The $\omega$-limit set of a graph map

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Received 8 September 2006; received in revised form 17 March 2007; accepted 20 March 2007

Abstract

Let $G$ be a graph and $f : G \to G$ be continuous. Denote by $P(f)$, $\overline{P(f)}$, $\omega(f)$ and $\Omega(f)$ the set of periodic points, the closure of the set of periodic points, $\omega$-limit set and non-wandering set of $f$, respectively. In this paper we show that: (1) $v \in \omega(f)$ if and only if $v \in P(f)$ or there exists an open arc $L = (v, w) \subset G$ such that every open arc $U = (v, c) \subset L$ contains at least 2 points of some trajectory; (2) $v \in \omega(f)$ if and only if every open neighborhood of $v$ contains at least $r + 1$ points of some trajectory, where $r$ is the valence of $v$; (3) $\omega(f) = \bigcap_{n=0}^{\infty} f^n(\Omega(f))$; (4) if $x \in \omega(f) - P(f)$, then $x$ has an infinite orbit.

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MSC: 37B40; 37E25; 54H20

Keywords: Graph map; Non-wandering set; $\omega$-limit set; Periodic point; Trajectory; Orbit

1. Introduction

In this paper, let $\mathbb{N}$ denote the set of all positive integers. Write $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$, $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and $\mathbb{Z}_n = \{0\} \cup \mathbb{N}_n$ for any $n \in \mathbb{N}$.

Let $X$ be a topological space and $C^0(X)$ the set of all continuous maps from $X$ to $X$. For $f \in C^0(X)$, we define the trajectory $y(x) = y(x, f)$ of a point $x \in X$ to be the sequence $f^k(x)$ ($k \in \mathbb{Z}^+$), where $f^i(x), f^j(x)$ with $i \neq j$ in the trajectory of $x$ yield distinct points even if $f^i(x) = f^j(x)$. The set $O(x, f) = \{f^k(x): k \in \mathbb{Z}^+\}$ is called the orbit of $x$. A point $x \in X$ is called a periodic point of $f$ with period $n$ if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n - 1$. We define the $\omega$-limit set of a point $x \in X$ to be the set

$$\omega(x, f) = \{y: \text{there exist positive integers } 1 \leq n_1 < n_2 < \cdots < n_k \cdots \text{ such that } f^{n_k}(x) \to y\}.$$ 

A point $x \in X$ is called a recurrent point of $f$ if $x \in \omega(x, f)$ and non-wandering point of $f$ if every open set containing $x$ contains at least two points of some trajectory. The set of periodic points, the set of recurrent points and the set of

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* Project supported by the Special Foundation of National Prior Basis Researches of China (G1999075108) and NNSF of China (10461001) and NSF of Guangxi (0640205).

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doi:10.1016/j.topol.2007.03.008
non-wandering points of $f$ are denoted by $P(f)$, $R(f)$ and $\Omega(f)$, respectively. Write $\omega(f) = \bigcup_{x \in X} \omega(x, f)$, which is called $\omega$-limit set of $f$. It is known that for any $f \in C^0(X)$,

$$P(f) \subset R(f) \subset \omega(f) \subset \Omega(f).$$

For any $A \subset X$, we use $\text{int} A$, $\partial A$ and $\overline{A}$ to denote the interior, the boundary and the closure of $A$, respectively.

It is well known that $\omega$-limit set plays an important role in dynamical systems (see [1–6]). There is a growing interest in studying the topological structure of $\omega$-limit sets for one-dimensional dynamical systems. In [7] (also see [1]), Sharkovskii studied $\omega$-limit sets for interval maps and obtained the following theorems.

**Theorem A.** Let $I$ be a closed real interval and $f \in C^0(I)$. If every open interval with left endpoint $v$ contains at least 2 points of some trajectory then $v \in \omega(f)$. The same conclusion holds if ‘left’ is replaced by ‘right’.

**Theorem B.** Let $I$ be a closed real interval and $f \in C^0(I)$, then $v \in \omega(f)$ if and only if every open interval containing $v$ contains at least 3 points of some trajectory.

The following example shows that conditions of Theorem A are not necessary for the point $v \in \omega(f)$.

