrow \Sigma_K$ is normally decomposable.

**Proof.** (E) ⇒ (D) and (G) ⇒ (F) are trivial. By the definition of pure sequence, (B) ⇔ (A) ⇔ (C) are clear.

(C) ⇒ (E) and (C) ⇒ (G) Assume that the condition (C) holds. Let $K = (K_1, K_2, \ldots)$ be a non-trivial normal decomposition of $K$. Then the condition (7) in Theorem 2.3 holds, and hence the other six conditions in Theorem 2.3 also hold. Write $K_n = \bigotimes_{i=1}^{\infty} K_i$. Then $K = K_n$. From the condition (4) in Theorem 2.3 we see that $\Sigma_K$ is isometrically isomorphic to $\Sigma_K = \prod_{i=1}^{\infty} \Sigma_{K_i}$. Hence, there exist compact subgroups $G_1, G_2, \ldots$ of $\Sigma_K$ such that $G_n$ is isometrically isomorphic to $\Sigma_{K_n}$, for all $n \in \mathbb{N}$, and therefore, $\Sigma_K$ is isometrically isomorphic to the direct product group $G_n = \prod_{i=1}^{\infty} G_i$. This means that the condition (E) holds. Analogously, from the condition (3) in Theorem 2.3 we see that $f_K$ is isometrically conjugate to $f_{K_n}$. Thus the condition (G) holds also.

(D) ⇒ (C) Assume that the condition (D) holds. Let $(Q_1, Q_2, \ldots)$ be a non-trivial decomposition of $\Sigma_K$. Then the conditions of Theorem 3.5 hold, and by the conclusions (1), (3), and (4) of Theorem 3.5, the condition (C) holds.

(F) ⇒ (A) Assume that the condition (F) holds. Let $K_1, K_2, \ldots$ and the compound adding machine $f_K = \prod_{n=1}^{\infty} f_{K_n}$ be the same as in (3) of Definition 3.7. Then $f_K$ is topologically conjugate to $f_{K_n}$. This means that the condition (1) in Theorem 2.3
holds, and hence the other six conditions in Theorem 2.3 also hold. Write $K_N = \bigotimes_{n=1}^{\infty} K_n$. From conditions (1) and (2) in Theorem 2.3 we see that $f_K$ and $f_{K_n}$ are topologically conjugate, and it follows from (1) of Lemma 3.4 that $K \sim K_N$. Since $f_{K_1}$ and $f_{K_2}$ are non-trivial adding machines, $K_1$ and $K_2$ are non-trivial sequences. By (7) in Theorem 2.3, $K_1$ and $K_2$ are relatively prime. Thus $K_N$ is not pure, and hence, $K$ is not pure, i.e. the condition (A) holds. Theorem 3.8 is proven. □

For a given $K \in \mathbb{N}^N$, it is easy to determine whether $K$ is a pure sequence. Thus, by Theorem 3.8, we can easily determine whether $\Sigma_K$ and $f_K$ are decomposable.

By Corollary 2.8 and Theorem 2.3, the following proposition is obvious.

**Proposition 3.9.** Any sequence $K \in \mathbb{N}^N$ and the corresponding topological group $\Sigma_K$ and adding machine $f_K$ have complete normal decompositions.

Note that $K$, $\Sigma_K$ and $f_K$ have complete normal decompositions does not imply that they are decomposable, since the complete normal decompositions may be trivial.

4. Adding machines having periodic orbit factors

Let $K, L \in \mathbb{N}^N$, and let $G$ be a compact subgroup of the $K$-adic topological group $\Sigma_K$. If $L$ is a factor of $K$ and $L$ itself is an essentially finite sequence, then $L$ is called a finite factor of $K$. If $G$ is a factor of $\Sigma_K$ and $G$ itself is a finite cyclic group, then $G$ is called a cyclic factor of $\Sigma_K$. If the adding machine $f_L$ is a factor of $f_K$, and $f_L$ itself is a cyclic permutation, then $f_L$ is called a periodic orbit factor of $f_K$. Note that any finite subgroup of $\Sigma_K$ is cyclic, and, conversely, any compact cyclic subgroup of $\Sigma_K$ is finite. By Theorems 2.3, 3.5 and 3.8, the following theorem is obvious.

