Methods for Constructing Choquet-Capacity Preserving and Ergodic Systems: Examples

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ABSTRACT

The Choquet capacity of a random closed set on a metric space is regarded as or related to a non-additive measure, an upper probability, a belief function, and a counterpart of the distribution functions of ordinary random vectors. Unlike the ordinary measures which are additive, Choquet capacities are generally non-additive. The purpose of this paper is to generalize the traditional measure preserving and measure ergodic systems to non-additive settings: Choquet-capacity preserving and Choquet-capacity ergodic systems, by constructing solid and standard capacity systems.

Keywords: Measure preserving system; Measure ergodic system; Choquet-capacity preserving system; Choquet-capacity ergodic systems; Construction methods

1. Choquet capacities as non-additive measures

1.1 Non-additive set functions

Non-additive set functions occur in various branches of mathematics, such as potential theory [5], harmonic analysis, fractal geometry [10], functional analysis, the theory of nonlinear differential equations, the theory of difference equations, and in optimization. Interest in non-additive set functions is growing: In artificial intelligence and fuzzy set theory [34, 48, 49], belief functions are utilized to model uncertainty [51]; belief functions, corresponding plausibility measures and other kinds of non-additive set functions are also used in statistics [21]; non-additive expected utility theory has been applied in multi-stage decision and economics [16].

Non-additive set functions, such as outer measures and semi-variations of vector measures, actually appeared earlier in classical measure theory concerning countable
(resp., finite) additive set functions, and have been increasingly used to manipulate non-additive physical quantities. Non-additivity in probability started with J. Bernoulli in the 17th century (also, J. H. Lambert in the 18th century [41]), but there were no substantial studies in the next two more centuries (until G. Choquet [6]).

1.2 Choquet capacities and their significance

Choquet capacities, introduced by G. Choquet, a pioneer of the theory of non-additive set functions, are non-additive set functions and have influences on many parts of mathematics and different areas of sciences and technology, e.g., all fields mentioned previously as well as decision theory and social choice problems [1, 7, 9, 10, 16, 21, 34, 48, 51]. In addition, Choquet capacities are widely applied to mathematical finance, e.g., cooperative games [42], stochastic dominance and risk measures [45], uncertainty in economic theory [25]. Specially, on the multi-criteria decision aiming at ordering multidimensional alternatives where the traditional weighted sum that relies on a rarely verified independence of criteria is re-placed by a Choquet integral with respect to a capacity [14-16, 23, 24]. This approach was further developed by Kojadinovic [22].

In terms of Choquet capacities (monotonically increasing but not necessarily additive), the Choquet integral [26, 29, 30] was introduced to generalize the traditional Lebesgue-Stieltjes integral (Lebesgue-Radon integral) where the latter (Lebesgue-Stieltjes integral) requires a regular Borel measure that is associated to functions of bounded variation on the real line (another integral based on monotonic set functions is Sugeno’s integral Sugeno [49]).

Let $E$ be a locally compact separable (LCS) metric space (throughout the paper), such as $\mathbb{R}$ and other manifolds encountered in applications of mathematics. The Choquet integral of a measurable function $g : E \to \mathbb{R}^+$ with respect to a Choquet capacity $T$ (more general, with respect to a non-decreasing non-additive set function) is defined by

$$\int_0^\infty T(\{x \in E \mid g(x) \geq t\}) \, dt,$$

compared to the well-known representation of the expectation of an ordinary non-negative random variable $X$: $\mathbb{E}X = \int_0^\infty X(\omega) \, dP(\omega) = \int_0^\infty P(\{\omega \in \Omega \mid X(\omega) \geq t\}) \, dt$. Choquet integral was rediscovered by Schmeidler in the framework of decision under uncertainty, where he was looking for representation theorems in utility theory. Schmeidler first put forward an axiomatic model of choice with non-additive beliefs. In order to axiomatize a decision model more flexible than subjective expected utility, Schmeidler weakened the classical independence axiom requiring it only for comonotonic functions and as a consequence, obtained a representation of
preferences by a functional which is linear only on comonotonic functions and is exactly a Choquet integral [4, 13, 38, 39].

Another important role of Choquet capacities in probability theory is the characterization of probability laws of random closed sets (RACS). RACS serve as general mathematical models for set-valued observations and irregular geometrical patterns of physical phenomena, generalize the traditional concept of ordinary random variables, which are regarded as models for point-valued observations, and play an important role in stochastic geometry. RACS have direct applications in the fields mentioned previously. Early applications of RACS in statistics include statisticians designing to understand the basic principles of (finite) population sampling (see [17]). In parameter estimation, if \((\Omega, A, P_\theta, \theta \in \Phi)\) is a statistical model, then a multi-valued mapping \(S : \Omega \longrightarrow 2^\Phi\) such that \(S^{-1}(\theta) = \{\omega | \theta \in S(\omega)\} \in A\) for all \(\theta \in \Phi\) is a confidence region estimator of the true model parameter, with confidence level \(1 - \alpha = \inf\{P_\theta(\omega) | \theta \in S(\omega)\}, \theta \in \Phi\}\) (see [18]). A recent application demonstrates again the usefulness of RACS to statistics. Recall that the sample quantile is a strongly consistent estimator of the population quantile ([40], p. 75), and the weak consistency of sample quantiles as estimators of population quantiles was shown by Smirnov ([44]; see also [36], p. 179). Das and Resnick in 2008 applied random set theory to statistics by considering the set \(S_n\) of points forming the quantile-quantile plot as a RACS in \(\mathbb{R}^2\): Under certain regularity conditions on the given heavy tailed distribution \(F, S_n\) converges in probability to a closed, non-random set [6].

2. Preliminaries: Choquet capacities and Choquet theorem

2.1 Hit-or-miss topology and its convergence

Let \(\mathcal{F}(E), \mathcal{G}(E),\) and \(\mathcal{K}(E)\) denote respectively the sets of all closed, open and compact subsets of \(E\), abbreviated as \(\mathcal{F}, \mathcal{G},\) and \(\mathcal{K}\) (\(\emptyset \in \mathcal{F}, \emptyset \in \mathcal{G},\) and \(\emptyset \in \mathcal{K}\)).

