Ordering results on extremes of scaled random variables with dependence and proportional hazards

Rui Fang, Chen Li & Xiaohu Li

To cite this article: Rui Fang, Chen Li & Xiaohu Li (2018): Ordering results on extremes of scaled random variables with dependence and proportional hazards, Statistics, DOI: 10.1080/02331888.2018.1425998

To link to this article: https://doi.org/10.1080/02331888.2018.1425998

Published online: 24 Jan 2018.

Article views: 9

View related articles

View Crossmark data
Ordering results on extremes of scaled random variables with dependence and proportional hazards

Rui Fang\textsuperscript{a}, Chen Li\textsuperscript{b} and Xiaohu Li\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Shantou University, Shantou, Guangdong, People’s Republic of China; \textsuperscript{b}School of Sciences, Tianjin University of Commerce, Tianjin, People’s Republic of China; \textsuperscript{c}Department of Mathematical Sciences, Stevens Institute of Technology, Hoboken, NJ, USA

ABSTRACT
This study deals with random variables equipped with Archimedean copulas and following scale proportional hazards (SPHs) or revered hazards models. We build the usual stochastic order both between minimums of two SPHs samples with Archimedean survival copulas and between maximums from two scale proportional reversed hazards (PRHs) samples with Archimedean copulas. The hazard rate order between minimums of independent SPHs samples and the reversed hazard rate order between maximums of independent scale PRHs samples are both derived. Also we have a discussion on the dispersive order between minimums from samples with a common Archimedean survival copula. The present results either generalize or improve some related ones in the recent literature.

1. Introduction
The sample extremes play an important role in statistics, applied probability and other related areas. For example, in reliability theory the sample minimum $X_{1:n}$ and the sample maximum $X_{n:n}$ of component lifetimes $X_1, \ldots, X_n$ denote lifetimes of series and parallel systems, respectively, and in economics the maximum $X_{n:n}$ and the minimum $X_{1:n}$ define the winner’s price of the first-price sealed-bid auction and the first-price procurement auction, respectively (see, e.g. [1,2]). Due to nice applications, the past several decades have witnessed considerable research articles on sample extremes, and a large number of them focus on stochastic comparison between heterogeneous and homogeneous samples. On the other hand, owing to the complex distribution theory for sample with interdependent observations, most of the existing research are performed for mutual independent random variables (r.v.’s). For comprehensive references one may refer to Balakrishnan and Zhao [3]. Recently, some efforts are made to investigate stochastic comparisons on order statistics of r.v.’s with Archimedean copulas (e.g. [4–8]).

Along this line, a large number of references compare heterogeneous samples with proportional hazards (PH), proportional reversed hazards (PRHs), and scale model, respectively. Specifically, some attentions were also paid to heterogeneous and homogeneous samples with independent observations. See for example [9–19] and references therein. Besides, there are also a few exceptions. For example, by relaxing the independence to some Archimedean survival copula, Li and Li [6] compared minimums from Weibull samples, and Li and Fang [7] compared maximums from two heterogeneous PH samples with not necessarily common Archimedean copulas. Later, by considering PH samples...
with Archimedean survival copulas (PRH samples with Archimedean copulas), Fang et al. [4] further discussed the impact on both the sample minimums due to heterogeneity of frailty vector and the sample maximums due to heterogeneity of resilience vector. As for the scale sample equipped with Archimedean survival copulas or Archimedean copulas, Li et al. [5] investigated the impact on sample extremes and their adjacent order statistics due to heterogeneity of scale vector. It should be remarked here that almost all these studies consider models with only one heterogeneity vector, such as the frailty vector in PH model, the resilience vector in PRH model and the scale vector for the scale model. However, a lot of commonly used distributions usually possess two or more parameters, for example, Weibull distribution, Gamma distribution and exponentiated Weibull (EW) distribution. So, it is of important interest to investigate the effect of sample heterogeneity for these bivariate-parameter distributions. Recently, for generalized exponential (GE) distribution and gamma distribution, Kundu et al. [20] studied the impact of heterogeneity on sample maximums with two parameter vectors of independent samples being both heterogeneous. Actually, both GE and gamma samples in [20] are special cases of the scale proportional reversed hazards (SPRH) model (see Section 3 for the definition), having resilience and scale vectors of parameters.

This paper further investigates the impact of dependence and heterogeneity of the sample through studying extremes from two dependent scale proportional hazards (SPH) or SPRH samples. In line with Li et al. [5] and Fang et al. [4], we derive the usual stochastic order, the dispersive order of extremes from both the SPH sample with Archimedean survival copulas and the SPRH sample with Archimedean copulas. The hazard rate order and reversed hazard rate order on sample extremes from mutually independent observations are also established.

The rest of this paper is rolled out as follows: Section 2 recalls some related concepts and presents several useful lemmas to be used in deriving the main results in the remaining sections. Section 3 deals with two SPH samples with heterogeneous frailty vector and scale vector. We first study samples with Archimedean survival copulas and derive the usual stochastic order and dispersive order between sample minimums, and then we investigate the hazard rate order between minimums from two independent samples. Section 4 presents the usual stochastic order between maximums from SPRH samples with Archimedean copulas, and the reversed hazard rate order between maximums from independent SPRH samples. Finally, Section 5 concludes this study and highlights the future research in this line. All proofs of lemmas are appended in Section 6.

Throughout this note, we denote $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{I}_n = \{1, \ldots, n\}$, and

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \lambda = (\lambda_1, \ldots, \lambda_n), \quad 1 = (1, \ldots, 1), \quad 0 = (0, \ldots, 0),$$

$$\mathbb{D} = \{(x_1, \ldots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n\}, \quad \mathbb{D}^+ = \{(x_1, \ldots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n > 0\},$$

$$\mathbb{I} = \{(x_1, \ldots, x_n) : x_1 \leq x_2 \leq \cdots \leq x_n\}, \quad \mathbb{I}^+ = \{(x_1, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \cdots \leq x_n\}.$$  

Also, by $\alpha \preceq \lambda$ we mean $\alpha_i \leq \lambda_i$ for $i = 1, \ldots, n$. All r.v’s are implicitly assumed to be non-negative and absolutely continuous, and the terms increasing and decreasing mean non-decreasing and non-increasing, respectively.

2. Preliminaries

For ease of reference, let us recall some concepts closely related to the theme of this study.

Let $X$ and $Y$ be r.v’s with distribution functions $F$, $G$, survival functions $\bar{F}$, $\bar{G}$, and denote $F^{-1}$ and $G^{-1}$ as their right continuous inverses, respectively.

**Definition 2.1:** An r.v. $X$ is said to be smaller than the other one $Y$ in the

(i) usual stochastic order (denoted as $X \leq_{st} Y$) if $\bar{F}(t) \leq \bar{G}(t)$ for all $t$;

(ii) hazard rate order (denoted as $X \leq_{hr} Y$) if $G(x)/\bar{F}(x)$ is increasing in $x$;

(iii) reversed hazard rate order (denoted as $X \leq_{rh} Y$) if $G(x)/\bar{F}(x)$ is increasing in $x$;
(iv) **dispersive order** (denoted as \( X \leq_{\text{disp}} Y \)) if \( F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \) for all \( 0 < \alpha \leq \beta < 1 \).

For more on stochastic orders one may refer to Müller and Stoyan [21], Shaked and Shanthikumar [22] and Li and Li [23].

**Definition 2.2:** An r.v. \( X \) with survival function \( \bar{F} \), hazard rate \( h \) and reversed hazard rate \( r \) is

(i) **increasing hazard rate** (IHR) if \( \bar{F}(x) \) is log-concave;
(ii) **decreasing hazard rate** (DHR) if \( \bar{F}(x) \) is log-convex;
(iii) **increasing proportional hazard rate** (IPHR) if \( xh(x) \) is increasing;
(iv) **decreasing reversed hazard rate** (DRHR) if \( r(x) \) is decreasing.

For more on aging properties, one may refer to Barlow and Proschan [24], Block et al. [25], Marshall and Olkin [26] and Righter et al. [27].

Denote \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) on \( \mathbb{R}^n \), and let \( x_{1:n}, \ldots, x_{n:n} \) be the increasing arrangement of \( x_1, \ldots, x_n \).

**Definition 2.3:** A real vector \( x \) is said to be

(i) **majorized** by \( y \) (denoted as \( x \preceq y \)) if \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \) and \( \sum_{i=1}^{j} x_{n:i} \geq \sum_{i=1}^{j} y_{n:i} \) for \( j = 1, \ldots, n-1; \)
(ii) **weakly submajorized** by \( y \) (denoted as \( x \preceq_w y \)) if \( \sum_{i=1}^{n} x_{n:i} \leq \sum_{i=1}^{n} y_{n:i} \) for \( j \in \mathcal{I}_n \);
(iii) **weakly supermajorized** by \( y \) (denoted as \( x \succeq_w y \)) if \( \sum_{i=1}^{n} x_{n:i} \geq \sum_{i=1}^{n} y_{n:i} \) for \( j \in \mathcal{I}_n \);
(iv) **\( p \)-smaller** than \( y \) on \( \mathbb{R}^n_+ \) (denoted as \( x \preceq_p y \)) if \( \prod_{i=1}^{j} x_{n:i} \geq \prod_{i=1}^{j} y_{n:i} \) for \( j \in \mathcal{I}_n \).

It is well-known that \( x \preceq y \iff x \preceq_w y \iff x \preceq_m y \iff \bar{x} \preceq_w \bar{y} \iff \bar{x} \preceq_m \bar{y} \) on \( \mathbb{R}^n_+ \).

**Definition 2.4:** For a \( n \)-monotone \( \psi : [0, +\infty) \to (0, 1] \) with \( \psi(0) = 1 \) and \( \psi(+\infty) = 0 \),

\[
C_{\psi}(u_1, \ldots, u_n) = \psi(\phi(u_1) + \cdots + \phi(u_n)), \quad \text{for all } u_i \in (0, 1), i \in \mathcal{I}_n,
\]

is an Archimedean copula with generator \( \psi \), where \( \phi = \psi^{-1} \) is the right continuous inverse of \( \psi \).

Archimedean family covers a wide range of copulas and the **independence copula** corresponding to the independent case is also a special Archimedean copula with generator \( \psi_0(t) = e^{-t} \). For more readers may refer to Nelsen [28].

Before proceeding to the main results, let us build several lemmas concerned with the following two functionals of generator \( \psi \) of the \( n \geq 2 \) dimension Archimedean copula

\[
J_1(\alpha, \lambda; x, \psi) = \psi\left( \sum_{k=1}^{n} \phi\left( F^{\alpha_k}(\lambda_k x) \right) \right), \quad J_2(\alpha, \lambda; x, \psi) = 1 - \psi\left( \sum_{k=1}^{n} \phi\left( F^{\alpha_k}(\lambda_k x) \right) \right),
\]

and we defer their proofs to Section 6 for the sake of smoothness.

**Lemma 2.5:** (i) \( J_1(\alpha, \lambda; x, \psi) \) is decreasing in \( \alpha_i \) and in \( \ln \alpha_i \) for \( i \in \mathcal{I}_n \), respectively;
(ii) For \( \alpha, \beta, \lambda \in \mathbb{I}_+ \) or \( \mathbb{D}_+ \), the \( \alpha \preceq_m \beta \) implies \( J_1(\alpha, \lambda; x, \psi) \geq J_1(\beta, \lambda; x, \psi) \) whenever \( \psi \) is log-convex;
(iii) For \( \alpha, \beta, \lambda \in \mathbb{I}_+ \) or \( \mathbb{D}_+ \), the \( \ln \alpha_1, \ldots, \ln \alpha_n \preceq_m (\ln \beta_1, \ldots, \ln \beta_n) \) implies \( J_1(\alpha, \lambda; x, \psi) \geq J_1(\beta, \lambda; x, \psi) \) whenever \( \ln(-\ln \psi) \) is concave.
Lemma 2.6: (i) \( J_1(\alpha, \lambda; x, \psi) \) is decreasing in \( \lambda_i \) for \( i \in I_n \);

(ii) For \( \alpha, \lambda, \mu \in \mathbb{I}_+ \) or \( \mathbb{D}_+ \), the \( \lambda \leq \mu \) implies \( J_1(\alpha, \lambda; x, \psi) \geq J_1(\alpha, \mu; x, \psi) \) whenever \( \psi \) is log-convex.

