CONSTRUCTION OF COMPACTLY SUPPORTED
CONJUGATE SYMMETRIC COMPLEX TIGHT
WAVELET FRAMES

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Two algorithms for constructing a class of compactly supported complex tight wavelet frames with conjugate symmetry are provided. Firstly, based on a given complex refinable function φ, an explicit formula for constructing complex tight wavelet frames is presented. If the given complex refinable function φ is compactly supported conjugate symmetric, then we prove that there exists a compactly supported conjugate symmetric/anti-symmetric complex tight wavelet frame Ψ = {ψ1, ψ2, ψ3} associated with φ. Secondly, under the conditions that both the low-pass filters and high-pass filters are unknown, we give a parametric formula for constructing a class of smooth conjugate symmetric/anti-symmetric complex tight wavelet frames. Free parameters in the algorithm are explicitly identified, and can be used to optimize the result with respect to other criteria. Finally, two examples are given to illustrate how to use our method to construct conjugate symmetric complex tight wavelet frames.

Keywords: Complex tight wavelet frame; compact support; smoothness; conjugate symmetry.

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1. Introduction

Recently, wavelet research has primarily focused on real-valued wavelet bases (see Refs. 1–6). However, complex-valued wavelet bases offer a number of potential advantageous properties. For example: (1) It has been recently suggested that the complex Daubechies wavelet can be made symmetric and orthogonal. This is impossible in real-valued wavelet. (2) In practical application, complex signals are often used to deal with Radar and Sonar. (3) It is very important for movements compressing and feature drawing that complex wavelets carry phase information and shift-invariance. In Refs. 7–11, researchers studied the complex compactly supported orthogonal wavelet and built a systematic theory. In Ref. 7,
the complex compactly supported orthogonal symmetric wavelet was constructed. As we know, real-valued symmetric wavelets can have a linear phase. But complex-valued symmetric wavelets cannot have the linear phase. Only the conjugate symmetric complex wavelet has a linear phase. Linear phase is desirable for reasons of both computational complexity and image quality. The conjugate symmetry of linear phase filters leads to a lower complexity hardware implementation because the multiplies involving origin conjugate symmetric coefficient pairs can be combined. This saving is especially important in image and video coding applications, where the required processing rate is quite high. Additionally, it is known that the phase distortions due to information loss that occurs in compression using nonlinear phase filters can lead to visually objectional artifacts. But it is unfortunate that the compactly supported conjugate symmetric and orthogonal complex wavelet does not exist. To preserve such properties for a linear phase while requiring compact support, the notion of tight frames is probably the only replacement of that of orthogonal wavelets. Then in this paper, we study the construction of complex tight wavelet frames with conjugate symmetry and linear phase, and give an explicit parametric formula for the construction. Furthermore, by selecting the proper parameters, we can find the optimal support-size of the filters according to the proposed cost function and get the lowest cost in terms of bit allocation. Then the decoder that works effectively in real time under a digital signal processor (DSP) can revert the codified string with a linear computational order of complexity in relation to each audio frame’s length.

The main goal of our paper is to construct a class of complex tight wavelet frames with linear phase. We consider only functions of one variable in the space \( L^2(\mathbb{R}) \) with the inner product

\[
\langle f, g \rangle := \int_{-\infty}^{+\infty} f(x) \overline{g(x)} \, dx.
\]

As usual we denote by \( \hat{f}(\omega) \) the Fourier transform of the function \( f(x) \in L^2(\mathbb{R}) \),

\[
\hat{f}(\omega) := \int_{-\infty}^{+\infty} e^{-i\omega x} f(x) \, dx.
\]

Suppose a function \( \varphi \in L^2(\mathbb{R}) \) satisfies the following conditions:

(a) \( \hat{\varphi}(2\omega) = m_0(\omega)\hat{\varphi}(\omega) \), where \( m_0 \) is an essentially bounded \( 2\pi \)-periodic function;

(b) \( \lim_{\omega \to 0} \hat{\varphi}(\omega) = (2\pi)^{-1/2} \); then the function \( \varphi \) is called refinable or scaling, \( m_0 \) is called a symbol of \( \varphi \), and the relation in item (a) is called a refinement equation.