The hit-or-miss topology \(\tau_f\) on \(\mathcal{F}\) (also known as H-topology [11], Fell topology [29, 32], Choquet-Matheron topology [46], or weak Vietoris topology [56]) is generated by the subbase

\[
\mathcal{F}_K, K \in \mathcal{K}; \quad \mathcal{F}_G, G \in \mathcal{G}
\]

where \(\mathcal{F}_K = \{F \in \mathcal{F} | F \cap K = \emptyset\}\) and \(\mathcal{F}_G = \{F \in \mathcal{F} | F \cap G \neq \emptyset\}\). A topological base of \(\tau_f\) is

\[
\mathcal{F}_G^{K_{1, \ldots, n}}, K \in \mathcal{K}, G_i \in \mathcal{G} \quad (1 \leq i \leq n), \quad n \geq 0,
\]

where \(G_1, \ldots, G_n\) are arbitrary sets of \(\mathcal{G}\) and \(K\) is a compact set of \(\mathcal{K}\).
where $\mathcal{F}_G^K = \mathcal{F}^K \cap \mathcal{F}_G \cap \ldots \cap \mathcal{F}_G$. Note that $\mathcal{F}^0 = \mathcal{F}$ and $\mathcal{F}_G^K$ means $\mathcal{F}^K$ when $n = 0$.

A sequence $\{F_n\}$ of closed subsets of $E$ converges to a closed subset $F$ of $E$ in $\mathcal{F}$ if and only if (see [26])

(i) If an open set $G$ hits $F$, then $G$ hits all the $F_n$'s except at the most a finite number of $F_n$'s.

(ii) If a compact set $K$ is disjoint of $F$, then it is disjoint of all the $F_n$'s except at the most a finite number of $F_n$'s.

The above (i) and (ii) are respectively equivalent to the following (i)' and (ii)'.

(i)' For every $x \in F$, there exists for each integer $n$ a point $x_n$ in $F_n$ (except, at the most, for a finite number of $F_n$'s) such that $\lim_{n \to \infty} x_n = x$ in $E$.

(ii)' If $F_{n_k}$ is a subsequence of $F_n$, and $x_{n_k} (x_{n_k} \in F_{n_k})$ is an arbitrary convergent sequence, then $\lim_{n \to \infty} x_{n_k}$ is in $F$.

For more properties of the hit-or-miss topology, such as the relations to Vietoris topology and Hausdorff metric, we refer to the references cited in Nguyen et al. [33], Wang and Wei [52], Wang et al. [54, 55], Wei and Wang [57, 58], and Wei et al. [60].

2.2 Choquet capacities

Choquet introduced the concept of capacitability of sets [5].

Let $\mathcal{P}$ denote the class of all subsets of $E$. $\mathcal{N}$ denotes the set of all positive integers. A capacity is a mapping $T : \mathcal{P} \to [0, 1]$ satisfying the following three conditions [26]:

(i) If $A, B \in \mathcal{P}$ and $A \subseteq B$, then $T(A) \leq T(B)$;

(ii) For $A, A_n \in \mathcal{P}$ ($n \in \mathcal{N}$), if $A_n \uparrow A$ in $\mathcal{P}$ (i.e., $A_n \subseteq A_{n+1}$ for $n \in \mathcal{N}$ and $\bigcup_{n=1}^{\infty} A_n = A$), then $T(A_n) \uparrow T(A)$;

(iii) For $K, K_n \in \mathcal{K}$ ($n \in \mathcal{N}$), if $K_n \downarrow K$ in $\mathcal{K}$ (i.e., $K_{n+1} \subseteq K_n$ for $n \in \mathcal{N}$ and $\bigcap_{n=1}^{\infty} K_n = K$), then $T(K_n) \downarrow T(K)$.

A Choquet capacity $T$ requires an additional probability condition, alternating of infinite order which is described as follows.

Let $K, K_1, K_2, \ldots$ be compact sets of $E$, and let $\Delta_n(K; K_1, \ldots, K_n)$ be the probability for the RACS $X$ to hit $K_1, \ldots, K_n$ but miss $K$. Then all functions $\Delta_n$ defined below must be non-negative (see [26]).

\[
\Delta_1(K; K_1) = T(K \cup K_1) - T(K) \\
\Delta_2(K; K_1, K_2) = \Delta_1(K; K_1) - \Delta_1(K \cup K_2; K_1)
\]

\[
\Delta_n(K; K_1, \ldots, K_n) = \Delta_{n-1}(K; K_1, \ldots, K_{n-1}) - \Delta_{n-1}(K \cup K_n; K_1, \ldots, K_{n-1})
\]
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Since

$$\Delta_n(K;K_1,\ldots,K_n) = \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} T(K \cup \bigcup_{i \in I} K_i) - T(K),$$

(4) is equivalent to the following condition: $T$ is monotone increasing on $\mathcal{K}$ and the inequality

$$T\left(\bigcap_{k=1}^{n} K_k\right) \leq \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} T\left(\bigcup_{i \in I} K_i\right)$$

holds for all $n \geq 2$ ([30], p. 117). Notice that for a probability measure, the inequality (6) becomes an equality (Poincaré's equality, see [30], p. 12).

Let $T$ be defined on $\mathcal{K}$, satisfy Conditions (i)-(iii), and be alternating of infinite order on $\mathcal{K}$. When $E$ is LCS metrizable, $T$ can be extended to $\mathcal{P}$ by first to any open subset $G$ and then to arbitrary subset $A$ of $E$ (refer to Definition 2.1).

**Definition 2.1**

Given a Choquet capacity $T$ on a space $E$ (so $T(K)$ is well-defined for all compact subsets $K$ of $E$), the interior capacity of any set $A$ of $E$ is defined as $T^*(A) = \sup\{T(K) \mid K \in \mathcal{K}, K \subseteq A\}$. In particular, for any open set $G$ of $E$, its interior capacity $T^*(G)$ has been defined. Then by using the defined interior capacities of open sets, the exterior capacity of $A$ is defined by $T^*(A) = \inf\{T^*(G) \mid G \in \mathcal{G}, G \supseteq A\}$.

$A$ is said to be capacitable if $T^*(A) = T^*(A)$.