**Example 1.** Let $I = [0, 1]$ and $f(x) = 1/2$ for any $x \in I$, then $1/2 \in \omega(f)$, but no open interval with left or right endpoint $1/2$ contains two points of the same trajectory.

Blokh [8] studied the set of non-wandering points and $\omega$-limit set for interval maps and obtained the following theorem.

**Theorem C.** Let $I$ be a closed real interval and $f \in C^0(I)$, then $\omega(f) = \bigcap_{n=0}^{\infty} f^n(\Omega(f))$.

But it is unclear if Theorem C holds for circle maps (see [1, p. 227]). In [2], Block and Coppel proved the following theorem (also see Theorem 5.4 of [9]).

**Theorem D.** Let $I$ be a closed real interval and $f \in C^0(I)$. If $x \in \omega(f) - \overline{P(f)}$, then $O(x, f)$ is an infinite set.

Recently, there has been a lot of work on the dynamics of continuous maps of a graph (see [10–14]), as this kind of research is related to the study of dynamics of a surface homeomorphism and the structure of attractors of a diffeomorphism (see [15,16]). In this paper, we shall show that analogs of Theorems A, B, C and D still hold for continuous maps of a graph.

By a graph we mean a compact connected one-dimensional polyhedron. Let $G$ be a graph. For any $x, y \in G$, define the metric $d(x, y)$ to be the minimal length of arcs in $G$ whose endpoints are $x$ and $y$. A subtree of $G$ is a subgraph of $G$ containing no simple closed curve. For $x \in G$ and a sequence of connected neighborhoods $\{V_i\}$ of $x$ with diam($V_i$) $\to$ 0, write $\text{val}_G(x) = \min\{\#\partial(V_i): i \in \mathbb{N}\}$, which is called the valence of $x$ (in $G$), where $\#(A)$ is the number of elements of a finite subset $A$ of $G$. $x$ is called an endpoint of $G$ if val$_G(x) = 1$ and a branch point of $G$ if val$_G(x) > 2$. We use $E(G)$ and $B(G)$ to denote the set of endpoints of $G$ and the set of branch points of $G$, respectively. A finite set $V(G) \supset B(G) \cup E(G)$ is a set of vertices of $G$ if for each simple closed curve $\Sigma$ of $G$, $\Sigma \cap V(G) \subset B(G) \cup E(G)$ when $\#(\Sigma \cap (B(G) \cup E(G))) \geq 3$ and $\#(\Sigma \cap V(G)) = 3$ when $\#(\Sigma \cap (B(G) \cup E(G))) < 3$, that is, we add some artificial points with valence 2 as vertices. In this way, each edge (the closure of some connected component of $G - V(G)$) is homeomorphic to [0, 1] and if $I$ and $J$ are two edges of $G$, then $\#(I \cap J) \leq 1$. For some edge $I$ of $G$ and any $a, b \in I$, we use $[a, b]$ (or simply $[a, b]$) if there is no confusion) to denote the smallest connected closed subset of $I$ containing $[a, b]$. Define $[a, b] = [a, b] - \{a\}$ and $(a, b) = (a, b) - \{b\}$. For any $x \in G$ and any $\varepsilon > 0$, write $B(x, \varepsilon) = \{y \in G: d(x, y) < \varepsilon\}$. Our main results are the following theorems.

**Theorem 1.** Let $G$ be a graph and $f \in C^0(G)$, then $v \in \omega(f)$ if and only if $v \in P(f)$ or there exists an open arc $L = (v, w)$ contained in some edge of $G$ such that every open arc $(v, c) \subset L$ contains at least 2 points of some trajectory.
Theorem 2. Let $G$ be a graph and $f \in C^0(G)$. Then $v \in \omega(f)$ if and only if every open neighborhood of $v$ contains at least $r+1$ points of some trajectory, where $r = \text{val}_G(v)$.

Theorem 3. Let $G$ be a graph and $f \in C^0(G)$, then $\omega(f) = \bigcap_{n=0}^\infty f^n(\Omega(f))$.

Theorem 4. Let $G$ be a graph and $f \in C^0(G)$. If $x \in \omega(f) - \overline{P(f)}$, then $O(x, f)$ is an infinite set.

2. Proofs of Theorems 1 and 2

In this section, we shall show Theorems 1 and 2. To do this we need the following propositions.