Lemma 2.7: (i) \( J_2(\alpha, \lambda; x, \psi) \) is increasing in \( \alpha_i \) and in \( \ln \alpha_i \) for \( i \in I_n \), respectively;

(ii) For \( \alpha, \beta \in \mathbb{I}_+, \lambda \in \mathbb{D}_+ \) or \( \alpha, \beta \in \mathbb{D}_+, \lambda \in \mathbb{I}_+ \), the \( \lambda \leq \mu \) implies \( J_2(\alpha, \lambda; x, \psi) \leq J_2(\beta, \lambda; x, \psi) \) whenever \( \psi \) is log-convex;

(iii) For \( \alpha, \beta \in \mathbb{I}_+, \lambda \in \mathbb{D}_+ \) or \( \alpha, \beta \in \mathbb{D}_+, \lambda \in \mathbb{I}_+ \), the \( \ln(\alpha_1, \ldots, \ln \alpha_n) \leq \ln(\beta_1, \ldots, \ln \beta_n) \) implies \( J_2(\alpha, \lambda; x, \psi) \leq J_2(\beta, \lambda; x, \psi) \) whenever \( \psi \) is log-convex and \( F \) is DRHR.

Lemma 2.8: (i) \( J_2(\alpha, \lambda; x, \psi) \) is decreasing in \( \lambda_i \) and in \( \ln \lambda_i \) for \( i \in I_n \), respectively;

(ii) For \( \alpha \in \mathbb{I}_+, \lambda, \mu \in \mathbb{D}_+ \) or \( \alpha \in \mathbb{D}_+, \lambda, \mu \in \mathbb{I}_+ \), the \( \lambda \leq \mu \) implies \( J_2(\alpha, \lambda; x, \psi) \leq J_2(\alpha, \mu; x, \psi) \) whenever \( \psi \) is log-convex and \( F \) is DRHR;

(iii) For \( \alpha \in \mathbb{I}_+, \lambda, \mu \in \mathbb{D}_+ \) and \( (\alpha_1, \lambda_1, \ldots, \alpha_n, \lambda_n), (\alpha_1, \mu_1, \ldots, \alpha_n, \mu_n) \in \mathbb{I}_+ \) or \( \alpha \in \mathbb{D}_+, \lambda, \mu \in \mathbb{I}_+ \) and \( (\alpha_1, \lambda_1, \ldots, \alpha_n, \lambda_n), (\alpha_1, \mu_1, \ldots, \alpha_n, \mu_n) \in \mathbb{D}_+ \), the \( \ln(\lambda_1, \ldots, \ln \lambda_n) \leq \ln(\mu_1, \ldots, \ln \mu_n) \) implies \( J_2(\alpha, \lambda; x, \psi) \leq J_2(\alpha, \mu; x, \psi) \) whenever \( \psi \) is log-convex and \( F \) is DRHR.

For more on log-convex generators of Archimedean copulas, one may refer to Bassan and Spizzichino [29] and Müller and Scarsini [30].

3. Scaled sample with proportional hazards

The r.v.s \( X_1, \ldots, X_n \) are said to follow the SPH model if, for \( i \in I_n \), \( X_i \) has the survival function \( \bar{F}_i(x) = \bar{F}^{\alpha_i}(\lambda_i x) \) for \( \alpha_i > 0, \lambda_i > 0, i \in I_n \), where \( \bar{F} \) is the baseline survival function, \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is the frailty vector and \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is the scale vector. In this section we consider two samples following SPH models and coupled by Archimedean survival copulas. Specifically, denote \( X = (X_1, \ldots, X_n) \sim \text{SPH}(\bar{F}, \alpha, \lambda, \psi) \) as the sample having the Archimedean survival copula with generator \( \psi \) and following a SPH model with \( X_i \sim \bar{F}^{\alpha_i}(\lambda_i x) \) for any \( i \in I_n \). It is routine to verify that the sample minimum \( X_{1:n} \) of \( X \) gets the survival function

\[
P(X_{1:n} > x) = P(X_i > x, i \in I_n) = J_1(\alpha, \lambda; x, \psi), \quad x \geq 0,
\]

where \( J_1 \) is defined in Equation (1). Throughout this section, for the baseline survival function \( \bar{F} \) with density function \( f \) we denote the hazard rate as \( h(x) = f(x)/\bar{F}(x) \) and let \( \tilde{h}(x) = xh(x) \).

3.1. The case of dependence

In this subsection we mainly consider two heterogeneous SPH samples with possibly different dependence structures and conduct the usual stochastic order between sample minimums.

Theorem 3.1: Suppose \( X \sim \text{SPH}(\bar{F}, \alpha, \lambda, \psi_1) \) and \( Y \sim \text{SPH}(\bar{F}, \beta, \lambda, \psi_2) \) with \( \alpha, \beta, \lambda \in \mathbb{I}_+ \) or \( \mathbb{D}_+ \). Then, \( \alpha \preceq_w \beta \) implies \( X_{1:n} \succeq_{st} Y_{1:n} \) if \( \phi_1 \circ \psi_2 \) is super-additive, and \( \psi_1 \) or \( \psi_2 \) is log-convex.

Proof: For \( \alpha, \beta, \lambda \in \mathbb{I}_+ \) with \( \alpha \preceq_w \beta \), in accordance with Theorem 5A.9 of Marshall et al. [31], there exists some \( \eta \in \mathbb{I}_+ \) such that \( \alpha \leq \eta \) and \( \eta \preceq_{m} \beta \). If \( \psi_1 \) is log-convex, due to Lemma 2.5(ii), \( \eta \preceq_{m} \beta \) implies \( J_1(\eta, \lambda; x, \psi_1) \geq J_1(\beta, \lambda; x, \psi_1) \). Also, according to Lemma 2.5(i), \( \alpha \preceq \eta \) implies \( J_1(\alpha, \lambda; x, \psi_1) \geq J_1(\eta, \lambda; x, \psi_1) \). As a consequence, we have

\[
J_1(\alpha, \lambda; x, \psi_1) \geq J_1(\beta, \lambda; x, \psi_1).
\]

For \( \alpha, \beta, \lambda \in \mathbb{D}_+ \), it can be proved in a very similar manner that \( \alpha \preceq_w \beta \) yields (3) also.
Suppose Theorem 3.2:
Theorem 3.1 and hence we omit the proof for brevity. Then, \( (\alpha, \lambda; x, \psi_1) \geq (\beta, \lambda; x, \psi_2) \). Similarly, if \( \psi_2 \) is log-convex, it also holds that
\[
J_1(\alpha, \lambda; x, \psi_1) \geq J_1(\beta, \lambda; x, \psi_1) \geq J_1(\beta, \lambda; x, \psi_2).
\]
According to Equation (2), \( X_{1:n} \) and \( Y_{1:n} \) get survival functions \( \bar{F}_1(x) = J_1(\alpha, \lambda; x, \psi_1) \) and \( \bar{G}_1(x) = J_1(\beta, \lambda; x, \psi_2) \) for \( x \geq 0 \), respectively. As a consequence, both Equations (4) and (5) imply \( \bar{G}_1(x) \leq \bar{F}_1(x) \) for \( x \geq 0 \). That is, \( X_{1:n} \geq_{st} Y_{1:n} \).

As is pointed out in [6], for the Archimedean survival copula, log-convexity of the generator leads to the RTIS (right tail increasing in sequence) property. Also, according to Denuit et al. [32] and Li and Fang [7], the super-additivity of \( \phi_1 \circ \psi_2 \) implies that the copula with generator \( \psi_1 \) is more positively dependent than that with \( \psi_2 \). Succinctly speaking, Theorem 3.1 stresses that, for two SPH samples with frailty vectors being arrayed in the same manner as the common scale vector, the sample with more symmetry and stronger positive dependence has a stochastically larger sample minimum.

For \( X \sim \text{SPH}(\bar{F}, \alpha, 1, \psi_1) \) and \( Y \sim \text{SPH}(\bar{F}, \beta, 1, \psi_2) \) with either \( \psi_1 \) or \( \psi_2 \) being log-convex and \( \phi_1 \circ \psi_2 \) being super-additive, \( \alpha \preceq \beta \) is proved to imply \( X_{1:n} \geq_{st} Y_{1:n} \) in Theorem 4.1 of Fang et al. [4]. Since these two samples are obviously special cases of SPH samples, Theorem 3.1 above serves as one of its generalization.

Through relaxing the dependence within samples Theorem 3.2 below obtains a result similar to Theorem 3.1 in the context of a stronger sub-majorization on the frailty vectors. By applying Theorem 5.A.9 of Marshall et al. [31] and Lemma 2.5 it can be verified in a similar manner to Theorem 3.1 and hence we omit the proof for brevity.

**Theorem 3.2:** Suppose \( X \sim \text{SPH}(\bar{F}, \alpha, \lambda, \psi_1) \) and \( Y \sim \text{SPH}(\bar{F}, \beta, \lambda, \psi_2) \) with \( \alpha, \beta, \lambda \in \mathbb{I}_+ \) or \( \mathbb{D}_+ \). Then, \( (\ln \alpha_1, \ldots, \ln \alpha_n) \leq_w (\ln \beta_1, \ldots, \ln \beta_n) \) implies \( X_{1:n} \geq_{st} Y_{1:n} \) if \( \phi_1 \circ \psi_2 \) is super-additive and \( \ln(-\ln \psi_1) \) or \( \ln(-\ln \psi_2) \) is concave.

Fang et al. [4, Theorem 4.2] showed, for \( X \sim \text{SPH}(\bar{F}, \alpha, 1, \psi_1) \) and \( Y \sim \text{SPH}(\bar{F}, \beta, 1, \psi_2) \) with log-concave \( -\ln \psi_1 \) or \( -\ln \psi_2 \) and super-additive \( \phi_1 \circ \psi_2 \), that \( X_{1:n} \geq_{st} Y_{1:n} \) whenever \( (\ln \alpha_1, \ldots, \ln \alpha_n) \leq_w (\ln \beta_1, \ldots, \ln \beta_n) \). Again, Theorem 3.2 above successfully generalizes Theorem 4.2 of Fang et al. [4] to a more general context.

The next theorem considers SPH samples with the same frailty vector and investigates the effect of dependence and symmetry on sample minimums. With the help of Lemma 2.6, Theorem 3.3 can also be proved in a similar manner to Theorem 3.1 and hence we omit the proof for the sake of briefness.

**Theorem 3.3:** For \( X \sim \text{SPH}(\bar{F}, \alpha, \lambda, \psi_1) \) and \( Y \sim \text{SPH}(\bar{F}, \alpha, \mu, \psi_2) \) with \( \alpha, \lambda, \mu \in \mathbb{I}_+ \) or \( \mathbb{D}_+ \). Then, the \( \lambda \preceq_w \mu \) implies \( X_{1:n} \geq_{st} Y_{1:n} \) if \( h(x) \) is increasing, \( \phi_1 \circ \psi_2 \) is super-additive, and \( \psi_1 \) or \( \psi_2 \) is log-convex.