Every refinable function generates multiresolution analysis (MRA) of the space \( L^2(\mathbb{R}) \), i.e. a nested sequence \( \cdots V^{-1} \subset V^0 \subset V^1 \subset \cdots \subset V^j \subset \cdots \) of closed linear subspaces of \( L^2(\mathbb{R}) \) such that:

(a) \( \bigcap_{j \in \mathbb{Z}} V^j = \{0\} \);

(b) \( \bigcup_{j \in \mathbb{Z}} V^j = L^2(\mathbb{R}) \);

(c) \( f(x) \in V^j \iff f(2x) \in V^{j+1} \).
Let us recall that a frame in a Hilbert space $H$ is a family of its elements $\{f_k\}_{k \in \mathbb{Z}}$ such that, for any $f \in H$, there exist $0 < A \leq B < \infty$ so that
\[ A \|f\|^2 \leq \sum_{k \in \mathbb{Z}} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \]
where optimal $A$ and $B$ are called frame constants. If $A = B$, the frame is called a tight frame. In the case when a tight frame has unit frame constants (e.g., if it is an orthonormal basis) for any function $f \in L^2(\mathbb{R})$, the expansion
\[ f = \sum_{k \in \mathbb{Z}} \langle f, f_k \rangle f_k \]
is valid.

The frame $\{\psi_{j,k}\}_{j,k=1}^n$, where $\psi_{j,k}(x) = 2^{j/2} \psi_j(2^j x - k), j, k \in \mathbb{Z}$, generated by translations and dilations of finite number of functions, is called an affine or wavelet frame.

An important property of wavelet system is the vanishing moments of wavelet function. It decides the smoothness of tight wavelet frame, whose definition is as follows:

**Definition 1.1.** We say that $\psi = \{\psi^1, \psi^2, \ldots, \psi^N\}$ has vanishing moments of order $m$ if
\[ \int_{-\infty}^{+\infty} t^k \psi^\ell(t) dt = 0, \quad k = 0, \ldots, m-1. \]

2. Explicit Construction of Complex Tight Wavelet Frames

Let $\phi$ be a refinable function with a symbol $m_0(\omega), \psi^k(\omega) = m_k(\omega/2)\hat{\phi}(\omega/2)$, where each symbol $m_k(\omega)$ is a $2\pi$-periodic and essentially bounded function for $k = 1, 2, \ldots, n$. It is well known that for constructing tight frames the matrix
\[ M(\omega) = \begin{pmatrix} m_0(\omega) & m_1(\omega) & \cdots & m_n(\omega) \\ m_0(\omega + \pi) & m_1(\omega + \pi) & \cdots & m_n(\omega + \pi) \end{pmatrix} \]
plays an important role.

In what follows we need Lemmas 2.1–2.3 (see Refs. 1 and 2) formulated below.

**Lemma 2.1.** If
\[ M(\omega)M^*(\omega) = I \] (2.1)
holds, then the functions $\{\psi^k\}_{k=1}^n$ generate a tight frame of $L^2(\mathbb{R})$.

**Lemma 2.2.** Equation (2.1) has a solution if and only if
\[ |m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 \leq 1 \quad (a.e.). \] (2.2)

**Lemma 2.3.** Let a $2\pi$-periodic function $m_0(\omega)$ satisfy (2.2). Then there exists a pair of $2\pi$-periodic measurable functions $m_1(\omega), m_2(\omega)$ which satisfy (2.1) for $n = 2$. 
Furthermore, any solution of (2.1) can be represented in the form of the first row of the matrix

\[ M_\psi(\omega) = S(\omega) D(\omega) T(\omega), \quad (2.3) \]

where

\[ S(\omega) = \begin{pmatrix} \frac{e^{i\omega} m_0(\omega + \pi)}{B(\omega)} & m_0(\omega) \frac{B(\omega)}{B(\omega)} \\ -\frac{e^{i\omega} m_0(\omega)}{B(\omega)} & m_0(\omega + \pi) \frac{B(\omega)}{B(\omega)} \end{pmatrix}, \quad (2.4) \]

\[ \Lambda(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - |m_0(\omega)|^2 - |m_0(\omega + \pi)|^2 \end{pmatrix}, \quad (2.5) \]

\[ |B(\omega)|^2 = |m_0(\omega)|^2 + |m_0(\omega + \pi)|^2, \]