Proposition 1. Let $G$ be a graph and $f \in C^0(G)$. Suppose that $Z$ is a connected closed subset of $G$ and $Y = \bigcup_{n=1}^\infty f^n(Z)$. If there exist $s, t \in \mathbb{Z}^+$ with $s < t$ such that $f^s(Z) \cap f^t(Z) \neq \emptyset$, then $Z \cap \overline{Y} \subset Y$.

Proof. Since $f^s(Z) \cap f^t(Z) \neq \emptyset$ for some $s, t \in \mathbb{Z}^+$ with $s < t$, $Y$ has only finitely many connected components. Note that $f(Y) \cup f(Z) = Y$, we have that $f(Y) \cap (\overline{Y} - Y) = \emptyset$. Since both $f(Y)$ and $f(Z)$ are closed, it follows that $f(Y) \cap f(Z) = \emptyset$. Thus, $\overline{Y} - Y \subset f(\overline{Y} - Y)$. Note that $\overline{Y} - Y$ is a finite set, from which we have that $\overline{Y} - Y = f(\overline{Y} - Y)$. Thus $\overline{Y} - Y \subset P(f)$. Let $v \in Z \cap \overline{Y}$. If $v \in P(f)$, then $v \in Y$. If $v \notin P(f)$, then $v \notin \overline{Y} - Y$, which implies $v \in Y$. Proposition 1 is proven. \[\square\]

Corollary 1. Let $G$ be a graph and $f \in C^0(G)$. Then $v \in \Omega(f)$ if and only if there exist points $x_k \to v$ and integers $n_k \to \infty$ such that $f^{n_k}(x_k) = v$.

Proof. Let $v \in \Omega(f)$. We may assume that $v \notin P(f)$, since if $v$ has period $m$, we can take $x_k = v$ and $n_k = km$. For any $k \in \mathbb{N}$, we have $v \in \bigcup_{i=1}^\infty f^i(B(v, 1/k))$. It follows from Proposition 1 that $v \in \bigcup_{i=1}^\infty f^i(B(v, 1/k))$, which implies that there exist $x_k \in B(v, 1/k)$ and $n_k \in \mathbb{N}$ such that $f^{n_k}(x_k) = v$. Thus $x_k \to v$. Moreover, since $v \notin P(f)$, by taking subsequence we may assume that $n_k \to \infty$.

The sufficiency is obvious. Corollary 1 is proven. \[\square\]

Proposition 2. Let $G$ be a graph and $f \in C^0(G)$. Suppose that $v \in G$ and $L = (v, w)$ is an open arc contained in some edge of $G$. If every open arc $(v, c) \subset L$ contains at least 2 points of some trajectory, then $v \in \omega(f)$. Furthermore, one of the following two statements holds:

(P.1) There exist $y \in L$ and positive integers $m_1 < m_2 < m_3 < \cdots$ such that $f^{m_i}(y) \in L$ for every $i \in \mathbb{N}$ and \[\lim_{i \to \infty} f^{m_i}(y) = v.\]

(P.2) $v \in P(f)$.

Proof. We may assume without loss of generality that $[v, w] = [0, 1/2]$ with $v = 0$.

A map $f$ is said to have Property P if for any $\varepsilon \in (0, 1/4)$, there exist $[a, b, c] \subset (0, \varepsilon]$ with $a < b$ and $k \in \mathbb{N}$ such that $f^k([a, b]) = [0, c]$.

Claim 1. If $f$ has Property P, then the conclusion (P.1) holds.

Proof. If $f$ has Property P, then there exist points $0 < \cdots < a_3 < b_3 < a_2 < b_2 < a_1 < b_1 \leq 1/4$ and positive integers $k_1, k_2, k_3, \ldots$ such that $\lim_{i \to \infty} b_i = 0$ and $f^{k_i}([a_i, b_i]) \supset [a_{i+1}, b_{i+1}]$ for every $i \in \mathbb{N}$. Choose $y \in [a_1, b_1] \cap f^{-k_1}([a_2, b_2]) \cap f^{-k_{1-2}}([a_3, b_3]) \cap \cdots$, then $v \in \omega(y, f) \subset \omega(f)$ and the conclusion (P.1) follows. Claim A is proven. \[\square\]

Claim 2. If $f$ has no Property P, then $v \notin P(f)$.