For Weibull samples \( X \sim \text{SPH}(e^{-x^\alpha}, 1, (\lambda_1, \ldots, \lambda_n), \psi) \) and \( Y \sim \text{SPH}(e^{-x^\alpha}, 1, (\mu_1, \ldots, \mu_n), \psi) \) with Archimedean survival copula generated by a log-convex \( \psi_1 \), Li and Li [6, Theorem 4.1] showed that \( (\lambda_1^\alpha, \ldots, \lambda_n^\alpha) \preceq_w (\mu_1^\alpha, \ldots, \mu_n^\alpha) \) implies \( X_{1:n} \geq_{st} Y_{1:n} \). Later, for samples \( W \sim \text{SPH}(\bar{F}, 1, \lambda, \psi_1) \) and \( Z \sim \text{SPH}(\bar{F}, 1, \mu, \psi_2) \), where \( \phi_1 \circ \psi_2 \) is super-additive and either \( \psi_1 \) or \( \psi_2 \) is log-convex, Li et al. [5, Theorem 3.1(ii)] further got the generalization: \( \lambda \preceq_w \mu \) and increasing \( h(x) \) together imply \( W_{1:n} \geq_{st} Z_{1:n} \). Note that Weibull sample follows SPH model, and for \( \alpha \geq 1 \), \( h(x) \) is increasing and \( \lambda \preceq_w \eta \) implies \( (\lambda_1^\alpha, \ldots, \lambda_n^\alpha) \preceq_w (\mu_1^\alpha, \ldots, \mu_n^\alpha) \). Theorem 3.3 generalizes Theorem 4.1 of Li and Li [6].
to SPH samples with neither a common dependence structure nor a homogeneous frailty vector, also it generalizes Theorem 3.1(ii) Li et al. [5] to SPH samples with some common frailty vector.

In what follows, we employ majorization of the frailty and scale vectors to describe the heterogeneity of a SPH model and have a discussion on how the heterogeneity of the sample affects the dispersiveness of its minimum.

**Theorem 3.4:** Suppose \( X \sim \text{SPH}(\bar{F}, \alpha, \lambda; \psi) \) and \( Y \sim \text{SPH}(\tilde{F}, \alpha I, \lambda I; \psi) \) with \( \alpha, \lambda \in \mathbb{I}_+ \) or \( \mathbb{I}_+ \). Then, \( \lambda \leq (\prod_{k=1}^{n} \lambda_k)^{1/n} \) along with \( \alpha \leq (1/n) \sum_{k=1}^{n} \alpha_k \) implies \( X_{1:n} \leq \text{disp} \ Y_{1:n} \) if \( \tilde{h}(e^x) \) is convex, \( \tilde{F} \) is both DHR and IPHR, and \( \psi / \psi' \) is decreasing and concave.

**Proof:** \( X_{1:n} \) and \( Y_{1:n} \) have respective distribution functions \( F_1(x) = 1 - \psi \left( \sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right) \) and \( H_1(x) = 1 - \psi \left( n\phi(\bar{F}_{\alpha}(\lambda x)) \right) \) for \( x \geq 0 \), and their respective density functions are

\[
f_1(x) = \psi' \left( \sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right) \sum_{k=1}^{n} \alpha_k \lambda_k h(\lambda_k x) \tilde{F}_{\alpha k}(\lambda_k x) \psi' \left( \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right),
\]

\[
h_1(x) = \psi' \left( n\phi(\bar{F}_{\alpha}(\lambda x)) \right) \frac{n\alpha \lambda h(\lambda x) \tilde{F}_{\alpha}(\lambda x)}{\psi' \left( \phi(\tilde{F}_{\alpha}(\lambda x)) \right)}.
\]

Denote \( L_1(x; \alpha, \lambda) = \bar{F}^{-1} (\psi^{-1} (\sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)))) \). Then, for \( x \geq 0 \), \( H_1^{-1}(F_1(x)) = (1/\lambda) L_1(x; \alpha, \lambda) \) and

\[
h_1 \left( H_1^{-1}(F_1(x)) \right) = \psi' \left( \sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right) \frac{n\alpha \lambda h(L_1(x; \alpha, \lambda)) \psi \left( \frac{1}{n} \sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right)}{\psi' \left( \frac{1}{n} \sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right)}.
\]

Due to the concavity of \( \psi / \psi' \) it holds that, for \( x \geq 0 \),

\[
\frac{\psi \left( \frac{1}{n} \sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right)}{\psi' \left( \frac{1}{n} \sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right)} \geq \frac{1}{n} \sum_{k=1}^{n} \psi \left( \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right).
\]

Since \( \tilde{h}(x) \) is increasing and \( \psi / \psi' \) is decreasing, \( \ln \bar{F}(e^x) \) is concave and \( \ln \psi \) is convex. In view of \( \lambda \leq (\lambda_1 \cdots \lambda_n)^{1/n}, \alpha \leq (1/n) \sum_{k=1}^{n} \alpha_k \) and the decreasing property of \( \ln \bar{F} \leq 0 \), we have

\[
\alpha \ln \bar{F}(\lambda x) \geq \alpha \ln \bar{F} \left( \left( \prod_{k=1}^{n} \lambda_k x \right)^{1/n} \right) \geq \frac{1}{n} \sum_{k=1}^{n} \alpha_k \cdot \ln \bar{F} \left( \exp \left( \frac{1}{n} \sum_{k=1}^{n} \ln(\lambda_k x) \right) \right).
\]

Note that \( \ln \bar{F}(e^x) \) is concave, \( \ln \psi \) is convex, and \( \alpha, \lambda \in \mathbb{I}_+ \) or \( \mathbb{I}_+ \). From Čebyšev’s inequality it follows that

\[
\alpha \ln \bar{F}(\lambda x) - \ln \psi \left( \frac{1}{n} \sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right)
\]

\[
\geq \frac{1}{n} \sum_{k=1}^{n} \alpha_k \cdot \ln \bar{F} \left( \exp \left( \frac{1}{n} \sum_{k=1}^{n} \ln(\lambda_k x) \right) \right) - \ln \psi \left( \frac{1}{n} \sum_{k=1}^{n} \phi(\bar{F}_{\alpha k}(\lambda_k x)) \right)
\]

\[
\geq \frac{1}{n} \sum_{k=1}^{n} \alpha_k \cdot \frac{1}{n} \sum_{k=1}^{n} \ln \bar{F}(\lambda_k x) - \frac{1}{n} \sum_{k=1}^{n} \alpha_k \ln \bar{F}(\lambda_k x) \geq 0.
\]
So, it holds that $L_1(x; \alpha, \lambda) \geq \lambda x$. Due to $\lambda \leq (\prod_{k=1}^{n} \lambda_k)^{1/n}$, decreasing $h(x)$, increasing $h(x)$ and convex $h(e^x)$ we have

\[
\lambda h(L_1(x; \alpha, \lambda)) \leq \frac{1}{x} \lambda x h(\lambda x) \leq \frac{1}{x} \left( \prod_{k=1}^{n} \lambda_k \right)^{1/n} \cdot h \left( \prod_{k=1}^{n} \lambda_k \right)^{1/n} \\
= \frac{1}{x} \exp \left\{ \frac{1}{n} \sum_{k=1}^{n} \lambda_k x \right\} \cdot h \left( \exp \left\{ \frac{1}{n} \sum_{k=1}^{n} \ln \lambda_k x \right\} \right) \\
\leq \frac{1}{n} \sum_{k=1}^{n} \lambda_k h(\lambda_k x).
\]  

(9)

Also, in light of $\alpha, \lambda \in \mathbb{I}_+ \text{ or } \mathbb{D}_+$, increasing $h$ and decreasing $\psi / \psi'$, by Čebyšev's inequality again we have

\[
\frac{1}{n} \sum_{k=1}^{n} \alpha_k \lambda_k h(\lambda_k x) \bar{F}^{\alpha_k}(\lambda_k x) \\
= \frac{1}{n} \sum_{k=1}^{n} \alpha_k \lambda_k h(\lambda_k x) \psi \left( \psi' \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right) \\
\leq \frac{1}{n} \sum_{k=1}^{n} \alpha_k \cdot \frac{1}{n} \sum_{k=1}^{n} \lambda_k h(\lambda_k x) \psi \left( \psi' \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right) \\
\leq \frac{1}{n} \sum_{k=1}^{n} \alpha_k \cdot \frac{1}{n} \sum_{k=1}^{n} \lambda_k h(\lambda_k x) \cdot \frac{1}{n} \sum_{k=1}^{n} \psi \left( \psi' \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right) \\
\leq \alpha \cdot \frac{1}{n} \sum_{k=1}^{n} \lambda_k h(\lambda_k x) \cdot \frac{1}{n} \sum_{k=1}^{n} \psi \left( \psi' \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right) \left( \text{due to } \alpha \leq \frac{1}{n} \sum_{k=1}^{n} \alpha_k, \frac{\psi(x)}{\psi'(x)} \leq 0 \right) \\
\leq \alpha \cdot \frac{1}{n} \sum_{k=1}^{n} \lambda_k h(\lambda_k x) \cdot \frac{\psi \left( \frac{1}{n} \sum_{k=1}^{n} \phi \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right)}{\psi' \left( \frac{1}{n} \sum_{k=1}^{n} \phi \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right)} \left( \text{due to (8)} \right) \\
\leq \alpha \lambda h(L_1(x; \alpha, \lambda)) \frac{\psi \left( \frac{1}{n} \sum_{k=1}^{n} \phi \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right)}{\psi' \left( \frac{1}{n} \sum_{k=1}^{n} \phi \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right)} \left( \text{due to (9), } \frac{\psi(x)}{\psi'(x)} \leq 0 \right)
\]

As a consequence of Equations (6) and (7), it follows from the above inequality that, for all $x \geq 0$,

\[
h_1 \left( H_{-1}^{-1}(F_1(x)) \right) - f_1(x) \\
= \frac{1}{n} \sum_{k=1}^{n} \alpha_k \lambda_k h(\lambda_k x) \bar{F}^{\alpha_k}(\lambda_k x) \\
= \frac{1}{n} \sum_{k=1}^{n} \alpha_k \lambda_k h(\lambda_k x) \bar{F}^{\alpha_k}(\lambda_k x) \\
= \frac{\alpha \lambda h(L_1(x; \alpha, \lambda)) \psi \left( \frac{1}{n} \sum_{k=1}^{n} \phi \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right)}{\psi' \left( \frac{1}{n} \sum_{k=1}^{n} \phi \left( \bar{F}^{\alpha_k}(\lambda_k x) \right) \right)} \leq 0.
\]

That is, $f_1(F_1^{-1}(\alpha)) \geq h_1(H_{-1}^{-1}(x))$ for all $x \in (0, 1)$. In accordance with (3B.11) of Shaked and Shanthikumar [22], this invokes $X_{1:n} \leq_{\text{disp}} Y_{1:n}$. □

According to Theorem 3.4, for positively dependent SPH samples with frailty and the scale vectors arrayed similarly, the minimum of the homogeneous one gets a higher variability. Namely,
the heterogeneity reduces variability of the sample minimum. For two independent Weibull samples
\[ X \sim \text{SPH}(e^{-x^\alpha}, (\lambda_1^\alpha, \ldots, \lambda_n^\alpha), 1, \psi_0) \quad \text{and} \quad Y \sim \text{SPH}(e^{-x^\alpha}, (\lambda_1^\alpha, \ldots, \lambda_n^\alpha), 1, \psi_0), \]

Fang and Zhang [11, Theorem 3.1] showed that \( \lambda^\alpha \geq (1/n) \sum_{i=1}^n \lambda_i^\alpha \) implies \( X_{1:n} \leq_{\text{disp}} Y_{1:n} \). Note that Weibull sample follows SPH model and the generator \( \psi_0(t) = e^{-t} \) gives rise to the independence copula. As a reinforcement, Theorem 3.4 derives the dispersive order between sample minimums from two more general SPH models with a common dependence structure. Further, for another sample \( Z \sim \text{SPH}(e^{-x^\alpha}, (\mu_1^\alpha, \ldots, \mu_n^\alpha), 1, \psi_0) \) with \( \mu \leq (\prod_{i=1}^n \lambda_i)^{1/n} \), by Theorem 3.4 we have
\[ Y_{1:n} \leq_{\text{disp}} X_{1:n} \leq_{\text{disp}} Z_{1:n}. \]

Naturally, one wonders whether this also holds for samples with dependent observations. According to Theorem 4.3(ii) of Fang et al. [4] the \( Y_{1:n} \leq_{\text{disp}} X_{1:n} \) holds for some Archimedean copulas with a log-convex generator. On the other hand, by Theorem 3.4 the second inequality holds for some Archimedean copulas with a log-concave generator. Obviously, the independence copula is the only member of Archimedean family with a log-convex and log-concave generator. So, the inequalities in Equation (10) cannot be extended to samples coupled by other Archimedean copulas.