\( D(\omega) \) is a diagonal matrix, \( D(\omega) \overline{D(\omega)} = \Lambda(\omega) \), and \( T(\omega) \) is an arbitrary unitary \((a.e.)\) matrix with \( \pi \)-periodic measurable components.

**Remark.** To describe all possible solutions to (2.1) for an arbitrary \( n \), we have to take an arbitrary \( n \times n \) unitary matrix \( T(\omega) \) with \( \pi \)-periodic elements and a \( 2 \times n \) matrix \( D' \) which is an extension of the matrix \( D \) by means of filling all new columns with zeros.

**Lemma 2.4.** Let \( a_0, \ldots, a_N \in \mathbb{R} \) and \( a_N \neq 0 \), such that \( A(\omega) := \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos k \omega \geq 0, \omega \in \mathbb{R} \). Suppose that the polynomial \( A(\omega) \) has complex roots with respect to \( z \), \( z = e^{-i\omega} \). Then there exists a polynomial \( B(z) = \sum_{k=1}^{N} b_k z^k \) with complex coefficients and exact degree \( N \) that satisfies

\[ |B(z)|^2 = A(\omega), \quad z = e^{-i\omega}. \]

**Proof.** Since the polynomial \( A(\omega) \) has complex roots, according to the proof of Riesz Lemma, \( A(\omega) \) has the decomposition as follows\(^{15}\):

\[ A(\omega) = \frac{1}{2} |a_N| \prod_{i=1}^{I} |r_i^{-1}| \prod_{j=1}^{J} |z_j|^{-2} \prod_{i=1}^{I} (z - r_i) \prod_{j=1}^{J} (z - z_j)(z - \overline{z}_j), \quad J \geq 1. \]

Then

\[ B'(z) = \left( \frac{1}{2} |a_N| \prod_{i=1}^{I} |r_i^{-1}| \prod_{j=1}^{J} |z_j|^{-2} \right)^{1/2} \prod_{i=1}^{I} (z - r_i) \prod_{j=1}^{J} (z - z_j)(z - \overline{z}_j), \]

\[ I + 2J = N. \]

Furthermore, for \( \forall i \ (1 \leq i \leq I) \), we substitute \( (r_i z - 1) \) for \( (z - r_i) \) in \( B'(z) \), at the same time for \( \forall j \ (1 \leq j \leq J) \), we substitute \( (\overline{z}_j z - 1) \) for \( (z - z_j) \) in \( B'(z) \).
Theorem 2.1. Let \( \phi \in L^2(\mathbb{R}) \) be a complex refinable function with a symbol \( m_0(\omega) \) that satisfies (2.2), \( A(\omega) = |m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 \). Suppose that the polynomial \( A(\omega) \) has complex roots with respect to \( z, z = e^{-i\omega} \). Then there exists a complex wavelet frame associated with \( \phi \) that satisfies

\[
\hat{\psi}_k(\omega) = m_k(\omega/2) \hat{\phi}(\omega/2), \quad k = 1, \ldots, n,
\]

where \( m_1(\omega), \ldots, m_n(\omega) \) are complex coefficients functions.