**Theorem 4.1.** For any $K = (k_1, k_2, \ldots) \in \mathbb{N}^N$, the following seven conditions are equivalent:

1. There exists a prime number $p$ such that the set $\{n \in \mathbb{N} : k_n/p \in \mathbb{N}\}$ is a non-empty finite set.
2. $K$ has a non-trivial finite factor $L$.
3. $K$ has an indecomposable normal non-trivial finite factor $L$.
4. $\Sigma_K$ has a non-trivial cyclic factor $G$.
5. $\Sigma_K$ has an indecomposable normal non-trivial cyclic factor $G$.
6. $f_K$ has a non-trivial periodic orbit factor $f_L$.
7. $f_K$ has an indecomposable normal non-trivial periodic orbit factor $f_L$.

**Lemma 4.2.** If $K = (k_1, k_2, \ldots) \in \mathbb{N}^N$ is an essentially infinite pure sequence, then $\Sigma_K - \{0_K\}$ contains no element of finite order.

From the equivalence of conditions (1) and (4) in Theorem 4.1 we see that if $K$ is an essentially infinite pure sequence, then $\Sigma_K - \{0_K\}$ contains no non-trivial cyclic factor. However, it is easy to give a sequence $L = (l_1, l_2, \ldots) \in \mathbb{N}^N$ such that $\Sigma_L - \{0_L\}$ contains an element $v$ of finite order and the cyclic group $\langle v \rangle$ generated by $v$ is not a cyclic factor of $\Sigma_L$. Thus it is still necessary to give

**Proof of Lemma 4.2.** Since $K$ is an essentially infinite pure sequence, there is a prime number $p$ such that $k_i = p^n$ for each $i \in \mathbb{N}$ with $n_i \in \mathbb{Z}_+$, and $\{i \in \mathbb{N} : n_i > 0\}$ is an infinite set. Let $L = (p, p, p, \ldots)$ be the sequence purely consists of $p$. Then by Definition 1.2 we have $K \sim L$, and by Lemmas 3.4 and 3.3 we know that the topological groups $\Sigma_K$ and $\Sigma_L$ are topologically isomorphic. Therefore, it suffices to show that $\Sigma_L - \{0_L\}$ contains no element of finite order. Consider any $v = (a_1, a_2, \ldots) \in \Sigma_L - \{0_L\}$ and any $n \in \mathbb{N}$. Let $\lambda = \min\{i \in \mathbb{N} : a_i > 0\}$, $\mu = \max\{i \in \mathbb{Z}_+: n/p^i\}$ is an integer and $c = n/p^\mu$. Then $a_\lambda \in \mathbb{N}_{p-1}$, $n = cp^\mu$, and $c$ and $p$ are relatively prime. Suppose that $nv = (b_1, b_2, \ldots)$. Then we have $b_1 = 0$ for $1 \leq i < \mu + \lambda - 1$ and $b_{\mu + \lambda} \in \mathbb{N}_{p-1}$ with $b_{\mu + \lambda} \equiv a_\lambda c \pmod{p}$. Thus $nv \neq 0_L$, and hence $\Sigma_L - \{0_L\}$ contains no element of finite order. □