One reviewer makes the following remarks on the finding of Theorem 3.4: (i) One can further reach \( \text{Var}(X_{1:n}) \leq \text{Var}(Y_{1:n}) \); (ii) By Theorem 3.4.13 of Shaked and Shanthikumar [22], \( X_{1:n} \leq_{\text{st}} Y_{1:n} \) holds if the support of the baseline distribution has a finite left endpoint; (iii) When the baseline is the exponential distribution and the samples are of independent observations, \( Y_{1:n} \) can be verified to be IHR and hence we reach \( X_{1:n} \leq_{\text{hr}} Y_{1:n} \). To the best of our knowledge, the likelihood ratio order for sample minimums of dependent samples remains unknown.

At the end of this subsection, we present an illustrating example for Theorem 3.4.

**Example 3.5:** It is easy to verify that the function \( \psi(x) = \begin{cases} e^{-2\arctan x}, & x \leq -1; \\ x^{-1} e^{-x/2}, & x > 1, \end{cases} \) is 3-monotone and thus generates a 3-dimensional Archimedean survival copula. Also, \( \psi(x)/\psi'(x) = \begin{cases} 1 + x^2, & x \leq 1; \\ 2/x, & x > 1, \end{cases} \) can be verified to be decreasing and concave. On the other hand, let \( F \sim \mathcal{W}(\beta, \gamma) \) with \( F(x) = e^{-(\gamma x)^\beta} \) for \( \beta \in (0, 1] \) and \( \gamma > 0 \). It is easy to verify that \( h(x) = \beta \gamma^\beta x^{\beta-1} \) is decreasing and \( h'(x) = \beta \gamma^\beta x^\beta \) is increasing. Also, \( h(e^x) = \beta \gamma^\beta e^{\beta x} \) is convex for \( \beta \in (0, 1] \) and \( \gamma > 0 \).

(i) For \( X \sim \text{SPH}(e^{-x^{0.5}}, (1, 2, 3), (2, 4, 8), \psi) \) and \( Y \sim \text{SPH}(e^{-x^{0.5}}, (1.5, 1.5, 1.5), (2, 2, 2), \psi) \), it is plain that \( 1.5 < (1 + 2 + 3)/3 \) and \( 2 < (2 \cdot 4 \cdot 8)^{1/3} \). Denote \( W_1(w_1) \) and \( W_2(w_2) \) the distribution (density) functions of \( X_{1:3} \) and \( Y_{1:3} \), respectively. Figure 1(a) plots the difference between \( w_1(x) \) and \( w_2((W_2^{-1} \circ W_1)(x)) \). As can be seen in Figure 1(a), the difference \( w_1(x) - w_2((W_2^{-1} \circ W_1)(x)) \) is positive, implying \( X_{1:3} \leq_{\text{disp}} Y_{1:3} \).

(ii) For a Weibull sample \( Z \sim \text{SPH}(e^{-x^{0.5}}, (2.5, 2.5, 2.5), (6, 6, 6), \psi) \), obviously \( 2.5 > (1 + 2 + 3)/3 \) and \( 6 > (2 \cdot 4 \cdot 8)^{1/3} \) violate the conditions in Theorem 3.4. Denote \( W_3 \) and \( W_4 \) distribution and density functions of \( X_{1:3} \), respectively. In Figure 1(b) \( w_1(x) - w_3((W_3^{-1} \circ W_1)(x)) \) changes the sign, and this negates \( X_{1:3} \leq_{\text{disp}} Z_{1:3} \).

(iii) For a Weibull sample \( U \sim \text{SPH}(e^{-x^{0.5}}, (2, 2, 2), (5, 5, 5), \psi) \), it is obvious that \( 2 = (1 + 2 + 3)/3 \) and \( 5 > (2 \cdot 4 \cdot 8)^{1/3} \). Denote \( W_4 \) and \( W_4 \) distribution and density functions of \( U_{1:3} \), respectively. As is seen in Figure 1(c), \( w_1(x) - w_4((W_4^{-1} \circ W_1)(x)) > 0 \) implies \( X_{1:3} \leq_{\text{disp}} U_{1:3} \).

### 3.2. The case of independence

Here we pay special attention to independent SPH models, that is, the Archimedean copula has generator \( \psi(t) = \psi_0(t) = e^{-t} \) in Equation (2). Our goal in this subsection is to understand how the degree
of symmetry of frailty scale vectors affects the sample minimums. Specifically, we derive sufficient conditions for the hazard rate order on sample minimums.

**Theorem 3.6:** For $X \sim \text{SPH}(\bar{F}, \alpha, \lambda, \psi_0)$ and $Y \sim \text{SPH}(\bar{F}, \beta, \lambda, \psi_0)$ with an increasing $\bar{h}(x)$,

1. the $\alpha \preceq_w \beta$ implies $X_{1:n} \preceq_{hr} Y_{1:n}$ if $\alpha, \beta, \lambda \in \mathbb{I}_+$, or $\mathbb{D}_+$;
2. the $\alpha \preceq_w \beta$ implies $X_{1:n} \preceq_{hr} Y_{1:n}$ if $\alpha, \beta \in \mathbb{I}_+$, or $\alpha, \beta \in \mathbb{D}_+$, $\lambda \in \mathbb{I}_+$.

**Proof:** We only prove (i), and (ii) can be verified similarly.

According to Equation (2), $X_{1:n}$ gets the survival function $\tilde{G}_1(\alpha, \lambda; x) = \prod_{k=1}^n \tilde{F}^{x_{k}}(\lambda_k x)$, density function $g_1(\alpha, \lambda; x) = \prod_{k=1}^n \tilde{F}^{x_{k}}(\lambda_k x) \sum_{k=1}^n a_k \lambda_k h(\lambda_k x)$ and hence the hazard rate function

$$h_1(\alpha, \lambda; x) = \frac{g_1(\alpha, \lambda; x)}{\tilde{G}_1(\alpha, \lambda; x)} = \sum_{k=1}^n a_k \lambda_k h(\lambda_k x). \tag{11}$$

It is plain that $\frac{\partial h_1(\alpha, \lambda; x)}{\partial \alpha_i} = \lambda_i h(\lambda_i x) \geq 0$ and hence $h_1(\alpha, \lambda; x)$ is increasing in $\alpha_i$, for $i \in \mathbb{I}_n$.

For $\alpha, \beta, \lambda \in \mathbb{I}_+$ with $\alpha \preceq_w \beta$, by Theorem 5.A.9 of Marshall et al. [31], there exists some $\eta \in \mathbb{I}_+$ such that $\alpha \preceq \eta \preceq \beta$, where $\alpha \preceq \eta$ means $\alpha_i \leq \eta_i$ for all $i \in \mathbb{I}_n$.

For $1 \leq i < j \leq n$, due to the increasing property of $\tilde{h}(x)$, we have, for $u \in (-\infty, 0]^n$,

$$\frac{\partial h_1(-u, \lambda; x)}{\partial u_i} - \frac{\partial h_1(-u, \lambda; x)}{\partial u_j} = -\frac{1}{x} [\lambda_i x h(\lambda_i x) - \lambda_j x h(\lambda_j x)] \geq 0,$$

implying that $\frac{\partial h_1(-u, \lambda; x)}{\partial u_k}$ is decreasing in $k = 1, \ldots, n$. Then, by Theorem 3.A.3 of Marshall et al. [31] we conclude that

$$\eta \preceq \beta \quad \text{on} \quad \mathbb{I}_+ \implies -\eta \preceq -\beta \quad \text{on} \quad \mathbb{D}, \quad \text{and} \quad -\eta, -\beta \leq 0 \implies h_1(\eta, \lambda; x) \leq h_1(\beta, \lambda; x).$$
From the monotonicity of $h_1(\alpha, \lambda; x)$ with respect to $\alpha$, for all $i \in \mathcal{I}_n$ it follows immediately that

$$\alpha \leq \eta \implies h_1(\alpha, \lambda; x) \leq h_1(\eta, \lambda; x).$$

So, we reach the conclusion that $\alpha \leq \eta$ implies $h_1(\alpha, \lambda; x) \leq h_1(\beta, \lambda; x)$, yielding $X_{1:n} \succeq_{hr} Y_{1:n}$.

The case of $\alpha, \beta, \lambda \in \mathbb{D}_+$ can be proved in a similar method.

Through a comparison on Weibull samples $X \sim \text{SPH}(e^{-x^\alpha}, (\lambda_1^\alpha, \ldots, \lambda_n^\alpha), 1, \psi_0)$ and $Y \sim \text{SPH}(e^{-y^\alpha}, (\beta_1^\alpha, \ldots, \beta_n^\alpha), 1, \psi_0)$, Khahedi and Kochar [14, Theorem 2.3] proved that $\lambda \succeq \beta$ implies $X_{1:n} \succeq_{hr} Y_{1:n}$ for $0 < \alpha < 1$ and $X_{1:n} \preceq_{hr} Y_{1:n}$ for $\alpha \geq 1$. Since $\lambda \succeq \beta$ implies $(\lambda_1^\alpha, \ldots, \lambda_n^\alpha) \succeq_w (\beta_1^\alpha, \ldots, \beta_n^\alpha)$ for $0 < \alpha < 1$ and $\lambda \succeq \beta$ implies $(\lambda_1^\alpha, \ldots, \lambda_n^\alpha) \succeq_w (\beta_1^\alpha, \ldots, \beta_n^\alpha)$ for $\alpha \geq 1$, Theorem 3.6 above extends Theorem 2.3 of Khahedi and Kochar [14] to SPH samples of independent observations.

**Theorem 3.7:** For $X \sim \text{SPH}(\tilde{F}, \alpha, \lambda, \psi_0)$ and $Y \sim \text{SPH}(\tilde{F}, \alpha, \mu, \psi_0)$ with increasing $\tilde{h}(x)$,

(i) the $\lambda \preceq_w \mu$ implies $X_{1:n} \preceq_{hr} Y_{1:n}$ if $\alpha \in \mathbb{D}_+, \lambda, \mu \in \mathbb{I}_+$, or $\alpha \in \mathbb{I}_+, \lambda, \mu \in \mathbb{D}_+$, and $\tilde{h}(x)$ is concave, and

(ii) the $\lambda \preceq_w \mu$ implies $X_{1:n} \preceq_{hr} Y_{1:n}$ if $\alpha, \lambda, \mu \in \mathbb{I}_+$, or $\mathbb{D}_+$, and $\tilde{h}(x)$ is convex.