Every probability measure $\mu$ on the Borel $\sigma$-algebra $\mathcal{B}$ generated by open sets of a given metric space $E$ is a special alternating Choquet capacity of infinite order (see e.g., [29]), and can be extended to a capacity by letting $T(A) = \inf\{\mu(B) \mid B \in \mathcal{B}(E), B \supseteq A\}$. Clearly, $T(\emptyset) = \mu(\emptyset) = 0$ and $0 \leq T \leq 1$.

Choquet’s main result in capacity theory is that for any LCS metrizable space $E$, every Borel set (and even every analytic set) is capacitable with respect to the Newtonian capacity, i.e., their inner and outer capacities coincide (see [5, 20]). Choquet’s definition of capacitability of a set is the equality of its inner and outer capacities, which bears a resemblance to the traditional measurability of a set as equality of its inner and outer measures. In the non-topological spaces, the Caratheodory measure and the relevant notion of measurability are also used. In such a case, a measure does not need to be a capacity. As consequences, theorems about capacities do not imply corresponding theorems about measurability, nor vice versa (see [43]).
2.3 Choquet theorem

Random closed sets serve as general mathematical models for set-valued observations and irregular geometrical patterns of physical phenomena, generalize the traditional concept of ordinary random variables, which are regarded as models for point-valued observations, and play an important role in stochastic geometry.

A RACS $X$ is an $A$-B($F(E)$) measurable function from a probability space $(\Omega, A, P)$ to the measurable space $(F(E), B(F(E)))$.

We will use the regular letter $P$ to represent a probability function, the italic $P$ to represent the induced probability measure ($P = PX^{-1}$), and $\mathbb{N}$ to denote the set of all non-negative integers.

The Choquet capacity $T$ of $X$ characterizes the probability law $P$ of $X$, and has been extensively investigated, see e.g., [5, 26-32, 46, 47, 56].

**Choquet Theorem:** (see [26]) Let $E$ be a LCS metric space and $F$ the space of all closed sets of $E$ equipped with the hit-or-miss topology. Then there exists a (necessarily unique) probability measure $P$ on the Borel $\sigma$-field $B(F)$ generated by the topology of $F$ satisfying $T(K) = P(X \cap K \neq \emptyset) = P(F_K)$ for $K \in \mathcal{K}$ if and only if $T$ is a Choquet capacity on $\mathcal{K}$ with $0 \leq T \leq 1$ and $T(\emptyset) = 0$.

Choquet Theorem is the foundation of random set theory. It reveals the relationship between the Choquet capacity $T$ of $X$ and the probability measure $P$ of $X$. In this theorem, the probability law $P$ describes the random evolution of $X$; the Choquet capacity $T$ ($T$ is subadditive only) plays the role of the distribution functions of ordinary random vectors; the hit-or-miss topology governs the convergence of closed sets, which is consistent with the standard convergence of closed sets (see [26, 59]). The theory of RACS originates from the hit-probability (miss-probability).

2.4 The Borel $\sigma$-algebra and the measure on the hyperspace $F$

Let $Y$ be a set. A class $\mathcal{S}$ of subsets of $Y$ is a semialgebra if 1) $\emptyset \in \mathcal{S}$, 2) $\mathcal{S}$ is closed for the finite intersection, and 3) the complement set of any $B \in \mathcal{S}$ is a finite union of disjoint sets in $\mathcal{S}$. A class $\mathcal{A}$ of subsets of $Y$ is an algebra if 1) and 2) above hold and 3) is replaced by: the complement set of any $B \in \mathcal{A}$ belongs to $\mathcal{A}$. $\mathcal{B}$ is a $\sigma$-algebra if a) $Y \in \mathcal{B}$, b) the complement set of any $B \in \mathcal{B}$ belongs to $\mathcal{B}$, and 3) if $B_n \in \mathcal{B}$, then their union $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$.

If $Y$ is actually a topological space, let $\mathcal{B}$ be the Borel $\sigma$-algebra generated by the open sets, and $\nu$ be a probability measure on $\mathcal{B}$. It is well-known that $f : Y \to Y$ preserves $\nu$ on all elements (measurable sets) of $\mathcal{B}$ if it preserves $\nu$ on a semialgebra that generates $\mathcal{B}$ (see e.g., [50, 62]).

We now begin the construction of the Borel $\sigma$-algebra in $F$ (see [26]).
Let $\mathcal{V}$ be the class of all sets of the form $V = G \cup K$, $G \in \mathcal{G}$, $K \in \mathcal{K}$. Clearly, $\mathcal{V}$ contains $\emptyset$ and $\mathcal{V}$ is closed for finite union. Let $\mathcal{S}$ denote the class of sets that is generated in $\mathcal{P}(\mathcal{F})$ by the classes $\mathcal{F}_V$, $\mathcal{F}^V (V \in \mathcal{V})$ and is closed for the finite intersection.

By the construction, $\mathcal{S}$ is closed for finite intersection, and all members of $\mathcal{S}$ hold the following form:

$$\mathcal{F}_{V_1, V_2, \ldots, V_n}, \ n \geq 0, \ V, V_1, V_2, \ldots, V_n \in \mathcal{V}. \tag{7}$$

Moreover, $\emptyset = \mathcal{F}_\emptyset \in \mathcal{S}$, and for any element $S = \mathcal{F}_{V_1, V_2, \ldots, V_n} \in \mathcal{S}$, its complement $S^c$ can be written as a union of disjoint elements of $\mathcal{S}$ as follows:

$$S^c = \mathcal{F}_V \cup \mathcal{F}^{V \cup V_1} \cup \mathcal{F}^{V \cup V_2} \cup \ldots \cup \mathcal{F}^{V \cup V_n}, \tag{8}$$

Hence, $\mathcal{S}$ is a semialgebra.

**Remark 2.2** The $\sigma$-algebra generated by $\mathcal{S}$ is precisely the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{F})$ in $\mathcal{F}$. In fact, $\mathcal{S}$ contains all the elements in the subbase of the hit-or-miss topology, which is observed by choosing $G = \emptyset$ or $K = \emptyset$. On the other hand, each element in $\mathcal{S}$ is a Borel set in $\mathcal{F}$. To see this, it suffices to show that $\mathcal{F}_V$ and $\mathcal{F}^V (V \in \mathcal{V})$ are Borel sets in $\mathcal{F}$ since every member of $\mathcal{S}$ is a finite intersection of such sets. Now, $\mathcal{F}_V = \mathcal{F}_{G \cup K} = \mathcal{F}_G \cup \mathcal{F}_K$ is a Borel set because $\mathcal{F}_G$ belongs to the subbase and $\mathcal{F}_K = \bigcap_{n \in \mathbb{N}} F_{G_n}$ is the intersection of countably many elements of the subbase (here, $G_n$ is chosen as a decreasing fundamental open neighborhood system of $K$; see [26]); $\mathcal{F}^V = \mathcal{F}^{G \cup K}$ is the complement of $\mathcal{F}_V = \mathcal{F}_{G \cup K}$ in $\mathcal{F}$ and thus a Borel set.