Recall that a continuous map $\varphi$ from a topological space $X$ to itself is called a periodic map if there is an $n \in \mathbb{N}$ such that $\varphi^n$ is the identity map $id_X$ of $X$; the smallest such $n$ is called the period of $\varphi$. A periodic map is said to be non-trivial if its period is greater than 1. Obviously, an element $v \in \Sigma_K$ is of order $n \in \mathbb{N}$ if and only if the translation $f_{kv}$ defined by (1.7) is a periodic map of period $n$. For any $i \in \mathbb{N}$, let $i_K = f^i_{K}(0_K)$ be defined as in (1.9). Then $f_{iK} = f^i_{K}$. For any $v \in \Sigma_K$, since the orbit $\text{Orb}(0 \in K, f_k) = \{i_K: i \in \mathbb{Z}_+\}$ is dense in $\Sigma_K$, there exists a sequence $i_1 < i_2 < i_3 < \cdots$ of positive integers such that $\lim_{i \to +\infty} (i_j)_K = v$. Note that a sequence $(v_1, v_2, v_3, \ldots)$ of points in $\Sigma_K$ converges to a point $v \in \Sigma_K$ if and only if the sequence $(f_{v_1K}, f_{v_2K}, f_{v_3K}, \ldots)$ of translations converges uniformly to $f_{vK}$. Therefore, from Theorem 4.1, Proposition 3.9 and Lemma 4.2 we obtain the following theorem, which is a supplement of Theorem 4.1.

**Theorem 4.1'.** For any $K = (k_1, k_2, \ldots) \in \mathbb{N}^N$, the following four conditions are equivalent:

4. $\Sigma_K$ has a non-trivial cyclic factor $G$.
8. There exists an element $v \in \Sigma_K$ such that the translation $f_{vK}$ is a non-trivial periodic map.
Proof. Let \(K \in \{0_k\}\) contains an element of finite order.

(10) The sequence \((f_k, f_K^2, f_K^3, \ldots)\) of iterates of the adding machine \(f_K\) has a subsequence converging uniformly to a non-trivial periodic map of \(\Sigma_K\).

From Theorems 2.3, 3.5 and 3.8 we also obtain the following two propositions at once.

**Proposition 4.3.** For any \(K = (k_1, k_2, \ldots) \in \mathbb{N}^\mathbb{N}\,\), the following seven conditions are equivalent:

1. For any prime number \(p\), the set \(\{n \in \mathbb{N}: k_n/p \in \mathbb{N}\}\) is a finite set.
2. \(K\) has a decomposition entirely consisting of finite factors.
3. \(K\) has a complete normal decomposition entirely consisting of finite factors.
4. \(\Sigma_K\) has a decomposition entirely consisting of cyclic factors.
5. \(\Sigma_K\) has a complete normal decomposition entirely consisting of cyclic factors.
6. \(f_K\) has a decomposition entirely consisting of periodic orbit factors.
7. \(f_K\) has a complete normal decomposition entirely consisting of periodic orbit factors.

**Proposition 4.4.**

1. Let \((Q_1, Q_2, \ldots)\) be a decomposition of \(\Sigma_K\). If \(Q_i\) and \(Q_m\) are finite cyclic subgroups of \(\Sigma_K\), for some \(m > i \geq 1\), then the orders of \(Q_i\) and \(Q_m\) are relatively prime.

2. Let \((f_{K_1}, f_{K_2}, \ldots)\) be a decomposition of \(f_K\). If \(f_{K_i}\) and \(f_{K_m}\) are cyclic permutations, for some \(m > i \geq 1\), then the periods of \(f_{K_i}\) and \(f_{K_m}\) are relatively prime.

5. Calculation of finite cyclic subgroups of \(\Sigma_K\)

For an explicitly given sequence \(K \in \mathbb{N}^\mathbb{N}\), it is easy to determine whether \(K\) has a non-trivial finite factor \(L\). If \(K\) has such a factor \(L\), then by Theorems 2.3 and 3.8 we know immediately that \(f_L\) is a non-trivial periodic orbit factor of the adding machine \(f_K\), and the topological group \(\Sigma_K\) has a non-trivial finite cyclic subgroup \(G\) of which the order is equal to the period of \(f_L\). However, in general, we cannot directly know what elements in \(\Sigma_K\) the non-trivial finite cyclic subgroup \(G\) consists of. In order to find these elements, or to find a generating element of \(G\), we need to do some calculation.