Proof. Since \( A(\omega) = |m_0(\omega)|^2 + |m_0(\omega + \pi)|^2, A(\omega) \) is a non-negative cosine polynomial. By Lemma 2.4, there exists a complex coefficients polynomial \( B(z) \) satisfying

\[
|B(z)|^2 = |m_0(\omega)|^2 + |m_0(\omega + \pi)|^2.
\]

Substitute \( B(z) \) into (2.4), then \( S(\omega) \) is a matrix with complex coefficients components. Hence \( m_1(\omega), \ldots, m_n(\omega) \) are complex coefficients functions. So there exists a complex wavelet frame \( \Psi = \{ \psi_k(\omega) \} \), where

\[
\hat{\psi}_k(\omega) = m_k(\omega/2) \hat{\phi}(\omega/2), \quad k = 1, \ldots, n.
\]

In what follows, the Laurent polynomial \( P(z) \) is specified by the \( z \)-transform of the symbol \( m_0(\omega) \), i.e. \( P(e^{-i\omega}) := m_0(\omega) \).

Theorem 2.2. Let \( \phi \in L^2(\mathbb{R}) \) be a compactly supported conjugate symmetric complex refinable function. If its two-scale symbol \( P(z) \) satisfies the following conditions:

\begin{enumerate}
\item [(C1)] \( |P(z)|^2 + |P(-z)|^2 \leq 1, |z| = 1; \)
\item [(C2)] \( 1 - |P(z)|^2 - |P(-z)|^2 \) has complex roots.
\end{enumerate}

Then there exists a compactly supported conjugate symmetric/anti-symmetric complex tight wavelet frame \( \Psi = \{ \psi_1, \psi_2, \psi_3 \} \) associated with \( \phi \).

Proof. Consider a Laurent polynomial matrix

\[
R(z) := \begin{pmatrix}
P(z) & z^{2l+1}P(-z) & Q(z) & z^{2l+1}Q(-z) \\
P(-z) & -z^{2l+1}P(z) & Q(-z) & -z^{2l+1}Q(z)
\end{pmatrix}, \quad k, l \in \mathbb{Z},
\]

((substitute \( z_j z - 1 \) for \( z - \tau_j \) in \( B'(z) \)). Then we get

\[
B(z) = \left( \frac{1}{2} a_N \prod_{i=1}^l \prod_{j=1}^m |z_j|^{-2} \right)^{1/2} \prod_{i=1}^l \left( r_i z - 1 \right) \prod_{j=1}^m \left( \tau_j z - 1 \right) (z - \tau_j).
\]

Since \( |z - r_i|^2 = |r_i z - 1|^2 \), and \( |z - \tau_j|^2 = |\tau_j z - 1|^2 \), we have

\[
|B(z)|^2 = |B'(z)|^2 = A(\omega), \quad z = e^{-i\omega}.
\]

So \( B(z) \) is the polynomial with complex coefficients and exact degree \( N \). \( \square \)
where \( Q(z) \) is some Laurent polynomial. It is easy to see that
\[
R(z)R^*(z) = |P(z)|^2 + |P(-z)|^2 + |Q(z)|^2 + |Q(-z)|^2 I_2. \tag{2.6}
\]
By Lemma 2.1, we only need to find a conjugate symmetric/anti-symmetric complex Laurent polynomial \( Q(z) \), such that
\[
|P(z)|^2 + |P(-z)|^2 + |Q(z)|^2 + |Q(-z)|^2 = 1, \quad |z| = 1. \tag{2.7}
\]
Suppose that
\[
Q(z) = C(z) \pm z^{4n+1}C(z), \tag{2.8}
\]
where
\[
C(z) := \sum_{j=0}^{n} (a_j + ib_j)z^{2j}, \quad a_j, b_j \in R; \quad b_nb_n \neq 0. \tag{2.9}
\]
Then \( Q(z) \) is a conjugate symmetric/anti-symmetric complex Laurent polynomial and satisfies
\[
|Q(z)|^2 + |Q(-z)|^2 = 4|C(z)|^2, \quad |z| = 1. \tag{2.10}
\]
By substituting (2.10) into (2.7), we get
\[
4|C(z)|^2 = 1 - (|P(z)|^2 + |P(-z)|^2), \quad |z| = 1. \tag{2.11}
\]
By Lemma 2.4, since \( P(z) \) satisfies the conditions (C1) and (C2), there exists a Laurent polynomial \( C(z) \) in the form of (2.9) which satisfies (2.11). Hence, \( Q(z) \) given by (2.8) is a conjugate symmetric/anti-symmetric complex Laurent polynomial. Consequently, \( \psi^1, \psi^2 \) and \( \psi^3 \) with two-scale symbols \( z^{2k+1}P(-z), Q(z) \) and \( z^{2k+1}Q(-z) \) respectively, generate a compactly supported conjugate symmetric/anti-symmetric complex tight wavelet frame associated with \( \phi \).