**Proof:** (i) Note that $\tilde{h}(x)$ is increasing and concave. Due to Theorem 5.A.2 of Marshall et al. [31], $\lambda \preceq_w \mu$ implies

$$(-\tilde{h}(\lambda_1 x), \ldots, -\tilde{h}(\lambda_n x)) \preceq_w (-\tilde{h}(\mu_1 x), \ldots, -\tilde{h}(\mu_n x)), \quad \text{for } x \geq 0. \quad (12)$$

For $\lambda, \mu \in \mathbb{I}_+$, the increasing $\tilde{h}$ gives rise to $(-\tilde{h}(\lambda_1 x), \ldots, -\tilde{h}(\lambda_n x)) \in \mathbb{D}$. Let $\zeta_k(y) = \alpha_k y/x$ for $k \in \mathcal{I}_n$. Then, $\alpha \in \mathbb{D}_+$ guarantees $\zeta_k'(a) = \alpha_k/x \geq \zeta_{k+1}'(b) = \alpha_{k+1}/x \geq 0$ for $\alpha \geq b, j = 1, \ldots, n-1$ and $\zeta_1(y) \geq \zeta_2(y) \geq \cdots \geq \zeta_n(y)$ for all $y$. As per Theorem 5.A.3 of Marshall et al. [31], $(-\tilde{h}(\lambda_1 x), \ldots, -\tilde{h}(\lambda_n x)) \preceq_w (-\tilde{h}(\mu_1 x), \ldots, -\tilde{h}(\mu_n x))$ implies $(\zeta_1(-\tilde{h}(\lambda_1 x)), \ldots, \zeta_n(-\tilde{h}(\lambda_n x))) \preceq_w (\zeta_1(-\tilde{h}(\mu_1 x)), \ldots, \zeta_n(-\tilde{h}(\mu_n x)))$. Consequently, by the definition of submajorization and Equation (11) we have

$$h_1(\alpha, \lambda; x) = \sum_{k=1}^{n} \zeta_k \left(-\tilde{h}(\lambda_k x)\right) \leq \sum_{k=1}^{n} \zeta_k \left(-\tilde{h}(\mu_k x)\right) = h_1(\alpha, \mu; x).$$

That is, $X_{1:n} \preceq_{hr} Y_{1:n}$.

On the other hand, $(-\tilde{h}(\lambda_1 x), \ldots, -\tilde{h}(\lambda_n x)) \in \mathbb{D}$ for $\lambda, \mu \in \mathbb{D}_+$. Since weak majorization is permutation invariant, by Equation (12) we have $(-\tilde{h}(\lambda_n x), \ldots, -\tilde{h}(\lambda_1 x)) \preceq_w (-\tilde{h}(\mu_n x), \ldots, -\tilde{h}(\mu_1 x))$. For $k \in \mathcal{I}_n$, let $\rho_k(y) = \alpha_{n-k+1} y/x$. Then $\alpha \in \mathbb{I}_+$ guarantees $\rho_j'(a) = \alpha_{n-j+1}/x \geq \rho_{j+1}'(b) = \alpha_{n-j}/x \geq 0$ for $\alpha \geq b, j = 1, \ldots, n-1$ and $\rho_1(\mu) \geq \rho_2(\mu) \geq \cdots \geq \rho_n(\mu)$ for all $\mu$. As a result, by Theorem 5.A.3 of Marshall et al. [31] we reach

$$h_1(\alpha, \lambda; x) = \sum_{k=1}^{n} \rho_k \left(-\tilde{h}(\lambda_{n-k+1} x)\right) \leq \sum_{k=1}^{n} \rho_k \left(-\tilde{h}(\mu_{n-k+1} x)\right) = h_1(\alpha, \mu; x),$$

and this yields $X_{1:n} \preceq_{hr} Y_{1:n}$ again.

(ii) can be verified in a similar manner by applying Theorem 5.A.2 of Marshall et al. [31] and Theorem 5.A.3 of Marshall et al. [31].

Recall that the gamma distribution $\mathcal{G}(\alpha, \lambda)$ with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$ has density function $f(x) = \lambda^\alpha x^{\alpha-1} e^{-\lambda x} / \Gamma(\alpha)$. Let $\tilde{F}$ be the survival function of $\mathcal{G}(\alpha, 1)$. 

For two independent gamma samples $X \sim \text{SPH}(\tilde{F}, 1, \lambda, \psi_0)$ and $Y \sim \text{SPH}(\tilde{F}, 1, \mu, \psi_0)$, Kundu et al. [20, Theorem 5.1] proved that $\lambda \preceq_w \mu$ implies $X_{1:n} \preceq_{hr} Y_{1:n}$ for $\alpha \in (0, 1]$ and $\lambda \preceq_w \mu$ implies $X_{1:n} \succeq_{hr} Y_{1:n}$ for $\alpha \in [1, +\infty)$. As is shown in the proof of Theorem 5.1 of Kundu et al. [20], $\tilde{h}(x)$ is increasing and concave (convex) for $\alpha \in (0, 1] (\alpha \in [1, +\infty))$. So, the above Theorem 3.7 generalizes Theorem 5.1 of Kundu et al. [20] to SPH samples.

Theorems 3.6 and 3.7 both assume the baseline survival function with an increasing $\hat{h}(x)$. In fact, many parametric distributions are IPHR, for example the generalized Pareto with survival function $(1 + \alpha x)^{-1/\alpha}$ for $\alpha > 0$. We refer readers to Hazra et al. [33] for more IPHR parametric distributions. For independent samples, Theorem 3.6(i) gets a slightly stronger requirement than Theorem 3.1 (the former assumes an IPHR baseline distribution) and hence presents a stronger stochastic ordering result. Similar discussion applies to conditions in Theorems 3.3 and 3.7(ii).

4. Scaled sample with PRHs

As a dual to the SPH model, the (r.v.’s) $X$ are said to follow the SPRH model if the distribution function of $X_i$ can be written as $F_i(x) = F^{\psi_i}(i x)$ for $\alpha > 0$, $\lambda_{i} > 0$, $i \in I_n$, where $F$ is the baseline distribution function, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the resilience vector and $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the scale vector. In this section, we mainly focus on two samples composed of r.v.’s having PRHs, scale parameters and Archimedean copulas. Specifically, denote $X = (X_1, \ldots, X_n) \sim \text{SPRH}(F, \alpha, \lambda, \psi)$ as the sample having the Archimedean copula with generator $\psi$ and following a SPRH model with $X_i \sim F^{\psi_i}(\lambda_i x)$, $i \in I_n$. It is routine to verify that the sample maximum $X_{n:n}$ of $X$ gets the survival function

$$P(X_{n:n} > x) = 1 - P(X_i \leq x, i \in I_n) = f_2(\alpha, \lambda; x, \psi), \quad x \geq 0,$$

where $f_2$ is defined in Equation (1). Throughout this section, we denote $f$ the density function and $r(x) = f(x)/F(x)$ the reversed hazard rate of the baseline function $F$ and let $\tilde{r}(x) = x r(x)$.

4.1. The case of dependence

In parallel to Theorems 3.1–3.3, we also present the next four theorems comparing the maximums of two samples with different dependence structures. Since the theorems can be verified in a similar manner to those of Theorems 3.1–3.3 based on Theorem 5.A.9 of Marshall et al. [31], Lemma A.1 of Li and Fang [7], Lemmas 2.7 and 2.8, we omit the proofs for brevity.

**Theorem 4.1:** For $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_1)$ and $Y \sim \text{SPRH}(F, \beta, \lambda, \psi_2)$ with $\alpha, \beta \in \mathbb{D}_+ \cap \mathbb{R}^n_+$ and $\lambda \in \mathbb{I}_+$, the $\alpha \preceq w \beta$ implies $X_{n:n} \preceq_{st} Y_{n:n}$ if $\phi_1 \circ \psi_2$ is super-additive, and $\psi_1$ or $\psi_2$ is log-convex.

**Theorem 4.2:** For $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_1)$ and $Y \sim \text{SPRH}(F, \beta, \lambda, \psi_2)$ with $\alpha, \beta \in \mathbb{D}_+ \cap \mathbb{R}^n_+$ and $\lambda \in \mathbb{I}_+$, the $\ln(\alpha_1, \ldots, \ln(\alpha_n) \preceq_w \ln(\beta_1, \ldots, \ln(\beta_n))$ implies $X_{n:n} \preceq_{st} Y_{n:n}$ if $\phi_1 \circ \psi_2$ is super-additive, and $\ln(-\ln \psi_1)$ or $\ln(-\ln \psi_2)$ is concave.

**Theorem 4.3:** For $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_1)$ and $Y \sim \text{SPRH}(F, \alpha, \mu, \psi_2)$ with $\alpha \in \mathbb{I}_+, \lambda, \mu \in \mathbb{D}_+$ or $\alpha \in \mathbb{D}_+, \lambda, \mu \in \mathbb{I}_+$, the $\lambda \preceq_w \mu$ implies $X_{n:n} \preceq_{st} Y_{n:n}$ if $\phi_1 \circ \psi_2$ is super-additive, $\psi_1$ or $\psi_2$ is log-convex, and $r(x)$ is decreasing.

**Theorem 4.4:** For $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_1)$ and $Y \sim \text{SPRH}(F, \alpha, \mu, \psi_2)$ with $\alpha = (\alpha_1, \ldots, \alpha_n \lambda_n)$, $(\alpha_1 \mu_1, \ldots, \alpha_n \mu_n) \in \mathbb{I}_+ \cap \mathbb{D}_+$ and $\lambda, \mu \in \mathbb{I}_+ \cap \mathbb{D}_+$, the $\lambda \preceq \mu$ implies $X_{n:n} \preceq_{st} Y_{n:n}$ if $\phi_1 \circ \psi_2$ is super-additive, $\psi_1$ or $\psi_2$ is log-convex, and $r(x)$ is decreasing.
4.2. The case of independence

In parallel to Section 3.2 we can obtain the following results on the independent SPRH model.

**Theorem 4.5:** For $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_0)$ and $Y \sim \text{SPRH}(F, \beta, \lambda, \psi_0)$ with decreasing $\tilde{r}(x)$,

(i) the $\alpha \preceq_w \beta$ implies $X_{n:n} \preceq_{rh} Y_{n:n}$ for $\alpha, \beta \in \mathbb{I}_+ \cap \lambda \in \mathbb{D}_+$ or $\alpha, \beta \in \mathbb{D}_+ \cap \lambda \in \mathbb{I}_+$, and

(ii) the $\alpha \preceq^w \beta$ implies $X_{n:n} \preceq_{rh} Y_{n:n}$ for $\alpha, \beta, \lambda \in \mathbb{I}_+$ or $\mathbb{D}_+$.

**Proof:** We prove (i) only, and (ii) can be verified in a similar method.

As per Equation (13), in the context of independence $X_{n:n}$ gets distribution function $G_n(\alpha, \lambda; x) = \prod_{k=1}^{n} F^{\alpha_k}(\lambda k x)$ and density function $g_n(\alpha, \lambda; x) = \prod_{k=1}^{n} \frac{\alpha_k \lambda_k}{\Gamma(\lambda_k)} \sum_{k=1}^{n} \alpha_k \lambda_k r(\lambda k x)$. Then, the corresponding reversed hazard rate

$$r_n(\alpha, \lambda; x) = \frac{g_n(\alpha, \lambda; x)}{G_n(\alpha, \lambda; x)} = \sum_{k=1}^{n} \alpha_k \lambda_k r(\lambda k x).$$ (14)

For any $i \in \mathcal{I}_n$, it is clear that $\frac{\partial r_n(\alpha, \lambda; x)}{\partial \alpha_i} = \lambda_i r(\lambda_i x) \geq 0$. Hence $r_n(\alpha, \lambda; x)$ is increasing in $\alpha_i$.