**Lemma 5.1.** Let \(K = (m, k_1, k_2, \ldots) \in (\mathbb{N} - \{1\})^\mathbb{N}\), i.e. \(K\) be a sequence of integers greater than 1. Suppose that \(m\) and \(k_n\) are relatively prime, for all \(n \in \mathbb{N}\). Then there exists a unique pair of sequences \((b_0, b_1, b_2, \ldots) \in \Sigma_K\) and \((c_0, c_1, c_2, \ldots) \in \mathbb{N}^\mathbb{N}\) with \(b_0 = c_0 = 1\) such that, for all \(n \in \mathbb{N}\),

\[
\begin{align*}
 b_n m + c_{n-1} &\equiv 0 \pmod{k_n}, \quad \text{and} \quad c_n = (b_n m + c_{n-1})/k_n. 
\end{align*}
\]

(5.1)

**Proof.** \(b_0 = c_0 = 1\) are uniquely determined. Assume that \(b_{n-1}\) and \(c_{n-1}\) are uniquely determined, for some \(n \in \mathbb{N}\). Then, since \(m\) and \(k_n\) are relatively prime, there is a unique \(b_n \in \mathbb{Z}_{k_n}\) such that \(b_n m + c_{n-1} \equiv 0 \pmod{k_n}\), and then \(c_n \in \mathbb{N}\) is uniquely determined by \(c_n = (b_n m + c_{n-1})/k_n\). Hence Lemma 5.1 holds.\(\square\)

The following proposition shows that Lemma 5.1 actually gives a procedure to calculate the element \(w = (1, b_1, b_2, \ldots)\) of order \(m\) in \(\Sigma_K\).

**Proposition 5.2.** Let \(K = (m, k_1, k_2, \ldots) \in (\mathbb{N} - \{1\})^\mathbb{N}\). Suppose that \(m\) and \(k_n\) are relatively prime, for all \(n \in \mathbb{N}\). Let \(w = w_K = (b_0, b_1, b_2, \ldots) \in \Sigma_K\) be the sequence determined by \(K\) as in Lemma 5.1. Then \(v = (1, a_1, a_2, \ldots)\) be an element in the topological group \(\Sigma_K\). Then \(v\) is an element of order \(m\) if and only if \(w = v\).

**Proof.** Let \(L = (m, 1, 1, \ldots)\), and \(M = (1, k_1, k_2, \ldots)\). Then \(K = L \otimes M\), and the topological group \(\Sigma_L\) is a cyclic group of order \(m\). Let \(1_L = (1, 0, 0, \ldots) \in \Sigma_L\) and \(0_M = (0, 0, 0, \ldots) \in \Sigma_M\).

Then \(1_L\) is a generating element of \(\Sigma_L, \Sigma_L \times \{0_M\}\) is a cyclic subgroup of the direct product group \(\Sigma_L \times \Sigma_M\), and \((1_L, 0_M)\) is a generating element of \(\Sigma_L \times \{0_M\}\) with order \(m\). Let \(\Psi = \Psi_{KL} : \Sigma_K \to \Sigma_L \times \Sigma_M\)

be the compound natural homomorphism of \(\Psi_{KL} : \Sigma_K \to \Sigma_L\) and \(\Psi_{KM} : \Sigma_K \to \Sigma_M\) defined as in (2.12). Then, by Lemma 2.7, \(\Psi\) is an isometric isomorphism. Let

\[
w_n = (b_0, b_1, \ldots, b_n, 0, 0, \ldots), \quad \text{for any} \ n \in \mathbb{N}.
\]
Then $\lim_{n \to \infty} w_n = w$. Write $\lambda_0 = 1$, $\lambda_1 = m$, $\lambda_n = m \cdot \prod_{i=1}^{n-1} k_i$, and $\beta_n = \sum_{i=0}^{n} b_i \lambda_i$, for all $n \geq 2$. Let $\beta_K = f_\beta^k(0_K)$ be defined as in (1.9). Then we have

$$w_n = (\beta_n)_K, \quad \Psi_{KL}(w_n) = (\beta_n)_L \quad \text{and} \quad \Psi_{KM}(w_n) = (\beta_n)_M.$$  \hfill (5.2)