Proof. Assume on the contrary that $v \notin P(f)$. Take $\varepsilon \in (0, 1/4)$ such that within $I = [0, \varepsilon]$ Property P does not hold. We claim that for every $m \in \mathbb{N}$ there exists $\delta_m > 0$ such that $(0, \delta_m) \cap f^m(I) = \emptyset$. Indeed, if not then for some $m \in \mathbb{N}$
and some $\delta > 0$, we have $[0, \delta] \subset f^m(I)$. Choose a sequence of points $z_i \in I$ so that $f^m(z_i) \in (0, \delta)$ and $f^m(z_i) \to 0$. By taking subsequence we may assume that $z_i \to z \in I$, then $f^m(z) = 0$ and $z \neq 0$. Let $z_i$ is close enough to $z$ such that $f^m([z, z_i]) \cap V(G) \subset [0]$. Choose $z' \in [z, z_i] \cap f^{-m}(0)$ such that $0 \notin f^{-m}((z', z_i])$, then $f^m([z', z_i]) = [0, \delta']$ for some $\delta' > 0$. Replacing $z_i$ by a point $z''$ very close to $z$, we will get points $z, z''$ and $\delta'$ playing the roles of $a, b, c$ from Property P, a contradiction.

Let $Y_n = \bigcup_{i=1}^n f^i(I)$ for every $n \in \mathbb{N}$ and $Y = \bigcup_{k=1}^\infty f^k(I)$. Write

$$W_1 = \left\{ z \in Y : \text{there exist positive integers } k_1 < k_2 < \cdots \text{ and points } w_i \in Y_{k_i+1} - Y_{k_i} \text{ such that } \lim_{i \to \infty} w_i = z \right\}.$$ 

Since for every $i \in \mathbb{N}, w_i \in Y_{k_i+1} - Y_{k_i} = f(Y_{k_i} \cup I) - f(Y_{k_i-1} \cup I)$, there exists $u_i \in Y_{k_i} - Y_{k_i-1}$ such that $f(u_i) = w_i$. Thus, $W_1 \cap f^{-1}(z) \neq \emptyset$ for any $z \in W_1$, which implies $f(W_1) \supset W_1$. Let $r_n$ be the number of the connected components of $Y_n$. Since $f^s(I) \cap f^t(I) \neq \emptyset$ for some $s, t \in \mathbb{Z}^+$ with $s < t$, we see that there exist $m \in \mathbb{Z}^+$ and $N \in \mathbb{N}$ such that for every $k \geq N, r_{k+1} - r_k = r_k$ and the number of the connected components of $Y_{k+1} - Y_k$ is less than $m$. Thus, $\#(W_1) \leq m$. Since $f(W_1) \supset W_1$, it follows that $f(W_1) = W_1 \subset P(f)$.

Since for every $m \in \mathbb{N}$ there exists $\delta_m > 0$ such that $(0, \delta_m) \cap f^m(I) = \emptyset$, it follows from the definition of $W_1$ that $0 \notin W_1$, a contradiction. Claim B is proven. \qed

From Claims A and B Proposition 2 follows. \qed

**Proof of Theorem 1.** The necessary is obvious. The sufficiency follows from Proposition 2. Theorem 1 is proven. \qed

By Theorem 1 and $P(f) \subset R(f) \subset \omega(f)$, we easily obtain the following corollary.

**Corollary 2.** Let $G$ be a graph and $f \in C^0(G)$, then

(i) $d(\Omega(f)) \subset \omega(f)$, where $d(\Omega(f))$ is the set of all accumulation points of $\Omega(f)$.

(ii) $\omega(f)$ is a closed subset of $G$.

(iii) $\overline{P(f)} \subset \overline{R(f)} \subset \omega(f)$.