For $\alpha, \beta \in \mathbb{I}_+$ and $\lambda \in \mathbb{D}_+$ with $\alpha \preceq_w \beta$, by Theorem 5.A.9 of Marshall et al. [31], there exists some $\eta \in \mathbb{I}_+$ such that $\alpha \preceq \eta \preceq \beta$. For $1 \leq i < j \leq n$, owing to the decreasing property of $\tilde{r}(x)$ we have

$$\frac{r_n(-u, \lambda; x)}{\partial u_i} - \frac{r_n(-u, \lambda; x)}{\partial u_j} = \frac{1}{x} [\tilde{r}(\lambda_i x) - \tilde{r}(\lambda_j x)] \geq 0,$$  

implying that $\frac{\partial r_n(-u, \lambda; x)}{\partial u_k}$ is decreasing in $k = 1, \ldots, n$. Therefore, as per Theorem 3.A.3 of Marshall et al. [31],

$$\eta \preceq \beta \text{ on } \mathbb{I}_+ \implies -\eta \preceq -\beta \text{ on } \mathbb{D} \implies r_n(\eta, \lambda; x) \leq r_n(\beta, \lambda; x).$$

Also, by the monotonicity of $r_n(\alpha, \lambda; x)$ with respect to $\alpha_i$ for all $i \in \mathcal{I}_n$, it holds that

$$\alpha \preceq \eta \implies r_n(\alpha, \lambda; x) \leq r_n(\eta, \lambda; x).$$

Thus, $\alpha \preceq_w \beta$ implies $r_n(\alpha, \lambda; x) \leq r_n(\beta, \lambda; x)$ for all $x$. That is, $X_{n:n} \preceq_{rh} Y_{n:n}$.

The case of $\alpha, \beta \in \mathbb{D}_+$ and $\lambda \in \mathbb{I}_+$ can be verified in a similar manner. \hfill \blacksquare

A r.v. is said to have the EW distribution $\text{EW}(\alpha, \beta, \lambda)$ with scale parameter $\lambda > 0$ and two shape parameters $\beta > 0, \lambda > 0$ if it has distribution function $(1 - e^{-(\lambda x)\theta})^\alpha$ (see [34]). For the two independent EW samples $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_0)$ and $Y \sim \text{SPRH}(F, \theta, \lambda, \psi_0)$ with $F \sim \text{EW}(1, \beta, 1)$, Kundu and Chowdhury [35, Theorem 3.3] showed that

(i) the $\alpha \preceq_w \theta$ implies $X_{n:n} \preceq_{rh} Y_{n:n}$ if either $\alpha, \theta \in \mathbb{I}_+ \cap \lambda \in \mathbb{D}_+$ or $\alpha, \theta \in \mathbb{D}_+ \cap \lambda \in \mathbb{I}_+$, and

(ii) the $\alpha \preceq^w \theta$ implies $X_{n:n} \preceq_{rh} Y_{n:n}$ if either $\alpha, \theta, \lambda \in \mathbb{I}_+$ or $\mathbb{D}_+$.

Obviously, the distribution $\text{EW}(\alpha, \beta, \lambda)$ with $\beta > 0$ is just a special case of the SPRH model. Thus, Theorem 3.3 of Kundu and Chowdhury [35] is covered by Theorem 4.5.

**Theorem 4.6:** For $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_0)$ and $Y \sim \text{SPRH}(F, \alpha, \mu, \psi_0)$ with $\alpha \in \mathbb{D}_+, \lambda, \mu \in \mathbb{I}_+$ or $\alpha \in \mathbb{I}_+, \lambda, \mu \in \mathbb{D}_+$, the $\lambda \preceq^w \mu$ implies $X_{n:n} \preceq_{rh} Y_{n:n}$ if $\tilde{r}(x)$ is decreasing and convex.

**Proof:** Note that $\tilde{r}(x)$ is decreasing and convex in $x$. In accordance with Theorem 5.A.2 of Marshall et al. [31], for any $x \geq 0, \lambda \preceq^w \mu$ implies $(\tilde{r}(\lambda_1), \ldots, \tilde{r}(\lambda_n)) \preceq_w (\tilde{r}(\mu_1), \ldots, \tilde{r}(\mu_n))$. By
For independent exponential samples $X \sim \text{SPRH}(e^{-x}, 1, \lambda, \psi_0)$ and $Y \sim \text{SPRH}(e^{-x}, 1, \mu, \psi_0)$, Dykstra et al. [36] proved that $\lambda \leq \mu$ implies $X_{nm} \leq_{\text{rh}} Y_{nm}$. It is easy to verify that, for $F(x) = 1 - e^{-x}$, $r(x)$ is decreasing and convex. Moreover, $\lambda \leq \mu$ implies $\lambda \leq_{w} \mu$. Hence, Theorem 4.6 is actually a generalization of the result of Dykstra et al. [36].

A r.v. is said to have the GE distribution $\text{GE}(\lambda, \theta)$ with scale parameter $\lambda$ and shape parameter $\theta > 0$ if it has distribution function $(1 - e^{-\lambda x})^{\theta}$ (see [37]). Let $F(x)$ be the distribution function of $\text{GE}(1, 1)$. Kundu et al. [20, Theorem 4.2] considered two independent GE samples $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_0)$ and $Y \sim \text{SPRH}(F, \alpha, \mu, \psi_0)$ and proved that $\lambda \leq \mu$ implies $X_{nm} \leq_{\text{rh}} Y_{nm}$ whenever either $\lambda, \mu \in \mathbb{D}_+$, $\alpha \in \mathbb{I}_+$ or $\lambda, \mu \in \mathbb{I}_+$, $\alpha \in \mathbb{D}_+$. For $F(x) \sim \text{GE}(1, 1)$ it is routine to check that $\tilde{r}(x) = x/(e^{\lambda x} - 1)$, $\tilde{r}'(x) \equiv -e^{-x}(x - 1) - 1 \leq 0$ and $\tilde{r}''(x) = x + 2 + e^{x}(x - 2) \geq 0$ for $x \geq 0$. That is, $F(x)$ also fulfills the conditions in Theorem 4.6. So, Theorem 4.6 serves as an essential extension of Theorem 4.2 of Kundu et al. [20].

Let $F(x; \beta)$ be the distribution function of $\text{EW}(1, \beta, 1)$ with $\beta > 0$. For two independent EW samples $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_0)$ and $Y \sim \text{SPRH}(F, \alpha, \mu, \psi_0)$, Kundu and Chowdhury [35] showed that $\lambda \leq \mu$ implies $X_{nm} \leq_{\text{rh}} Y_{nm}$ if either $\lambda, \mu \in \mathbb{D}_+$, $\alpha \in \mathbb{I}_+$ or $\lambda, \mu \in \mathbb{I}_+$, $\alpha \in \mathbb{D}_+$. One can verify that for $F(x; \beta) \sim \text{EW}(1, \beta, 1)$ with $\beta \leq 1$, $\tilde{r}(x)$ is decreasing and convex, which satisfies the condition in Theorem 4.6. Thus, Theorem 4.6 also partially generalizes the result of Kundu and Chowdhury [35] in the context of $\beta \leq 1$.

**Theorem 4.7:** For $X \sim \text{SPRH}(F, \alpha, \lambda, \psi_0)$ and $Y \sim \text{SPRH}(F, \alpha, \mu, \psi_0)$ with $\alpha \in \mathbb{I}_+$, $\lambda, \mu \in \mathbb{D}_+$ or $\alpha \in \mathbb{D}_+$, $\lambda, \mu \in \mathbb{I}_+$, the $\lambda \leq \mu$ implies $X_{nm} \leq_{\text{rh}} Y_{nm}$ if $\tilde{r}(x)$ is decreasing and $x \tilde{r}'(x)$ is increasing.

**Proof:** Note that $\lambda \leq \mu$ is equivalent to $(\ln \lambda_1, \ldots, \ln \lambda_n) \leq_{w} (\ln \mu_1, \ldots, \ln \mu_n)$. By applying Theorem 3.A.3 of Marshall et al. [31] and Theorem 5.A.9 of Marshall et al. [31], the proof can be completed in a similar manner to that of Theorem 4.5.

At the end of this section we illustrate Theorem 4.7 through the following example.

**Example 4.8 (Fréchet distribution):** Consider two size three samples from Fréchet distribution $X \sim \text{SPRH}(e^{-1/x}, (1, 2, 3), (8, 4, 2), \psi_0)$ and $Y \sim \text{SPRH}(e^{-1/x}, (1, 2, 3), (3, 2, 1), \psi_0)$.

It is clear that $(8, 4, 2) \leq_{w} (3, 2, 1)$ and $\tilde{r}(x) = 1/x$ satisfies the concerned assumption in Theorem 4.7. As can be seen in Figure 2, the distribution ratio $P(X_{3,3} \leq x)/P(Y_{3,3} \leq x)$ is indeed decreasing, and this confirms the reversed hazard rate order between $X_{3,3}$ and $Y_{3,3}$.

Also we remark that, for independent samples, Theorem 4.5(i) has a slightly stronger assumption than Theorem 4.1 (the former requires a decreasing $\tilde{r}(x)$) and hence gets the reversed hazard rate ordering result. Similar discussion applies to conditions in Theorems 4.3 and 4.6.

**5. Concluding remarks**

We have a study of stochastic comparison on minimums (maximums) from SPH (SPRH) samples with Archimedean survival copulas (Archimedean copulas). For two heterogeneous dependent SPH samples with a common scale vector, it is shown that the minimum from a more symmetric in the sense of weakly sub-majorization order and more positively dependent sample is larger in stochastic
Figure 2. Ratio between distribution functions of \(X_{3:3}\) and \(Y_{3:3}\).

sense; as for samples with a common frailty vector, the minimum from a more symmetric in the sense of weakly sub-majorization order and more positively dependent sample is stochastically larger. And as for a positively dependent heterogeneous SPH sample and a homogeneous SPH sample with the same dependence structure, the latter is found to get a more dispersive sample minimum whenever the sum of its frailty parameters is less than that of the heterogeneous one and the product of its scale parameters is smaller than that of the heterogeneous one. Similar comparison results are obtained for SPRH samples of dependent observations.

The present results only provide sufficient conditions for stochastic comparison on sample extremes. It is of interest to conduct similar comparison on other stochastic orders such as the ones adjacent to sample extremes and to investigate the existence of other stochastic orders such as the likelihood ratio order. Also, ordering properties for sample spacings from dependent and heterogeneous observations remain open and deserve for a further study.

6. Proofs of Lemmas

Lemma 2.5

Note that the log-convexity of \(\psi\) implies that \(\psi/\psi'\) is decreasing. Since \(\phi(\bar{F}^y(zx))\) is increasing both in \(y\) and in \(z\), then \(\psi(\phi(\bar{F}^y(zx)))/\psi'(\phi(\bar{F}^y(zx)))\) is decreasing in \(y\) and in \(z\). Taking the survival function \(\bar{F}(x)\) into account, we conclude that

\[
\eta_1(y, z, x) = \ln \bar{F}(zx) \frac{\psi(\phi(\bar{F}^y(zx)))}{\psi'(\phi(\bar{F}^y(zx)))}
\]

is increasing both in \(y\) and in \(z\).

(i) Since \(\psi\) is \(n\)-monotone, it holds that \(\psi'(x) \leq 0\) for \(x \geq 0\) and then, for \(i \in \mathcal{I}_n\),

\[
\frac{\partial f_1(\alpha, \lambda; x, \psi)}{\partial \alpha_i} = \psi' \left( \sum_{k=1}^{n} \phi(\bar{F}^{\alpha_k}(\lambda_kx)) \right) \eta_1(\alpha_i, \lambda_i, x) \leq 0.
\]

That is, \(f_1(\alpha, \lambda; x, \psi)\) is decreasing in \(\alpha_i\) for any \(i \in \mathcal{I}_n\). Note that \(\ln x\) is strictly increasing, the monotonicity with respect to \(\ln \alpha_i\) follows directly.