Suppose that $(\beta_n)_M = (0, b_{n1}, b_{n2}, b_{n3}, \ldots)$. For any $n \geq 2$, from (5.1) we get

$$\beta_n = (1 + b_m) + b_2 m k_1 + b_3 m k_1 k_2 + \cdots + b_{n} m k_1 k_2 \cdots k_{n-1}$$

$$= (c_1 k_1 + b_2 m k_1) + b_3 m k_1 k_2 + \cdots + b_{n} m k_1 k_2 \cdots k_{n-1}$$

$$= c_{n-1} k_1 k_2 \cdots k_{n-1} + b_{n1} m k_1 k_2 \cdots k_{n-1}$$

$$= c_{n1} k_1 k_2 \cdots k_n.$$  

Thus $b_{n1} = b_{n2} = \cdots = b_{nm} = 0$, and hence $\Psi_{KM}(w) = \lim_{n \to \infty} \Psi_{KL}(w_n) = \lim_{n \to \infty} (\beta_n)_M = 0_M$. Noting that $\beta_n \equiv 1 \pmod{m}$, we have $(\beta_n)_L = 1_L$. This implies that $\Psi_{KL}(w) = \lim_{n \to \infty} \Psi_{KL}(w_n) = \lim_{n \to \infty} (\beta_n)_L = 1_L$. Therefore, we obtain $\Psi(w) = (\Psi_{KL}(w), \Psi_{KM}(w)) = (1_L, 0_M)$, and hence, if $v = w$ then $v = w^{-1}(1_L, 0_M)$ is an element of order $m$.

On the other hand, if $v \neq w$ then there exists the least $n \in \mathbb{N}$ such that $a_{0n} - b_{mn} \neq 0$, and we have $0 < |a_{0n} - b_{mn}| < k_v$. Suppose that $m(v - w) = (a_0, a_1, a_2, \ldots)$. Then $a_0 = a_3 = \cdots = a_{n-2} = 0$ and $a_0 \neq 0$ since $a_0 \equiv m(a_{0n} - b_{mn}) \neq 0 \pmod{k_v}$. Thus we have $m(v - w) \neq 0_K$, which with $mw = 0_K$ implies that $mv \neq 0_K$, and hence $v$ is not an element of order $m$.

Proposition 5.2 is proven. □

Recall that a sequence $A = (a_0, a_1, a_2, \ldots)$ of integers is said to be eventually periodic if there exist $p$ and $q \in \mathbb{N}$ such that

$$a_{n+p} = a_n \quad \text{for all} \quad n \geq q.$$  \hfill (5.3)

The least $p \in \mathbb{N}$ for which (5.3) holds is called the eventual period of $A$. Note that each eventually periodic sequence is bounded.

**Proposition 5.3.** Let $K = (m, k_1, k_2, \ldots) \in (\mathbb{N} - \{1\})^{\mathbb{N}}$ and the element $w = (1, b_1, b_2, \ldots) \in \Sigma_K$ of order $m$ be the same as in Proposition 5.2. If $K$ is an eventually periodic sequence, then $w$ is also an eventually periodic sequence.

Furthermore, let $p$ be the eventual period of $K$. Take $q \in \mathbb{N}$ such that $k_q = \min\{k_q, k_{q+1}, \ldots, k_{q+p}\}$ and $k_{q+p} = k_n$ for all $n \geq q$.

Then the eventual period of $w$ is not greater than $pk_q(m - 1)$.