The Choquet capacity $T$ and the hyperspace probability measure $P$ of a RACS $X$ hold the following relation, where $\Delta_n$'s are defined in terms of $T$ (see [3]):

$$P(\mathcal{F}_{V_1, V_2, \ldots, V_n}) = \Delta_n(V; V_1, V_2, \ldots, V_n), \tag{9}$$

where $\Delta_n(V; V_1, V_2, \ldots, V_n)$ is the probability for $X$ hitting all $V_i$'s but missing $V$. Here, the relation given in (9) defines a single valued function: If an element of $\mathcal{S}$ admits another representation $\mathcal{F}_{V_1', V_2', \ldots, V_n'}$, then it must hold $\Delta_n(V; V_1, V_2, \ldots, V_n) = \Delta_n(V'; V_1', V_2', \ldots, V_n')$, and $P$ is $\sigma$-additive (see [26]).

It is well-known that a finite additive non-negative real-valued function defined on a semialgebra can be uniquely extended to the algebra generated by the semialgebra, and a countably additive non-negative real-valued function defined on an algebra can be uniquely extended to the $\sigma$-algebra generated by the algebra.
2.5 Cumulative distribution functions as special Choquet capacities

In particular, any ordinary random variable taking point values in a metric space \( E \) can be viewed as a special RACS, a random singleton set, and hence it is completely characterized by its associated Choquet capacity. See e.g., [29, 59].

Specially, if \( E = \mathbb{R} \), any ordinary random variable \( \alpha : (\Omega, \mathcal{A}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) induces a RACS \( X : (\Omega, \mathcal{A}, P) \to (\mathcal{F}(\mathbb{R}), \mathcal{B}(\mathcal{F}(\mathbb{R}))) \) by \( X(\omega) = [\alpha(\omega), \infty) \). The relation between the distribution function \( F \) of \( \alpha \) and the Choquet capacity \( T \) of \( X \) is given by

\[
T([a, b]) = P(X \cap [a, b] \neq \emptyset) = P(\alpha \leq b) = F(b).
\]

If \( E = \mathbb{R}^n \) and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) is a random vector, by letting \( X(\omega) = \prod_{j=1}^n [\alpha_j(\omega), \infty) \), it holds that

\[
T(\prod_{j=1}^n [a_j, b_j]) = P(X \cap \prod_{j=1}^n [a_j, b_j] \neq \emptyset) = P(\alpha_1 \leq b_1, \alpha_2 \leq b_2, \ldots, \alpha_n \leq b_n) = F(b_1, b_2, \ldots, b_n).
\]

2.6 Differences between Choquet capacities and measures

Choquet capacities of RACS are often interpreted through non-additive measures, upper probabilities, belief functions (see [29, 30]), and a counterpart of distribution functions of ordinary (point-valued) random vectors [59].

Upper semicontinuity and continuity from above are principal mathematical concepts. For instance, cumulative distribution functions of ordinary point-valued random variables are upper semicontinuous (u.s.c.), and finite measures are continuous from above (e.g., above the empty set). In particular, for Choquet capacities on LCS metric spaces, these two concepts coincide on \( \mathcal{F} \) [60].

If \( X \) is a random compact set, its Choquet capacity \( T \) is u.s.c. on \( \mathcal{F} \); however, for a general RACS \( X \) defined on a LCS metric space \( E \), its Choquet capacity is not necessarily u.s.c. on \( \mathcal{F} \). On the other hand, upper semicontinuity of \( T \) is highly desired in probability theory [29, 37]. This controversial situation has been recently resolved within the probabilistic context: the induced RACS of \( X \) is defined on the same probability space of \( X \), holds the same probability law as \( X \), takes values in a space that is homemorphic to the space of \( X \), and in particular, the Choquet capacity of the induced RACS is always u.s.c. (see [60]).

It is known that if a probability function \( P \) satisfies the countable additivity, then \( P \) is continuous from above and from below; conversely, if a function \( P \)
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satisfies $P(A) \geq 0$ and $P(\Omega) = 1$, is finitely additive, and is continuous from above at the empty set $\emptyset$ (or continuous from below), then $P$ is countably additive. Of course, Choquet capacities as upper probabilities satisfy finite subadditivity [29, 59], but fail the finite additivity in general.

Generally, a capacity describes the ability of a body to hold an electrical charge or mass for maintaining a specified equilibrium potential. It is known that for any compact set $K \subseteq E$, there exists a unique admissible measure $\mu$ defined on $K$ such that the resulting potential is 1 almost everywhere in $K$, except at most at points of a set of exterior capacity 0. This $\mu$ is the equilibrium distribution and its potential is the equilibrium potential of $K$. The equilibrium distribution is the only admissible measure on $K$ whose total charge or mass is equal to $T(K)$ (see [5]). The mathematical formulation of this kind of physical phenomena is the Choquet capacity, which in fact holds a strong subadditivity:

$$T(K_1 \cup K_2) + T(K_1 \cap K_2) \leq T(K_1) + T(K_2).$$

3. Choquet-capacity preserving and Choquet-capacity ergodic systems: New concepts

3.1 Traditional measure preserving and measure ergodic systems

Let $(E, \mathcal{B}(E), \mu, f)$ be a measure-preserving system, where $E$ is a topological space, $\mathcal{B}(E)$ is the Borel $\sigma$-field over $E$, $\mu$ is a probability measure on $\mathcal{B}(E)$, and $f : E \to E$ is a measurable transformation preserving the measure $\mu$ (i.e., every measurable set $A \subseteq E$ satisfies $\mu(f^{-1}(A)) = \mu(A)$; also we say $\mu$ is invariant under $f$).