**Remark 1.** When $G$ is an interval, Corollary 2 follows from [7].

**Proof of Theorem 2.** The necessary is obvious. Now, suppose that the sufficient condition is satisfied.

Assume on the contrary that $v \notin \omega(f)$. By Theorem 1 choose $r$ open pairwise disjoint wandering intervals $(v, a_1) = I_1, \ldots, (v, a_r) = I_r$ such that $(\bigcup_{j=1}^r I_j) \cap (O(v, f) = \emptyset$. Since $v$ is not periodic, it follows $f^{-i}(v) \cap f^{-j}(v) = \emptyset$ for any $i > j \geq 0$. Also $f^n(I_j) \neq \{v\}$ for any $n \in \mathbb{N}$ and any $1 \leq j \leq r$ because $v \notin P(f)$. Hence by Sierpinski’s theorem (see Theorem 6 of [17], p. 173) each interval $I_j$ contains points which are not preimages of $v$, and so we may assume that $f^n(a_j) \neq v$ for every $n \in \mathbb{N}$ and $1 \leq j \leq r$. Set $U = \{v\} \cup (\bigcup_{j=1}^r I_j)$. We may assume that $n_1 < n_2 < \cdots < n_{r+1}$ are such that $f^{n_1}(z) \in U$ for some $z$. Since all $I_j$ are wandering, it follows $f^{n_s}(z) = v$ for some $1 \leq s \leq r+1$. Note that $U \cap f(v, f) = \emptyset$, then $s = r+1$. We may assume that $f^{n_j}(z) \in I_j, j = 1, \ldots, r$. Consider the interval $(v, f^{n_j}(z)) \subset I_j$. Let $J_1 = \{y, f^{n_j}(z)\}$ be the component of the set $f^{-\{n_1, \ldots, n_j\}}(v) \cap [v, f^{n_j}(z)]$ containing $f^{n_j}(z)$, then $f^{n_j-n_1}(J_1) \subset I_j$ for $1 \leq j \leq r$. By continuity of $f$ and choice of $y$ we may take $w$ to be close enough to $y$ such that $\{f^{n_j-n_1}(w): j = 1, \ldots, r+1\} \subset U - \{v\}$, a contradiction. Theorem 2 is proven. \qed

**3. Proofs of Theorems 3 and 4**

In this section, we shall give the proofs of Theorems 3 and 4.

**Proposition 3.** Let $G$ be a graph and $f \in C^0(G)$. If $x \in \Omega(f) - R(f)$, then there exist $\varepsilon > 0$ with $B(x, \varepsilon) \cap V(G) \subset \{x\}$ and $y \in \partial B(x, \varepsilon)$ such that $B(x, \varepsilon) \cap (\bigcup_{n=1}^\infty f^n([x, y])) = \emptyset$.
**Proof.** Let \( \text{val}_G(x) = k \) and \( c = \min\{d(x, O(f(x, f)), d(x, \partial f), d(x, V(G) - \{x\}) \} \). For any \( c > \varepsilon > 0 \), write \( \partial B(x, \varepsilon) = \{y_{1, \ldots, k} \ldots, y_{k, \ldots, 1} \} \) and \( W_i = \bigcup_{n=1}^\infty f^n(x, y_{1, \ldots, k}) \).

Assume on the contrary that for any \( i \in \mathbb{N}_k \) and any \( \varepsilon \) in \( (0, c) \), we have that \( \overline{B(x, \varepsilon)} \cap W_i \neq \emptyset \). Then for any \( i \in \mathbb{N}_k \) there exists points \( x_i \in (x, y_{i, \ldots, k}) \) and positive integers \( m_i \to \infty \) such that \( f^{m_i}(x_i) \to x \), from which we have that \( f^s((x, y_{i, \ldots, k}) \cap f^t((x, y_{i, \ldots, k})) \neq \emptyset \) for some \( 0 \leq s < t \) and \( x \in W_{i, \ldots, k} \). Thus, it follows from Proposition 1 that \( x \in W_{i, \ldots, k} \). Let \( a_i \in f^{-m_i}(x) \cap (x, y_{i, \ldots, k}) \) for some \( m_i \in \mathbb{N} \) with \( (x, a_i) \cap f^{-m_i}(x) = \emptyset \) and \( b_i \in (x, a_i) \cap f^{-m_i}(\partial B(x, \varepsilon)) \) with \( (b_i, a_i) \cap f^{-m_i}(\partial B(x, \varepsilon)) = \emptyset \), then \( f^{m_i}([a_i, b_i]) = [x, y_{i, \ldots, k}] \) for some \( j_i \in \mathbb{N}_k \). For \( i_0 = 1 \), there exist \( i_1, \ldots, i_k \in \mathbb{N}_k \), \( m_0, m_1, \ldots, m_{k-1} \in \mathbb{N} \) and \( a_j, b_j \in (x, y_{j, \ldots, k}) \) for every \( j \in Z_{k-1} \) satisfying
\[
 f^{m_i}([a_i, b_i]) = [x, y_{j, \ldots, k}] \quad \text{for every} \ j \in Z_{k-1}.
\]
Since \( \#(N, k, y_{j, \ldots, k}) = k \), it follows that \( i_j = i_r \) for some \( 0 \leq j < r \leq k \). Then there exist \( u, v \in [x, y_{j, \ldots, k}] \) such that
\[
 f^{m_r + \cdots + m_i}([u, v]) = [x, y_{j, \ldots, k}].
\]
This yields \( [x, y_{j, \ldots, k}] \cap P(f) \neq \emptyset \). A contradiction. Proposition 3 is proven. □