3. Parametrization of Complex Wavelet Tight Frames

In the above argumentation, we describe a construction method of a class of complex tight wavelet frames with conjugate symmetry/anti-symmetry, which is based on a given complex refinable function \( \phi \). In this section, under the conditions that both the low-pass filters and high-pass filters are unknown, we present another parametric formula for constructing a class of smooth conjugate symmetric/anti-symmetric complex tight wavelet frames.

Let \( \phi \) be a complex refinable function, and \( \psi^1, \psi^2, \psi^3 \) generate a complex tight wavelet frame corresponding to \( \phi \); \( P(z), Q_1(z), Q_2(z) \) and \( Q_3(z) \), respectively, be the two-scale symbols of \( \phi, \psi^1, \psi^2, \psi^3 \). In order to get compactly supported conjugate symmetric complex tight wavelet frames, we only consider that the symbols \( P(z), Q_1(z), Q_2(z) \) and \( Q_3(z) \) are complex Laurent polynomials. For simplicity, suppose that \( \phi \) is a conjugate symmetric complex refinable function, among \( \psi^1, \psi^2, \psi^3 \), one function is conjugate symmetric, another two are conjugate anti-symmetric.
Now we give two theorems which show the concrete algorithm for constructions of compactly supported complex wavelet frames with \( n = 2N + 1 \) and \( n = 2N \), respectively.

**Theorem 3.1.** Let \( P(z) = \sum_{j=0}^{2N} h_j z^j, h_j \in \mathbb{C} \). Then there exists \( h_0, h_1, \ldots, h_{2N} \), such that \( P(z) \) is a conjugate symmetric complex low-pass filter with \( N - 1 \) vanishing moments and two parametric variables. And \( P(z) \) can generate a refinable function \( \phi \). Furthermore, there exists three conjugate symmetric/anti-symmetric complex high-pass filters \( Q_1(z), Q_2(z), Q_3(z) \) corresponding to \( P(z) \), and \( Q_1(z), Q_2(z), Q_3(z) \) have two parametric variables and \( (2N + 1) \) length, respectively. Every parameter in the corresponding closed interval will give a complex tight wavelet frame.

**Proof.** Let \( h_0, \ldots, h_{2N} \) be low-pass filter coefficients with conjugate symmetry. Then they satisfy normalized condition:
\[
\sum_{j=0}^{2N} h_j = 1
\]
and conjugate symmetric conditions:
\[
h_j = \overline{h}_{2N-j}, \quad j = 0, 1, \ldots, N - 1.
\]
Hence we have
\[
\sum_{j=0}^{N-1} (h_j + \overline{h}_j) = 1 - h_N.
\]
We will add \( N - 1 \) vanishing moments, which satisfy the following equations\(^{16,17}\):
\[
\sum_{n} (-1)^n n^k h_{1-n} = 0, \quad k = 1, 2, \ldots, N - 1.
\]
By putting the above conditions together, we have
\[
\begin{cases}
\sum_{j=0}^{N-1} (h_j + \overline{h}_j) = 1 - h_N, \\
\sum_{j=0}^{N-1} (-1)^j [(1 - j)^k h_j + (1 + j - 2N)^k \overline{h}_j] = -(1 - N)^k (-1)^N h_N.
\end{cases}
\]
Let \( h_j = a_j + ib_j, j = 0, 1, \ldots, N - 1 \). By substituting it into the above equations, we get
\[
\begin{cases}
\sum_{j=0}^{N-1} 2a_j = 1 - h_N, \\
\sum_{j=0}^{N-1} (-1)^j [(1 - j)^k + (1 + j - 2N)^k] a_j + \sum_{j=0}^{N-1} (-1)^j [(1 - j)^k - (1 + j - 2N)^k] b_j i \\
= -(1 - N)^k (-1)^N h_N.
\end{cases}
\]
That is
\[
\begin{align*}
    \sum_{j=0}^{N-1} 2a_j &= 1 - h_N, \\
    \sum_{j=0}^{N-1} (-1)^j[(1 - j)^k + (1 + j - 2N)^k]a_j &= -(1 - N)^k(-1)^Nh_N, \quad (3.1) \\
    \sum_{j=0}^{N-1} (-1)^j[(1 - j)^k - (1 + j - 2N)^k]b_j &= 0.
\end{align*}
\]