A measure-preserving system $(E, \mathcal{B}(E), \mu, f)$ (or simply $f$) is ergodic if the only measurable sets $A$ invariant under $f$ (i.e., $f^{-1}(A) = A$, or more general, the difference between $f^{-1}(A)$ and $A$ is at most a set of measure 0) have either measure 0 or measure 1.

Here, the condition $f^{-1}(A) = A$ implies $f(A) \subseteq A$ and $f(E \setminus A) \subseteq E \setminus A$, and hence the system is decomposed into two smaller systems $(A, \mathcal{B}(E) \mid_A, \mu \mid_A, f \mid_A)$ and $(E \setminus A, \mathcal{B}(E) \mid_{E \setminus A}, \mu \mid_{E \setminus A}, f \mid_{E \setminus A})$. If $\mu(A) > 0$ and $\mu(E \setminus A) > 0$, the original system is simplified by the decomposition. However, if $\mu(A) = 0$ or $\mu(E \setminus A) = 0$, the original system has not been simplified as a set of measure 0 can be ignored in measure theory [50]. In the latter case, the original system is indecomposable, and this leads to the concept of ergodic systems.

There are several equivalent characterizations of ergodicity, see e.g., Brucks and Bruin [3], Walters [50], Ye et al. [62]. One of them is: For any $B \in \mathcal{B}$ with $\mu(B) > 0$, $\mu(\bigcup_{n \in \mathbb{N}} f^{-n}(B)) = 1$; another is via invariant functions (see e.g., Brin and Stuck [2], Hasselbott and Katok [19], Pollicott and Yuri [35]).
3.2 From measure systems to capacity systems

Instead of the traditional setting of a probability measure space \((E, \mathcal{B}(E), \mu)\), we introduce the so-called Choquet capacity space \((E, \mathcal{C}(E), T)\), where \(E\) is a locally compact separable (LCS) metric space, \(\mathcal{C}(E)\) is the family of all \(T\)-capacitable sets of \(E\), and \(T\) is the Choquet capacity of a random closed set.

By analogy with the traditional \(\mu\)-measurable sets, the concept of \(T\)-capacitable sets was previously given (Definition 2.1). Similarly, by analogy with the usual \(\mu\)-measurable transformations, \(T\)-capacitable transformations on \(E\) can be introduced: \(f : E \to E\) is a \(T\)-capacitable transformation if \(f^{-1}(B)\) is \(T\)-capacitable whenever \(B\) is.

Let \(f : E \to E\) be a \(T\)-capacitable transformation.

Although \(\mathcal{C}(E)\) is \(T\)-dependent, it always contains all Borel sets of \(E\) for any Choquet capacity \(T\). Namely, Borel sets are always capacitable with respect to every Choquet capacity. As every probability measure \(\mu\) is a special Choquet capacity, this setting is more general than a probability measure space.

By putting all together, \((E, \mathcal{C}(E), T, f)\) will be called a Choquet capacity system.

\(\mu\)-measurable sets and \(\mu\)-measurable transformations are special \(T\)-capacitable sets and \(T\)-capacitable transformations, respectively. Hence, when \(T\) is actually a probability measure and \(f\) a measurable transformation, the Choquet capacity system \((E, \mathcal{C}(E), T, f)\) degenerates to the traditional probability measure system \((E, \mathcal{B}(E), \mu, f)\), where the latter is a fundamental structure of mathematics especially in dynamical systems and ergodic theory (see [8, 50]).

Like measure-ergodic systems, for a capacity-ergodic system \((E, \mathcal{C}, T, f)\), if \(B \in \mathcal{C}(E)\) is \(f\)-invariant, i.e., \(f^{-1}(B) = B\), then \(T(B) = 0\) or \(T(B) = 1\). The condition \(f^{-1}(B) = B\) implies \(f(B) \subseteq B\) and \(f(E \setminus B) \subseteq E \setminus B\). Hence, we can simplify the study to two smaller capacity systems: \((B, \mathcal{B}(E) |_{B}, T|_{B}, f|_{B})\) and \((E \setminus B, \mathcal{B}(E) |_{E \setminus B}, T|_{E \setminus B}, f|_{E \setminus B})\) (unless the original system is ergodic).

**Remark 3.1** Remarkably, for a capacity-ergodic system, the condition \(f^{-1}(B) = B\) also implies that \(\mathcal{F}_{B}\) and \(\mathcal{F}^{B}\) are invariant sets of \(2^{f}\), where \(2^{f} : \mathcal{F} \to \mathcal{F}\) defined by \(2^{f}(\mathcal{F}) = f(\mathcal{F})\) is the so-called induced hyperspace map (see [52-55]): \((2^{f})^{-1}(\mathcal{F}_{B}) = \mathcal{F}_{f^{-1}(B)} = \mathcal{F}_{B}\) and \((2^{f})^{-1}(\mathcal{F}^{B}) = \mathcal{F}^{f^{-1}(B)} = \mathcal{F}^{B}\). Moreover, it holds that \(P(\mathcal{F}_{B}) = T(B) = 0\) or \(1\), and \(P(\mathcal{F}^{B}) = 1 - P(\mathcal{F}_{B}) = 1\) or \(0\).

The intersection of a collection of invariant sets is again invariant, and so is the union. If \(B_{1}\) and \(B_{2}\) are \(f\)-invariant, then both \(\mathcal{F}_{B_{1}} \cap \mathcal{F}_{B_{2}}\) and \(\mathcal{F}_{B_{1} \cup B_{2}}\) are \(2^{f}\)-invariant. Recall that \(\mathcal{F}_{B_{1}} \cap \mathcal{F}_{B_{2}} \supseteq \mathcal{F}_{B_{1} \cup B_{2}}\), but the converse inclusion does not hold because a closed set can hit both \(B_{1}\) and \(B_{2}\) without hitting their intersection (the intersection may even be empty).

For the first study on the ergodicity of the hyperspace measure system \((\mathcal{F}, \mathcal{B}(E), P, 2^{f})\) induced by the Choquet-capacity system \((E, \mathcal{C}(E), T, f)\), please refer to our recent work [61].
4. Constructing examples

4.1 Special case: Measure preserving and measure ergodic systems

This first example is a measure preserving and ergodic system, along with its induced hyperspace system that is also measure preserving and ergodic (in Wei, Wang and Li [61], the equivalence regarding ergodicity between a system and its induced hyperspace system is provided).