**Proof.** Let $(c_0, c_1, c_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$ be as in Lemma 5.1. For all $n \in \mathbb{N}$, noting that $b_{nm} \in \mathbb{Z}_{b_m}$, from (5.1) we get $c_n \in \mathbb{N}_{b_m}$. Therefore, there exist integers $0 \leq i < j < k_q(m - 1)$ such that $b_{q+jp} = b_{q+ip}$ and $c_{q+ip} = c_{q+jp}$, which with (5.1) implies that $b_{n+(j-i)p} = b_{n}$ and $c_{n+(j-i)p} = c_{n}$ for all $0 \geq q + ip$. Thus $w$ is an eventually periodic sequence, and the eventual period of $w$ is a factor of $(j - i)p$, which is not greater than $pk_q(m - 1)$. □

**Remark 5.4.** Let $K = (m, k_1, k_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$ be as in Proposition 5.2 and its proof. By Lemma 2.7, the compact natural homomorphism $\Psi_{KL} : \Sigma_K \to \Sigma_L \times \Sigma_M$ is an isometric isomorphism. Comparing the $K$-adic topological group $\Sigma_K$ and the direct product group $\Sigma_L \times \Sigma_M$, we observe the following interesting phenomena:

1. $\Sigma_L \times \Sigma_M$ has a cyclic subgroup of order $m$, which can be easily described by a simple and explicit expression, that is $\Sigma_L \times \{0_M\}$.
2. However, in general, the corresponding cyclic subgroup $\Psi_{KL}^{-1}(\Sigma_L \times \{0_M\})$ of $\Sigma_K$ with order $m$ is obscure, and it may be difficult even impossible to precisely describe the elements of $\Psi_{KL}^{-1}(\Sigma_L \times \{0_M\})$ in the form of sequences. In fact, let the generating element $w = (1, b_1, b_2, \ldots) \in \Psi_{KL}^{-1}(\{1\}, 0_M)$ of $\Psi_{KL}^{-1}(\Sigma_L \times \{0_M\})$ be as in Proposition 5.2 and its proof. If $K$ is an eventually periodic sequence and the number $pk_q(m - 1)$ appearing in Proposition 5.3 is large, then the calculation of precise value of $w$ may be quite troublesome. If $K$ is not an eventually periodic sequence, then, in general, it will be impossible to find an expression of $b_{n}$ via $m, k_1, k_2, \ldots$, $k_n$ for all $n \in \mathbb{N}$. In this case, we can calculate approximate values of $w$ but we cannot obtain its absolutely precise value.

**Example 5.5.** Let the sequence $K = (m, k_1, k_2, \ldots)$ and the element $w_K = (1, b_1, b_2, \ldots)$ of order $m$ in $\Sigma_K$ be as in Proposition 5.2. If $K$ is an eventually periodic sequence and the number $pk_q(m - 1)$ appearing in Proposition 5.3 is relatively small, then by (5.1) we can easily calculate $b_n$ for all $n \in \mathbb{N}$. For example,

- if $K_1 = (3, 2, 2, 2, \ldots)$ then $w_{K_1} = (1, 1, 0, 1, 0, 1, 0, \ldots)$,
- if $K_2 = (5, 2, 3, 2, 3, \ldots)$ then $w_{K_2} = (1, 1, 0, 1, 0, 1, 0, \ldots)$.  

Author's personal copy
(note that $w_{K_1}$ and $w_{K_2}$ are not the same elements since they are in different topological groups), and if

$$K = (5, 4, 3, 4, 3, 3, 4, 3, 4, 3, 3, 3, 4, 3, 3, 4, 3, 3, 4, 3, 3, 4, 3, \ldots),$$

then

$$w_K = (1, 3, 1, 1, 2, 0, 1, 2, 0, 3, 1, 1, 2, 0, 1, 2, 0, \ldots).$$

Acknowledgment

We would like to thank the referee for a variety of helpful suggestions concerning this paper.

References