The first and second equations of (3.1) have \(N + 1\) variables \(a_0, \ldots, a_{N-1}, h_N\) and \(N\) independent equations, so there exists just one free variable. Let \(h_N\) be the free variable. Then we can write the linear equations in matrix form
\[
A(a_0, a_1, \ldots, a_{N-1})^T = c_1,
\]
where \(A\) is the matrix formed by the left coefficients of the equations, and \(c_1\) is the coefficients vectors related to \(h_N\). The third equation of (3.1) has \(N\) variables \(b_0, \ldots, b_{N-1}\) and \(N - 1\) independent equations, so there exists just one free variable. Let \(b_{N-1}\) be the free variable. Then we can also write the linear equations in matrix form
\[
B(b_0, b_1, \ldots, b_{N-2})^T = c_2,
\]
where \(B\) is the matrix formed by the left coefficients of the equations, and \(c_2\) is the coefficients vectors related to \(b_{N-1}\). We get \(h_j = h_j(h_N, b_{N-1}), \quad j = 0, \ldots, N - 1, \)

where \(h_N \in [e,N, f_N]\), \(b_{N-1} \in [p_{N-1}, q_{N-1}]\). Thus we get a conjugate symmetric complex low-pass filter \(P(z)\) with \(N - 1\) vanishing moments and two parametric variables \(h_N, b_{N-1}\) as follows:
\[
P(z) = \sum_{j=0}^{N-1} (h_j(h_N, b_{N-1})z^j + \bar{h}_j(h_N, b_{N-1})z^{2N-j}) + h_Nz^N. \quad (3.2)
\]

By selecting \(h_N, b_{N-1}\) in \([e_N, f_N]\) and \([p_{N-1}, q_{N-1}]\), respectively, we can get the low-pass filters. For a given cost function, we can select the proper \(h_N, b_{N-1}\) and get the best low-pass filters.

Let \(P(z)\) constructed in the above be a conjugate symmetric complex low-pass filter. Next we are going to construct the corresponding high-pass filters. From Theorem 2.2, we know that there exists a \((2N + 1)\) length and conjugate anti-symmetric Laurent polynomial \(Q_2(z)\) satisfying
\[
|P(z)|^2 + |P(-z)|^2 + |Q_2(z)|^2 + |Q_2(-z)|^2 = 1, \quad |z| = 1. \quad (3.3)
\]

Hence let
\[
Q_2(z) = \sum_{j=0}^{N-1} (g_jz^j - \bar{g}_jz^{2N-j}) + g_Nz^N, \quad (3.4)
\]
where \( g_j = u_j + iv_j, \ j = 0, \ldots, N-1, \) and \( g_N = v_Ni. \) By substituting (3.4) into (3.3), we get \( g_j := g_j(h_N, b_{N-1}), \ j = 0, \ldots, N. \) Hence we have

\[
Q_2(z) = \sum_{j=0}^{N-1} (g_j(h_N, b_{N-1})z^j - \overline{g}_j(h_N, b_{N-1})z^{2N-j}) + g_N(h_N, b_{N-1})z^N.
\]

Again applying Theorem 2.2, another two high-pass filters are, respectively,

\[
Q_1(z) = zP(-z), \quad Q_3(z) = zQ_2(-z).
\]  

Similarly, we have the following Theorem 3.2.