Example 4.1 Let \( \mathbb{Z} \) be the set of all integers equipped with the discrete topology and \( \mathbb{Z}_\infty = \mathbb{Z} \cup \{x_\infty\} \) be the one-point compactification of \( \mathbb{Z} \). Every finite subset and every (infinite) subset containing \( x_\infty \) are closed sets of \( \mathbb{Z}_\infty \). The measure equipped on \( \mathbb{Z}_\infty \) is defined by \( \mu(\{x_\infty\}) = 1 \) and \( \mu(\mathbb{Z}) = 0 \).

Let \( f : \mathbb{Z} \to \mathbb{Z} \) be the translation defined by \( f(x) = x + 1 \). Extend \( f \) to \( f : \mathbb{Z}_\infty \to \mathbb{Z}_\infty \) by letting \( f(x) = x_\infty \), \( x \in \mathbb{Z} \), and \( f(x_\infty) = x_\infty \). \( f \) is a continuous, injective and surjective mapping, and it preserves \( \mu \).

\( \{x_\infty\}, \mathbb{Z} \) and \( \mathbb{Z}_\infty \) are the only (non-empty) invariant sets of \( f \). Notice that \( \mathbb{Z} \) is an open invariant set of \( \mathbb{Z}_\infty \), but not a closed one. Under this invariant measure, the measures of \( \{x_\infty\}, \mathbb{Z} \) and \( \mathbb{Z}_\infty \) are 1, 0 and 1 respectively, and hence \( f \) is ergodic with respect to \( \mu \).

For the induced hyperspace measure system \( (\mathcal{F}(\mathbb{Z}_\infty), \mathcal{B}, P, 2f) \), it can be proved that \( 2f \) preserves \( P \) and \( 2f \) is \( P \)-ergodic. In fact, in the case of additive measures, the induced hyperspace measure system is trivial (from the view of probability) since \( P(\mathcal{F}_1) = 1 \), where \( \mathcal{F}_1 \) is the subspace of \( \mathcal{F} \) that consists of all the singleton sets, and the restriction of \( 2f \) on \( \mathcal{F}_1 \) is essentially identical to \( f \) (topologically conjugate and measure-preserving under the conjugation).

Ergodic transformations capture a common phenomenon in statistical physics. Let \( E \) represent a closed box containing a finite number of point gases/masses such as molecules that are interacted each other, with elastic boundary condition. Relevant differential equations describing the dynamics of such a system are Hamiltonian [12]. Suppose that \( E \) is a finite set of positions that the particles fill at any time, \( \mu \) is the counting measure (or Liouville measure [12, 63]) on \( E \), and \( f(x) \) is the position of the particle \( x \) after one unit of time.

Then, the ergodicity of \( f \) on \( \mu \) means that for any subset \( A \) of \( E \) with \( 0 < \mu(A) < 1 \), it holds that \( \mu(f(A) \cap A^c) > 0 \) (i.e., any part of the gas which is neither empty nor the whole recipient is mixed with its complement during one unit of time).

Example 4.2 Rotation of circle with respect to Haar measure. Consider the translation of the unit circle: \( f_\lambda : S^1 \to S^1 \) where \( \lambda \) is a real number and \( f_\lambda(e^{i\theta}) = e^{i(\theta + 2\pi \lambda)} \), \( 0 \leq \theta < 2\pi \). The measure on \( S^1 \) is the Haar measure \( \mu \), which is \( f_\lambda \)-invariant. Here the Haar measure can be interpreted as the probability measure of the continuous uniform variable on the circle.

If \( \lambda \) is an irrational number, then the two-sided orbit of every point \( x = e^{i\theta} \), i.e., \( orb(x) = \{e^{i(\theta + 2\pi z)} \mid z \in \mathbb{Z}\} \), is dense in \( S^1 \) and is \( f_\lambda \) invariant. Hence, the only non-empty
closed invariant set is $S^i$ itself. It is known that every measurable invariant set, closed or not, holds measure 0 or 1, and hence the measure system $(S^i, B, \mu, f_i)$ is ergodic (see [3, 50, 62]).

On the other hand, when $\lambda$ is a rational number, the original dynamical system $(S^i, f_i)$ is not ergodic (respectively, not transitive). In fact, let $\lambda = \frac{p}{q}$ be any rational number, in the irreducible form. For each $x = e^{i\theta} \in S^i$, we have $f^q_i(x) = e^{i(q+2\pi)p} = x$. Namely, every $x \in S^i$ is a periodic point with a period $q$.

Again, for the induced hyperspace measure system $(\mathcal{F}(S^i), B, P, 2^\lambda)$ where $\lambda$ is irrational, it can be proved that $2^\lambda$ preserves $P$ and $2^\lambda$ is $P$-ergodic.

4.2 General case: Capacity preserving and capacity ergodic systems

Let begin with Choquet’s construction of capacities.

Example 4.3 (Choquet) Let $\varphi : E \to [0, 1]$ be usc, where $E = \mathbb{R}$. Define

$$T(\emptyset) = 0, \ T(K) = \max\{\varphi(x) \mid x \in K\}, \ \emptyset \neq K \in \mathcal{K}. \quad (10)$$

Such a $T$ is a (non-additive) Choquet capacity, and we will actually find out its associated RACS $X$ below. Let $\alpha$ be the continuous uniform variable on $[0, 1]$, with the corresponding probability space $(\Omega, \Sigma, P)$. Define

$$X(\omega) = \{x \in E \mid \varphi(x) \geq \alpha(\omega)\}, \ \omega \in \Omega. \quad (11)$$

We claim that $X$ is the RACS whose Choquet capacity is the above $T$. To verify, it suffices to show the following equality:

$$P(X \cap K \neq \emptyset) = T(K), \ K \in \mathcal{K}, \quad (12)$$

which is proved as follows (see [56]):

$$P(X \cap K \neq \emptyset) = P(\{\omega \in \Omega \mid X(\omega) \cap K \neq \emptyset\})$$

$$= P(\{\omega \in \Omega \mid \exists x \in K, \varphi(x) \geq \alpha(\omega)\})$$

$$= P(\{\omega \in \Omega \mid \max\{\varphi(x) \mid x \in K\} \geq \alpha(\omega)\})$$

$$= P(\{\omega \in \Omega \mid \alpha(\omega) \leq T(K)\})$$

$$= T(K). \quad \text{(since } \alpha \sim \text{Unif}[0, 1])$$
Now, we present our constructions of capacity preserving and capacity ergodic systems.