(ii) For \(1 \leq i < j \leq n\) and \(u, \lambda \in \mathcal{D}_+\), we have

\[
-\frac{\partial f_1(u, \lambda; x, \psi)}{\partial u_i} - \frac{\partial f_1(u, \lambda; x, \psi)}{\partial u_j} = -\psi' \left( \sum_{k=1}^{n} \phi(\bar{F}^{\alpha_k}(\lambda_kx)) \right) \left[ \eta_1(u_i, \lambda_i, x) - \eta_1(u_j, \lambda_j, x) \right] \geq 0,
\]
implying that \(-\partial J_1(u, \lambda; x, \psi)/\partial u_k\) is decreasing in \(k = 1, \ldots, n\). Then, according to Theorem 3.A.3 of Marshall et al. [31],

\[
\alpha \preceq \beta \text{ on } \mathbb{D}_+ \implies -J_1(\alpha, \lambda; x, \psi) \preceq -J_1(\beta, \lambda; x, \psi) \iff J_1(\alpha, \lambda; x, \psi) \geq J_1(\beta, \lambda; x, \psi).
\]

Also, for \(1 \leq i < j \leq n, \lambda, \mu \in \mathbb{D}_+, u \in \mathbb{D}\) and \(u \leq 0\), we have

\[
\frac{-\partial J_1(-u, \lambda; x, \psi)}{-\partial u_i} - \frac{-\partial J_1(-u, \lambda; x, \psi)}{-\partial u_j} = \psi' \left( \sum_{k=1}^{n} \phi \left( \hat{F}^{-u_k}(\lambda_i x) \right) \right) \left[ \eta_1(-u_i, \lambda_i; x) - \eta_1(-u_j, \lambda_j; x) \right] \geq 0,
\]

implying that \(-\partial J_1(-u, \lambda; x, \psi)/\partial u_k\) is decreasing in \(k = 1, \ldots, n\). As per Theorem 3.A.3 of Marshall et al. [31] again we have

\[-\alpha \preceq -\beta \text{ on } \mathbb{D} \text{ with } -\alpha, -\beta \leq 0 \implies -J_1(\alpha, \lambda; x, \psi) \preceq -J_1(\beta, \lambda; x, \psi).
\]

Since it is plain that \(\alpha \preceq \beta \text{ on } \mathbb{D}_+\) implies \(-\alpha \preceq -\beta \text{ on } \mathbb{D}\) and \(-\alpha, -\beta \leq 0\), actually we get

\[\alpha \preceq \beta \text{ on } \mathbb{D}_+ \implies J_1(\alpha, \lambda; x, \psi) \geq J_1(\beta, \lambda; x, \psi).
\]

(iii) The concavity of \(\ln(-\ln \psi)\) implies that \((\psi/\psi') \ln \psi\) is increasing. Since \(\phi(\hat{F}'(zx))\) is increasing both in \(y\) and in \(z\), for \(\eta_1\) in (15),

\[y\eta_1(y, z, x) = y \ln \psi \left( \phi \left( \hat{F}'(zx) \right) \right) \frac{\psi \left( \phi(\hat{F}'(zx)) \right)}{\psi' \left( \phi(\hat{F}'(zx)) \right)}
\]

is increasing both in \(y\) and in \(z\).

For \(1 \leq i < j \leq n\) and \(u, \lambda \in \mathbb{D}_+,\) it holds that

\[
\frac{-\partial J_1(u, \lambda; x, \psi)}{-\partial \ln u_i} - \frac{-\partial J_1(u, \lambda; x, \psi)}{-\partial \ln u_j} = -\psi' \left( \sum_{k=1}^{n} \phi \left( \hat{F}'(\lambda_k i x) \right) \right) \left[ u_i \eta_1(u_i, \lambda_i; x) - u_j \eta_1(u_j, \lambda_j; x) \right] \geq 0,
\]

implying that \(-\partial J_1(u, \lambda; x, \psi)/\partial \ln u_k\) is decreasing in \(k = 1, \ldots, n\). Then, by Theorem 3.A.3 of Marshall et al. [31] we have

\[(\ln \alpha_1, \ldots, \ln \alpha_n) \preceq (\ln \beta_1, \ldots, \ln \beta_n) \text{ on } \mathbb{D}\]

\[\implies -J_1(\alpha, \lambda; x, \psi) \preceq -J_1(\beta, \lambda; x, \psi) \iff J_1(\alpha, \lambda; x, \psi) \geq J_1(\beta, \lambda; x, \psi).
\]

Also, for \(1 \leq i < j \leq n\) and \(u, \lambda \in \mathbb{D}_+,\) it holds that

\[
\frac{-\partial J_1(u, \lambda; x, \psi)}{-\partial \ln u_i} - \frac{-\partial J_1(u, \lambda; x, \psi)}{-\partial \ln u_j} = \psi' \left( \sum_{k=1}^{n} \phi \left( \hat{F}'(\lambda_k i x) \right) \right) \times \left[ u_i \eta_1(u_i, \lambda_i; x) - u_j \eta_1(u_j, \lambda_j; x) \right] \geq 0.
\]

Likewise, as per Theorem 3.A.3 of Marshall et al. [31],

\[(-\ln \alpha_1, \ldots, -\ln \alpha_n) \preceq (-\ln \beta_1, \ldots, -\ln \beta_n) \text{ on } \mathbb{D} \implies J_1(\alpha, \lambda; x, \psi) \geq J_1(\beta, \lambda; x, \psi).
\]
Since \((\ln \alpha_1, \ldots, \ln \alpha_n) \leq (\ln \beta_1, \ldots, \ln \beta_n)\) on \(\mathbb{I}\) implies \((-\ln \alpha_1, \ldots, -\ln \alpha_n) \geq (-\ln \beta_1, \ldots, -\ln \beta_n)\) on \(\mathbb{D}\), actually we conclude that \((\ln \alpha_1, \ldots, \ln \alpha_n) \leq (\ln \beta_1, \ldots, \ln \beta_n)\) on \(\mathbb{I}\) implies \(J_1(\alpha, \lambda; x, \psi) \geq J_1(\beta, \lambda; x, \psi)\), the desired conclusion.

**Lemma 2.6**

Due to the log-convex \(\psi\) and \(\phi(\widehat{F}'(zx))\) increasing both in \(y\) and in \(z\), it holds that \(\psi(\phi(\widehat{F}'(zx)))/\psi'(\phi(\widehat{F}'(zx)))\) is decreasing in \(y\) and in \(z\). Let \(h(x)\) be the hazard rate function corresponding to the survival function \(\widehat{F}\). Due to the IHR property, \(h(x)\) is increasing. As a result we conclude that

\[
\eta_2(y, z, x) = y x h(zx) \frac{\psi(\phi(\widehat{F}'(zx)))}{\psi'(\phi(\widehat{F}'(zx)))}
\]

is decreasing both in \(y\) and in \(z\).

(i) Since \(\psi\) is \(n\)-monotone, \(\psi'(x) \leq 0\) for \(x \geq 0\) and hence

\[
\frac{\partial J_1(\alpha, \lambda; x, \psi)}{\partial \lambda_i} = -\psi' \left( \sum_{k=1}^{n} \phi(\widehat{F}^\alpha(\lambda_kx)) \right) \eta_2(\alpha_i, \lambda, x) \leq 0.
\]

That is, \(J_1(\alpha, \lambda; x, \psi)\) is decreasing in \(\lambda_i\) for any \(i \in \mathbb{I}_n\).

(ii) For \(1 \leq i < j \leq n\) and \(\alpha, u \in \mathbb{D}_+\), it holds that

\[
\frac{\partial J_1(\alpha, \lambda; x, \psi)}{-\partial u_i} - \frac{\partial J_1(\alpha, \lambda; x, \psi)}{-\partial u_j} = \psi' \left( \sum_{k=1}^{n} \phi(\widehat{F}^\alpha(u_kx)) \right) \left[ \eta_2(\alpha_i, u, x) - \eta_2(\alpha_j, u, x) \right] \geq 0,
\]

implying that \(-\partial J_1(\alpha, \lambda; x, \psi)/\partial u_k\) is decreasing in \(k = 1, \ldots, n\). Then, from Theorem 3.A.3 of Marshall et al. [31] it stems that

\[
\lambda \leq \mu \text{ on } \mathbb{D}_+ \implies -J_1(\alpha, \lambda; x, \psi) \leq -J_1(\alpha, \mu; x, \psi) \iff J_1(\alpha, \lambda; x, \psi) \geq J_1(\alpha, \mu; x, \psi).
\]

Also, for \(1 \leq i < j \leq n\) and \(\alpha \in \mathbb{I}_+, u \in \mathbb{D}\) with \(u \leq 0\), it holds that

\[
\frac{\partial J_1(\alpha, -u; x, \psi)}{-\partial u_i} - \frac{\partial J_1(\alpha, -u; x, \psi)}{-\partial u_j} = -\psi' \left( \sum_{k=1}^{n} \phi(\widehat{F}^\alpha(-u_kx)) \right) \left[ \eta_2(\alpha_i, -u, x) - \eta_2(\alpha_j, -u, x) \right] \geq 0,
\]

implying that \(-\partial J_1(\alpha, -u; x, \psi)/\partial u_k\) is decreasing in \(k = 1, \ldots, n\). As per Theorem 3.A.3 of Marshall et al. [31] again we have

\[
\lambda \leq \mu \text{ on } \mathbb{I}_+ \implies -\lambda \leq -\mu \text{ on } \mathbb{D} \text{ with } -\lambda, -\mu \leq 0 \implies -J_1(\alpha, \lambda; x, \psi) \leq -J_1(\alpha, \mu; x, \psi) \iff J_1(\alpha, \lambda; x, \psi) \geq J_1(\alpha, \mu; x, \psi).
\]

That is the desired result.
Lemma 2.7

Due to the log-convex \( \psi \) and \( \phi(F^\gamma(zx)) \) increasing in \( y \) and decreasing in \( z \),

\[
\eta_3(y, z, x) = \ln F(zx) \frac{\psi'(\phi(F^\gamma(zx)))}{\psi'(\phi(F^\gamma(zx)))}
\]

is both increasing in \( y \) and decreasing in \( z \).

(i) Since \( \psi \) is \( n \)-monotone, it holds that \( \psi'(x) \leq 0 \) for \( x \geq 0 \) and hence

\[
\frac{\partial J_2(\alpha, \lambda; x, \psi)}{\partial \alpha_i} = -\psi' \left( \sum_{k=1}^{n} \phi \left( F^{\alpha_k}(\lambda_i x) \right) \right) \eta_3(\alpha_i, \lambda_i, x) \geq 0.
\]

That is, \( J_2(\alpha, \lambda; x, \psi) \) is increasing in \( \alpha_i \) for any \( i \in \mathcal{I}_n \). Due to the increasing property of \( \ln x \), \( J_2(\alpha, \lambda; x, \psi) \) is also increasing in \( \ln \alpha_i \) for any \( i \in \mathcal{I}_n \).