**Theorem 3.2.** Let \( P(z) = \sum_{i=0}^{2N-1} h_jz^j, h_j \in C. \) Then there exists \( h_0, h_1, \ldots, h_{2N-1}, \) such that \( P(z) \) is a conjugate symmetric complex low-pass filter with \( N-1 \) vanishing moments and one parametric variable. And \( P(z) \) can generate a refinable function \( \phi. \) Furthermore, there exists three conjugate symmetric/antisymmetric complex high-pass filters \( Q_1(z), Q_2(z), Q_3(z) \) corresponding to \( P(z), \) and \( Q_1(z), Q_2(z), Q_3(z) \) have one or two parametric variables and \( 2N \) length respectively. Every parameter in the corresponding closed interval will give a complex tight wavelet frame.

4. Examples

Next we give two examples with concrete parametrization form for \( n = 5 \) and \( n = 4. \)

**Example 4.1.** Let \( P(z) = \sum_{k=0}^{4} h_kz^k, \) where \( h_0 = \overline{h}_4, \ h_1 = \overline{h}_3. \) Furthermore let \( h_j = a_j + ib_j, \ j = 0, 1. \) According to Theorem 3.1, we can get the parametrization form as follows:

\[
\begin{align*}
P(z) &= \left[ \frac{1}{4} - \frac{1}{2}h_2 + \frac{1}{2}b_1 \right] z + \left[ \frac{1}{4} + b_1 \right] z^2 + \left[ \frac{1}{4} - b_1 \right] z^3, \\
Q_1(z) &= \left[ \frac{1}{4} - \frac{1}{2}h_2 - \frac{1}{2}b_1 \right] z - \left[ \frac{1}{4} - b_1 \right] z^2 + \left[ \frac{1}{4} + b_1 \right] z^3, \\
Q_2(z) &= [u_0 + iv_0] + [u_1 + iv_1]z + v_2iz^2 - [u_1 - iv_1]z^3 - [u_0 - iv_0]z^4, \\
Q_3(z) &= [u_0 - iv_0]z - [u_1 - iv_1] - v_2iz^{-1} + [u_1 + iv_1]z^{-2} - [u_0 + iv_0]z^{-3},
\end{align*}
\]

where \( g_j = u_j + iv_j, \ j = 0, 1, g_2 = v_2i. \) Among these variables, \( u_0, v_0, u_1, v_1, v_2 \) can be represented by \( h_2 \) and \( b_1. \)
Choose $h_2 = \frac{3}{8}$, $b_1 = \frac{1}{16}$, by (3.2), we get

\begin{align*}
P(z) &= [0.0625 + 0.03125i]z + [0.2500 + 0.0625i]z^2 + [0.2500 - 0.0625i]z^3 + [0.0625 - 0.03125i]z^4.
\end{align*}

By solving Eq. (3.3), we can get

\begin{align*}
Q_2(z) &= [0.0625 + 0.03125i]z + [0.3454 + 0.0450i]z + 0.1887iz^2 - [0.3454 - 0.0450i]z^3 - [0.0625 - 0.03125i]z^4.
\end{align*}

Applying (3.5), we have

\begin{align*}
Q_1(z) &= [0.0625 - 0.03125i]z - [0.2500 - 0.0625i] + 0.3750z^{-1} - [0.2500 + 0.0625i]z^{-2} + [0.0625 + 0.03125i]z^{-3},
Q_3(z) &= [0.0625 - 0.03125i]z - [0.3454 - 0.0450i] - 0.1887iz^{-1} + [0.3454 + 0.0450i]z^{-2} - [0.0625 + 0.03125i]z^{-3}.
\end{align*}

Fig. 1. The real parts of $\phi$, $\psi^1$, $\psi^2$ and $\psi^3$ from Example 4.1.
Compactly Supported Conjugate Symmetric Complex Tight Wavelet Frames

Fig. 2. The imaginary parts of $\phi$, $\psi^1$, $\psi^2$ and $\psi^3$ from Example 4.1.