**Example 4.4** Let \( f_l : [0, 1) \rightarrow [0, 1) \) be the translation defined by \( f_l(x) = x + \lambda \pmod{1} \), where \( 0 < \lambda < 1 \). Let both \( \alpha_1 \) and \( \alpha_2 \) be the uniform variable on \([0, 1)\), independent each other. Define \( \alpha_{\min} = \min\{\alpha_1, \alpha_2\} \) and \( \alpha_{\max} = \max\{\alpha_1, \alpha_2\} \). Put \( X = [\alpha_{\min}, \alpha_{\max}] \). This \( X \) is a RACS (a random interval). The cumulative distribution functions of \( \alpha_{\min} \) and \( \alpha_{\max} \) are respectively

\[
F_{\min}(x) = P(\alpha_{\min} \leq x) = \begin{cases} 0, & \text{if } x < 0 \\ 2x - x^2, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x \geq 1 \end{cases}
\]

and

\[
F_{\max}(x) = P(\alpha_{\max} \leq x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x \geq 1 \end{cases}
\]

Now, let \( T \) be the associated Choquet capacity of \( X \). Then for any interval \([a, b] \subseteq [0, 1)\), it holds that

\[
T([a, b]) = P(X \cap [a, b] \neq \emptyset) = 1 - P(X \cap [a, b] = \emptyset) = 1 - [P(\alpha_{\max} < a) + P(\alpha_{\min} > b) - P(\alpha_{\max} < a \text{ and } \alpha_{\min} > b)]
\]

\[
= 1 - [P(\alpha_{\max} < a) + P(\alpha_{\min} > b) - 0] = 1 - [a^2 + (1 - (2b - b^2)) - 0] = 2b - a^2 - b^2.
\]

Particularly, \( T([0]) = 0 \), \( T([1]) = 0 \) (i.e., \( T([a]) \rightarrow 0 \) as \( a \rightarrow 1 \)), \( T([\frac{1}{2}]) = \frac{1}{2} \) and \( T([0, 1)) = 1 \). \( T \) is not translation invariant as the capacity around \( \frac{1}{2} \) is larger than the capacity around 0 or 1, which is also observed from the following: \( T([0, \frac{1}{2}]) = \frac{5}{9} \) but \( T([\frac{1}{3}, \frac{2}{3}]) = 2 \cdot (\frac{2}{3})^2 - (\frac{2}{3})^2 = \frac{2}{9} \).

\( T \) is non-additive, for \( T([0, \frac{1}{3}]) + T([\frac{2}{3}, 1]) = 2 \cdot (\frac{1}{3})^2 - (\frac{1}{3})^2 + 2(1 - (\frac{2}{3})^2 - 1^2) = \frac{8}{3} + \frac{5}{9} = \frac{10}{3} > 1 \) and (of course) \( T([0, \frac{1}{2}] \cup [\frac{2}{3}, 1]) \leq 1 \). A further calculation shows that \( T([0, \frac{1}{3}] \cup [\frac{2}{3}, 1]) = 1 - P(X \subseteq (\frac{1}{3}, \frac{2}{3})) = 1 - P(\alpha_{\min} > \frac{1}{3} \text{ and } \alpha_{\max} < \frac{2}{3}) = 1 - P(\alpha_{\min} > \frac{1}{3}) P(\alpha_{\max} < \frac{2}{3}) \alpha_{\min} = \frac{1}{3} - (1 - \frac{1}{3}) \cdot \frac{2}{3} = \frac{5}{9} \).

**Example 4.5** Rotation of circle with respect to Choquet capacity. Let \( f_\lambda : [0, 1) \rightarrow [0, 1) \) be the translation defined by \( f_\lambda(x) = x + \lambda \pmod{1} \), where \( 0 < \lambda < 1 \). Alternatively, the space \([0, 1)\) and the transformation \( f_\lambda \) here can be regarded as the
circle and the rotation of the circle. Namely, the space \([0, 1)\) represents the quotient space \(\mathbb{R}/\mathbb{Z}\) (so 1 and 0 are identified as a same point). Now, any pair of random points on the circle form a RACS — the shorter closed arc between the two points. More precisely, let both \(\alpha_1\) and \(\alpha_2\) be the continuous uniform variable on \([0, 1)\), independent each other, with the probability space \((\Omega, \Sigma, P)\). Now, we define a RACS \(X\) as the shorter closed arc between \(\alpha_1\) and \(\alpha_2\) over the same probability space as follows:

\[
X(\omega) = \begin{cases} 
[\alpha_1(\omega), \alpha_2(\omega)], & \text{if } 0 < \alpha_2(\omega) - \alpha_1(\omega) \leq \frac{1}{2} \\
[\alpha_2(\omega), 1) \cup [0, \alpha_1(\omega)], & \text{if } \alpha_2(\omega) - \alpha_1(\omega) > \frac{1}{2} \\
\{\alpha_1(\omega)\} \cup \{\alpha_2(\omega)\}, & \text{if } \alpha_2(\omega) = \alpha_1(\omega) \\
[\alpha_2(\omega), \alpha_1(\omega)], & \text{if } 0 < \alpha_1(\omega) - \alpha_2(\omega) \leq \frac{1}{2} \\
[\alpha_1(\omega), 1) \cup [0, \alpha_2(\omega)], & \text{if } \alpha_1(\omega) - \alpha_2(\omega) > \frac{1}{2}.
\end{cases}
\]

(Notice that \(P(\alpha_1 = \alpha_2) = 0\) and \(P(\alpha_2 - \alpha_1 = \frac{1}{2}) = 0\).)

Let \(T\) be the associated Choquet capacity of \(X\). For \([a, b]\) \(\subseteq [0, 1)\), the RACS \(X\) hits \([a, b]\) in three possibilities, as described in the following (Recall \([0, 1)\) represents the circle).

(i) Both \(\alpha_1\) and \(\alpha_2\) enter \([a, b]\). As \(\alpha_1\) and \(\alpha_2\) are independent, it holds that \(P(\alpha_1 \in [a, b] \text{ and } \alpha_2 \in [a, b]) = P(\alpha_1 \in [a, b]) \cdot P(\alpha_2 \in [a, b]) = (b - a)^2\).