The key lemma for the proof of Theorem 3 will be the following.

**Lemma 1.** Let \( G \) be a graph, \( I \) is an edge of \( G \) and \( f \in C^0(G) \). If \( x \in \bigcap_{n=0}^\infty f^n(\Omega(f)) \cap \text{int} I \), then \( x \in \omega(f) \).

**Proof.** Assume on the contrary that \( x \in \bigcap_{n=0}^\infty f^n(\Omega(f)) \cap \text{int} I \). Then \( x \notin \text{int} \Omega(f) \) and for every \( n \in \mathbb{N} \) there exists \( x_n \in \Omega(f) \) such that \( f^n(x_n) = x \). Let \( (c, d) \subset \text{int} I \) with \( x \in (c, d) \) and \( (c, d) \cap P(f) = \emptyset \). By Propositions 2 and 3, we may assume that \( (c, x) \cap \bigcup_{n=1}^\infty f^n((c, x)) = \emptyset \) and \( (c, d) \cap \bigcup_{n=1}^\infty f^n([x, d]) = \emptyset \). Write
\[
 W = \bigcup_{n=1}^\infty f^n((c, d)).
\]
It is obvious that the following two statements hold:

(i) \( f(W) \subset W \), \( f(\overline{W}) \subset \overline{W} \) and \( W \) has finite connected components since \( x \in \Omega(f) \).

(ii) \( (c, x) \cap W = \emptyset \) and \( x \in W \).

We claim that \( x_n \in W \) for every \( n \in \mathbb{N} \). Indeed, for any \( n \in \mathbb{N} \), choose \( \delta_n > 0 \) such that \( f^n(B(x_n, \delta_n)) \subset (c, d) \). Since \( x_n \in \Omega(f) \), it follows from Corollary 1 that there exist \( w_n \in B(x_n, \delta_n) \) and \( m > n \) such that \( f^m(w_n) = x_n \), which implies \( x_n = f^{m-n}(f^n(w_n)) \in W \).

Since \( x \notin P(f) \), it follows that \( x_j = x_j \) for any \( i \neq j \). Note that \( \partial W \) is a finite set, we see that \( x_j \in \text{int} W \) for some \( N \in \mathbb{N} \). Let \( \varepsilon > 0 \) such that \( B(x_N, \varepsilon) \subset W \) and \( f^N(B(x_N, \varepsilon)) \subset (c, d) \). It follows from (i) and (ii) that \( f^N(B(x_N, \varepsilon)) \subset (x, d) \). According to Corollary 1, there exist \( u \in B(x_N, \varepsilon) \) and \( m \in \mathbb{N} \) such that \( f^m(u) = x_N \). Thus, \( z = f^N(u) \in [x, d] \) and \( f^m(z) = z \). A contradiction. Lemma 1 is proven. □

**Proof of Theorem 3.** Since \( f(\omega(f)) = \omega(f) \) and \( \omega(f) \subset \Omega(f) \), we have \( \omega(f) \subset \bigcap_{n=0}^\infty f^n(\Omega(f)) \). In the following we show that \( \omega(f) \supseteq \bigcap_{n=0}^\infty f^n(\Omega(f)) \).