Then we get the real part figure and imaginary part figure of the complex wavelet frame as follows: (see Figs. 1 and 2, respectively).

**Example 4.2.** Let $P(z) = \sum_{k=0}^{3} h_k z^k$, where $h_0 = \overline{h}_3$, $h_1 = \overline{h}_2$. Furthermore let $h_j = a_j + ib_j$, $j = 0, 1$. According to Theorem 3.2, we can get the parametrization form as follows:

\[
\begin{cases}
    P(z) = \left[\frac{1}{8} - ib_1\right]z + \left[\frac{3}{8} + ib_1\right]z^2 + \left[\frac{3}{8} - ib_1\right]z^3, \\
    Q_1(z) = \left[\frac{1}{8} + ib_1\right]z - \left[\frac{3}{8} - ib_1\right] + \left[\frac{3}{8} + ib_1\right]z^{-1} - \left[\frac{1}{8} - ib_1\right]z^{-2}, \\
    Q_2(z) = [u_0 + iv_0] + [u_1 + iv_1]z - [u_1 - iv_1]z^2 - [u_0 - iv_0]z^3, \\
    Q_3(z) = [u_0 - iv_0]z - [u_1 - iv_1] - [u_1 + iv_1]z^{-1} - [u_0 + iv_0]z^{-2},
\end{cases}
\]

where $g_j = u_j + iv_j$, $j = 0, 1$. Among these variables, $u_0, v_0, v_1$ can be represented by $u_1$ and $b_1$. 
Choose $u_1 = \frac{1}{16}$, $b_1 = \frac{1}{16}$, by (3.2), we get

$$P(z) = [0.1250 - 0.0625i] + [0.3750 + 0.0625i]z + [0.3750 - 0.0625i]z^2$$
$$+ [0.1250 + 0.0625i]z^3.$$ 

By solving Eq. (3.3), we can get

$$Q_2(z) = [0.1237 + 0.1896i] + [0.0625 - 0.2176i]z - [0.0625 + 0.2176i]z^2$$
$$- [0.1237 - 0.1896i]z^3.$$ 

Applying (3.5), we have

$$Q_1(z) = [0.1250 + 0.0625i]z - [0.3750 - 0.0625i] + [0.3750 + 0.0625i]z^{-1}$$
$$- [0.1250 - 0.0625i]z^{-2},$$

$$Q_3(z) = [0.1237 - 0.1896i]z - [0.0625 + 0.2176i] - [0.0625 - 0.2176i]z^{-1}$$
$$+ [0.1237 + 0.1896i]z^{-2}.$$ 

Fig. 3. The real parts of $\phi$, $\psi^1$, $\psi^2$ and $\psi^3$ from Example 4.2.
Then we get the real part figure and imaginary part figure of the complex wavelet frame as follows: (see Figs. 3 and 4, respectively).

5. Conclusion

This paper gives two algorithms for constructing compactly supported complex tight wavelet frames which are based on the complex refinable function $\phi$ being known and unknown respectively. As we know, though complex tight wavelet frames lose orthogonal property, it allows the filters short length as well as the scaling and wavelet functions more smoothness. In the first algorithm, a class of compactly supported conjugate symmetric complex tight wavelet frames associated with $\phi$ is constructed. Consequently, the second algorithm gives a parametric formula for constructing a class of smooth conjugate symmetric complex tight wavelet frames. The parametrization of complex filters can provide an effective structure and algorithm for implementation. It can be used as an effective method to choose optimal complex wavelet filters for efficient image compression and processing complex signals such as in radar and sonar. Furthermore, by using the adaptive search for finding
the best filters’ length, the wavelets as a pattern to encode all audio clips do not increase the computational complexity significantly and a real time implementation of the decoder works efficiently.

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References