(ii) For \( 1 \leq i < j \leq n \), \( \mathbf{u} \in \mathbb{D}_+ \) and \( \lambda \in \mathbb{I}_+ \), it holds that

\[
\frac{\partial J_2(\mathbf{u}, \lambda; x, \psi)}{\partial u_i} - \frac{\partial J_2(\mathbf{u}, \lambda; x, \psi)}{\partial u_j} = -\psi' \left( \sum_{k=1}^{n} \phi \left( F^{\alpha_k}(\lambda_i x) \right) \right) \left[ \eta_3(u_i, \lambda_i, x) - \eta_3(u_j, \lambda_i, x) \right] \geq 0,
\]

implying that \( \partial J_2(\mathbf{u}, \lambda; x, \psi)/\partial u_k \) is decreasing in \( k = 1, \ldots, n \). Then, by Theorem 3.A.3 of Marshall et al. [31] we have

\[
\alpha \preceq \beta \quad \text{on} \quad \mathbb{D}_+ \implies J_2(\alpha, \lambda; x, \psi) \leq J_2(\beta, \lambda; x, \psi).
\]

Also, for \( 1 \leq i < j \leq n \) and \( \mathbf{u} \in \mathbb{D}, \mathbf{u} \leq 0, \lambda \in \mathbb{I}_+ \), it holds that

\[
\frac{\partial J_2(-\mathbf{u}, \lambda; x, \psi)}{\partial u_i} - \frac{\partial J_2(-\mathbf{u}, \lambda; x, \psi)}{\partial u_j} = \psi' \left( \sum_{k=1}^{n} \phi \left( F^{\alpha_k}(\lambda_i x) \right) \right) \left[ \eta_3(-u_i, \lambda_i, x) - \eta_3(-u_j, \lambda_i, x) \right] \geq 0,
\]

implying that \( \partial J_2(-\mathbf{u}, \lambda; x, \psi)/\partial u_k \) is decreasing in \( k = 1, \ldots, n \). As per Theorem 3.A.3 of Marshall et al. [31] again, we have

\[
\alpha \preceq \beta \quad \text{on} \quad \mathbb{I}_+ \implies -\alpha \preceq -\beta \quad \text{on} \quad \mathbb{D} \quad \text{and} \quad -\alpha, -\beta \leq 0 \implies J_2(\alpha, \lambda; x, \psi) \leq J_2(\beta, \lambda; x, \psi).
\]

(iii) Since \( \phi(F^\gamma(zx)) \) is both increasing in \( y \) and decreasing in \( z \) and the concavity of \( \ln(-\ln \psi) \) implies the increasing \( (\psi'/\psi) \ln \psi \), it holds that

\[
\eta_4(y, z, x) = \ln \psi \left( \phi(F^\gamma(zx)) \right) \frac{\psi(\phi(F^\gamma(zx)))}{\psi'(\phi(F^\gamma(zx)))} \psi'(\phi(F^\gamma(zx)))
\]

is increasing in \( y > 0 \) and decreasing in \( z > 0 \).
For \(1 \leq i < j \leq n\) and \(u \in \mathbb{D}_+, \lambda \in \mathbb{I}_+\),

\[
\frac{\partial J_2(u, \lambda; x, \psi)}{\partial \ln u_i} - \frac{\partial J_2(u, \lambda; x, \psi)}{\partial \ln u_j} = -\psi' \left( \sum_{k=1}^{n} \phi \left( F_{\lambda k}(\lambda_k x) \right) \right) \left[ \eta_5(u_i, \lambda_i, x) - \eta_5(u_j, \lambda_j, x) \right] \geq 0,
\]

implying that \(\partial J_2(u, \lambda; x, \psi)/\partial \ln u_k\) is decreasing in \(k \in \mathbb{I}_n\). According to Theorem 3.A.3 of Marshall et al. [31],

\[
(\ln \alpha_1, \ldots, \ln \alpha_n) \overset{m}{\leq} (\ln \beta_1, \ldots, \ln \beta_n) \text{ on } \mathbb{D} \implies J_2(\alpha, \lambda; x, \psi) \leq J_2(\beta, \lambda; x, \psi).
\]

Also, for \(1 \leq i < j \leq n\) and \(u \in \mathbb{I}_+, \lambda \in \mathbb{D}_+, \) it holds that

\[
\frac{\partial J_2(u, \lambda; x, \psi)}{-\partial \ln u_i} - \frac{\partial J_2(u, \lambda; x, \psi)}{-\partial \ln u_j} = \psi' \left( \sum_{k=1}^{n} \phi \left( F_{\lambda k}(\lambda_k x) \right) \right) \left[ \eta_5(u_i, \lambda_i, x) - \eta_5(u_j, \lambda_j, x) \right] \geq 0,
\]

implying that \(\partial J_2(u, \lambda; x, \psi)/-\partial \ln u_k\) is decreasing in \(k \in \mathbb{I}_n\). Again by Theorem 3.A.3 of Marshall et al. [31] we have

\[
(\ln \alpha_1, \ldots, \ln \alpha_n) \overset{m}{\leq} (\ln \beta_1, \ldots, \ln \beta_n) \text{ on } \mathbb{I} \implies (-\ln \alpha_1, \ldots, -\ln \alpha_n) \overset{m}{\leq} (-\ln \beta_1, \ldots, -\ln \beta_n) \text{ on } \mathbb{D} \implies J_2(\alpha, \lambda; x, \psi) \leq J_2(\beta, \lambda; x, \psi).
\]

This completes the proof.

**Lemma 2.8**

The log-convexity of \(\psi\) implies a decreasing \(\psi/\psi'\), and \(\phi(F'(zx))\) is both increasing in \(y\) and decreasing in \(z\), then \(\psi(\phi(F'(zx)))/\psi'(\phi(F'(zx)))\) is both decreasing in \(y\) and increasing in \(z\). Due to DRHR property, the reversed hazard rate \(r(x)\) corresponding to \(\bar{F}(x)\) decreases. Thus, we conclude that

\[
\eta_5(y, z, x) = yx r(z) \frac{\psi(\phi(F'(zx)))}{\psi'(\phi(F'(zx)))}
\]

is both decreasing in \(y > 0\) and increasing in \(z > 0\).

(i) Since \(\psi\) is \(n\)-monotone, it holds that \(\psi'(x) \leq 0\) for \(x \geq 0\) and hence

\[
\frac{\partial J_2(\alpha, \lambda; x, \psi)}{\partial \lambda_i} = -\psi' \left( \sum_{k=1}^{n} \phi \left( F_{\lambda k}(\lambda_k x) \right) \right) \eta_5(\alpha_i, \lambda_i, x) \leq 0.
\]

That is, \(J_2(\alpha, \lambda; x, \psi)\) is decreasing in \(\lambda_i\) for \(i \in \mathbb{I}_n\).

(ii) For \(1 \leq i < j \leq n, \alpha \in \mathbb{I}_+\) and \(u \in \mathbb{D}_+, \) it holds that

\[
\frac{\partial J_2(\alpha, u; x, \psi)}{u_i} - \frac{\partial J_2(\alpha, u; x, \psi)}{u_j} = -\psi' \left( \sum_{k=1}^{n} \phi \left( F_{\lambda k}(u_k x) \right) \right) \left[ \eta_5(\alpha_i, u_i, x) - \eta_5(\alpha_j, u_j, x) \right] \geq 0,
\]

implying that \(\partial J_2(\alpha, u; x, \psi)/\partial u_k\) is decreasing in \(k \in \mathbb{I}_n\). Then, according to Theorem 3.A.3 of Marshall et al. [31] we have

\[
\lambda \overset{m}{\leq} \mu \text{ on } \mathbb{D}_+ \implies J_2(\alpha, \lambda; x, \psi) \leq J_2(\alpha, \mu; x, \psi).
\]
For $1 \leq i < j \leq n$ and $\alpha \in \mathbb{D}_+ \cap \mathbb{I}$ with $u \leq 0$, it holds that

$$\frac{\partial J_2(\alpha, -u; x, \psi)}{\partial u_i} - \frac{\partial J_2(\alpha, -u; x, \psi)}{\partial u_j} = \psi' \left( \sum_{k=1}^{n} \phi \left( F^{z_k}(-u_k x) \right) \right) \left[ \eta_5(\alpha_i, -u_i, x) - \eta_5(\alpha_j, -u_j, x) \right] \geq 0,$$

implying that $\partial J_2(\alpha, -u; x, \psi)/\partial u_k$ is decreasing in $k \in \mathbb{I}_n$. Thus, by Theorem 3.A.3 of Marshall et al. [31] again we have

$$\lambda \leq \mu \iff -\lambda \leq -\mu \quad \text{on } \mathbb{D} \quad \text{and} \quad -\lambda, -\mu \leq 0 \implies J_2(\alpha, \lambda; x, \psi) \leq J_2(\alpha, \mu; x, \psi).$$

(iii) Due to the log-convex $\psi$ and $\phi(F^{y}(zx))$ both increasing in $y$ and decreasing in $z$, it holds that $\psi(\phi(F^{y}(zx))) / \psi'(\phi(F^{y}(zx)))$ is decreasing in $y$ and increasing in $z$. Note that the DRHR aging property of $\bar{F}$ guarantees the decreasing reversed hazard rate $r(x)$. We conclude that

$$\eta_6(y, z, x) = x r(zx) \frac{\psi(\phi(F^{y}(zx)))}{\psi'(\phi(F^{y}(zx)))}$$

is both decreasing in $y > 0$ and increasing in $z > 0$.

For $1 \leq i < j \leq n$, $\alpha \in \mathbb{I}_+ \cap \mathbb{I}$, $u \in \mathbb{D}_+$ and $(\alpha_1 u_1, \ldots, \alpha_n u_n) \in \mathbb{I}_+$,

$$\frac{\partial J_2(\alpha, u; x, \psi)}{\partial \ln u_i} - \frac{\partial J_2(\alpha, u; x, \psi)}{\partial \ln u_j} = -\psi' \left( \sum_{k=1}^{n} \phi \left( F^{z_k}(u_k x) \right) \right) \left[ \alpha_i u_i \eta_6(\alpha_i, u_i, x) - \alpha_j u_j \eta_6(\alpha_j, u_j, x) \right] \geq 0,$$

implying that $\partial J_2(\alpha, u; x, \psi)/\partial \ln u_k$ is decreasing in $k \in \mathbb{I}_n$. Then, from Theorem 3.A.3 of Marshall et al. [31] it follows that, for $\alpha \in \mathbb{I}_+, \lambda, \mu \in \mathbb{D}_+$, and $(\alpha_1 \lambda_1, \ldots, \alpha_n \lambda_n), (\alpha_1 \mu_1, \ldots, \alpha_n \mu_n) \in \mathbb{I}_+$,

$$(\ln \lambda_1, \ldots, \ln \lambda_n) \leq (\ln \mu_1, \ldots, \ln \mu_n) \quad \text{on } \mathbb{D} \implies J_2(\alpha, \lambda; x, \psi) \leq J_2(\alpha, \mu; x, \psi).$$

For $1 \leq i < j \leq n$ and $\alpha \in \mathbb{D}_+, u \in \mathbb{I}_+, (\alpha_1 u_1, \ldots, \alpha_n u_n) \in \mathbb{D}_+$,

$$\frac{\partial J_2(\alpha, u; x, \psi)}{-\partial \ln u_i} - \frac{\partial J_2(\alpha, u; x, \psi)}{-\partial \ln u_j} = \psi' \left( \sum_{k=1}^{n} \phi \left( F^{z_k}(u_k x) \right) \right) \left[ \alpha_i u_i \eta_6(\alpha_i, u_i, x) - \alpha_j u_j \eta_6(\alpha_j, u_j, x) \right] \geq 0,$$

implying that $\partial J_2(\alpha, u; x, \psi)/-\partial \ln u_k$ is decreasing in $k \in \mathbb{I}_n$. Again by Theorem 3.A.3 of Marshall et al. [31] we have

$$(\ln \lambda_1, \ldots, \ln \lambda_n) \geq (\ln \mu_1, \ldots, \ln \mu_n) \quad \text{on } \mathbb{I} \implies (-\ln \lambda_1, \ldots, -\ln \lambda_n) \geq (-\ln \mu_1, \ldots, -\ln \mu_n) \quad \text{on } \mathbb{D} \implies J_2(\alpha, \lambda; x, \psi) \leq J_2(\alpha, \mu; x, \psi),$$

for $\alpha \in \mathbb{D}_+, \lambda, \mu \in \mathbb{I}_+$, and $(\alpha_1 \lambda_1, \ldots, \alpha_n \lambda_n), (\alpha_1 \mu_1, \ldots, \alpha_n \mu_n) \in \mathbb{D}_+$.

**Acknowledgments**

The authors would like to thank the two anonymous reviewers for their valuable comments, which help improve the presentation of this paper.
Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

Dr Fang’s research is supported by National Natural Science Foundation of China [11601303] and Guangdong Natural Science Foundation [2016A030310076].

ORCID

Rui Fang http://orcid.org/0000-0001-6569-5462

References