Since \( \Omega(f) \supseteq f(\Omega(f)) \supseteq \cdots \) and \( f^n(\Omega(f)) \) is a closed subset of \( G \) for every \( n \geq 0 \), we have
\[
 f\left( \bigcap_{n=0}^\infty f^n(\Omega(f)) \right) = \bigcap_{n=0}^\infty f^n(\Omega(f)).
\]
Let \( x \in \bigcap_{n=0}^\infty f^n(\Omega(f)) \), then there exist \( x_1 = x, x_2, \ldots \in \bigcap_{n=0}^\infty f^n(\Omega(f)) \) such that \( f(x_{i+1}) = x_n \) for every \( n \in \mathbb{N} \).

If \( x_i = x_j \) for some \( 1 \leq i < j \), then \( x \in P(f) \subset \omega(f) \).

If \( x_i \neq x_j \) for any \( 0 \leq i < j \), then there exist an edge \( I \) of \( G \) and some \( i \in \mathbb{N} \) such that \( x_i \in \bigcap_{n=0}^\infty f^n(\Omega(f)) \cap \text{int} I \).

It follows from Lemma 1 that \( x_i \in \omega(f) \), which implies \( x \in \omega(f) \). Theorem 3 is proven. □

To show Theorem 4, we need the following lemma.

**Lemma 2.** Let \( f \in C^0(G) \) and \( \#V(G) = n \). Suppose that \( G - V(G) \) have \( m \) connected components and \( Z \) is an invariant closed set of \( f \) with \( k \) connected components. If \( v \in \Omega(f) - Z \) such that \( O(v, f) \cap Z \neq \emptyset \) and \( f^{-(2m+2k+n)}(v) \cap \Omega(f) \neq \emptyset \), then \( v \in P(f) \).
Proof. Choose $x \in f^{-(2m+2k+n)}(v) \cap \Omega(f)$ and write $x_i = f^i(x)$ for all $i \in \mathbb{Z}^+$. If $x_i = x_j$ for some $0 \leq i < j \leq (m+k)+n$, then $v \in P(f)$. If $x_i \neq x_j$ for any $0 \leq i < j \leq (m+k)+n$, then there exists a connected component $L$ of $G - Z - V(G)$ such that $\#(\{x_i: i \in \mathbb{Z}^{(k+m)+n}\} \cap L) \geq 3$. Let $x_i, x_j, x_r \in \{x_i: i \in \mathbb{Z}^{(k+m)+n}\} \cap L$ with $x_i = (x_j, x_r)$. For any neighborhoods $W$ of $v$, take a closed connected neighborhood $U$ of $x$ satisfying the following conditions:

(i) $U \cap V(G) \subset \{x\}$ and $f^{2(m+k)+n}(U) \subset W$;
(ii) $f^j(U), f^i(U), f^r(U) \subset L$ and $f^a(U) \cap f^b(U) = \emptyset$ for $a, b \in \{i, j, r\}$ with $a \neq b$.

Since $O(v, f) \cap Z \neq \emptyset$, $x \in \Omega(f)$ and $f(Z) \subset Z$, there exist $s, t \in \mathbb{N}$ with $2(m+k)+n < s < t$ and $y \in U$ such that $f^i(y) \in U$ and $f^s(x) \in Z$. Let $f^i(U) = [p, q]$ and $f^s(U) = [c, d]$, then there exist $\alpha, \beta \in [x, y]$ such that $f^{t+i}([\alpha, \beta]) = [c, d]$ or $f^{m+i}([\alpha, \beta]) = [p, q]$. Thus $f^j(U) \cap P(f) \neq \emptyset$ or $f^r(U) \cap P(f) \neq \emptyset$, which implies $W \cap P(f) \neq \emptyset$. Lemma 2 is proven. \(\square\)

Proof of Theorem 4. Assume on the contrary that there exists $k \in \mathbb{N}$ such that $x_k = f^k(x) \in P(f)$. Then $x \in \omega(f) - O(x_k, f) \subset \Omega(f) = O(x_k, f)$ and $f^{-n}(x) \cap \omega(f) \neq \emptyset$ for any $n \in \mathbb{N}$ since $f(\omega(f)) = \omega(f)$. By Lemma 2, we have that $x \in P(f)$. A contradiction. Theorem 4 is proven. \(\square\)

Remark 2. Theorem 4 follows also from [11–13], which was pointed out to us by the referee.

Acknowledgement

We wish to thank the referee for several valuable suggestions, especially the simplifications of proofs of Proposition 2 and Theorem 2 that have been adopted in this paper.

References