(ii) One of \(\alpha_1\) and \(\alpha_2\) enters \([a, b]\) and the other does not. The relevant probability is \(P(\alpha_1 \in [a, b] \text{ and } \alpha_2 \not\in [a, b]) + P(\alpha_1 \not\in [a, b] \text{ and } \alpha_2 \in [a, b]) = P(\alpha_1 \in [a, b]) \cdot P(\alpha_2 \not\in [a, b]) + P(\alpha_1 \not\in [a, b]) \cdot P(\alpha_2 \in [a, b]) = 2 \cdot (b - a)(1 - (b - a))\).

(iii) Neither \(\alpha_1\) nor \(\alpha_2\) enters \([a, b]\). Consider two cases: \(b - a \geq \frac{1}{2}\) and \(b - a < \frac{1}{2}\). In the former case, \(X\) does not hit \([a, b]\) since \(X\) is the shorter closed arc that lies outside \([a, b]\). For the latter case, we have the following calculation for the probability of \(X\) hitting \([a, b]\). First, observe that if \(\alpha_1\) takes a point in the interval \((b - \frac{1}{2}, a)\) and \(x\) denotes the distance from \(b - \frac{1}{2}\) to that point, then \(\alpha_2\) can assume values in the interval \((b - x, \frac{1}{2})\) and ensure the shorter arc between \(\alpha_1\) and \(\alpha_2\) to hit \([a, b]\). By letting this point change from \(b - \frac{1}{2}\) to \(a\) (so \(x\) changes from 0 to \(a - (b - \frac{1}{2})\)), such a probability is calculated as \(\int_0^{a-(b-\frac{1}{2})} x dx = \frac{1}{2} \left(\frac{3}{2} - (b - a)\right)^2\). Switching between \(\alpha_1\) and \(\alpha_2\), the desired probability is \(2 \cdot \frac{1}{2} \left(\frac{3}{2} - (b - a)\right)^2 = \left(\frac{3}{2} - (b - a)\right)^2\).

Hence, we have

\[
T([a, b]) = P(X \cap [a, b] \neq \emptyset) = \begin{cases} 
(b-a)^2 + 2(b-a)(1-(b-a)), & \text{if } b-a \geq \frac{1}{2}; \\
(b-a)^2 + 2(b-a)[1-(b-a)] + \left(\frac{1}{2} - (b-a)\right)^2, & \text{if } b-a < \frac{1}{2}.
\end{cases}
\]

(17)

\[
T([a, b]) = \begin{cases} 
(b-a)^2[2 - (b-a)], & \text{if } b-a \geq \frac{1}{2}; \\
(b-a)[2 - (b-a)] + \left(\frac{1}{2} - (b-a)\right)^2, & \text{if } b-a < \frac{1}{2}
\end{cases}
\]

(18)
where \((b - a)[2 - (b - a)] + (\frac{1}{4} - (b - a))^2 = (b - a) + \frac{1}{4}\).

Particularly, by the above formula, we have

\[
T([a, b]) = \begin{cases} 
0.25, & \text{if } b - a = 0 \text{ (i.e., } T(\{a\}) = 0.25) \\
0.50, & = 0.25 \\
0.75, & = 0.50 \\
0.9375, & = 0.75 \\
1.0, & = 1^{-} (\text{i.e., } T([a, b]) \to 1.0 \text{ as } b - a \to 1^{-}).
\end{cases} \tag{19}
\]

It can be calculated that the expected length of \(X\) is \(\frac{1}{2}\), and hence on average in four trials, one of the resulting arcs covers a given point, which explains \(T([a]) = 0.25\).

Let \(x_1, x_2, \ldots, x_{i_1}, x_{i_2}, \ldots, x_n\) be uniformly distributed on \([0, 1)\), e.g., \(0, \frac{1}{n}, \ldots, \frac{n-1}{n}\). Put \(F = \{x_1, x_2, \ldots, x_{i_1}, x_{i_2}, \ldots, x_n\}\). Notice that \(X \cap F = \emptyset\) if and only if the shorter closed arc between \(\alpha_1\) and \(\alpha_2\) is completely contained in an interval formed by two adjacent points, i.e., \((x_i, x_{i+1})\) for some \(i\) \((1 \leq i \leq n); (x_n, x_1)\) represents \((x_n, 1)\); for each of such intervals, the probability that both \(\alpha_1\) and \(\alpha_2\) belong to this interval is the product of \((x_{i+1} - x_i)\) with itself as \(\alpha_1\) and \(\alpha_2\) are independent.

Hence, it holds that

\[
P(X \cap F = \emptyset) = \sum_{i=1}^{n} (x_{i+1} - x_i)^2 = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 = \frac{1}{n}, \text{ and } P(X \cap F \neq \emptyset) = 1 - \frac{1}{n}, \tag{20}
\]

where the latter approaches 1.

\(T\) is non-additive because \(T([0, 0.75] \cup \{0.9\}) \leq 1.0\) but \(T([0, 0.75]) + T(\{0.9\}) = 0.9375 + 0.25 = 1.1875\).

Because the space is the circle, intervals \([a, b]\) where \(b < a\) can occur. In this case, we have \([a, b] = [a, 1) \cup [0, b]\) which has the length of \(1 - a + b\), or written as \(1 + (b - a)\) (which is less than 1 since \(b - a\) is negative).

Because \(\alpha_1\) and \(\alpha_2\) are continuous variables, the calculation formula for \(T([a, b])\) remains the same (when \(b > a\)).

A set \(B \subseteq [0, 1)\) is \(f_\lambda\)-invariant if and only if \(x \in B \iff x + \lambda \pmod{1} \in B\). The whole space \([0, 1)\) is the only non-empty invariant of the form of an interval. There are invariant sets of other forms, e.g., each (two sided) orbit is an invariant set.

Since \(T([a, b])\) is a function of \(b - a\), \(T\) is translation invariant for all intervals (open, closed, half open, singleton), e.g., \(T(f_\lambda^{-1}(a, b]) = T([a, b])\) and \(T(f_\lambda^{-1}(a, b)) = T([a, b])\).

More general, since \(X\) is a uniform variable, \(T\) is homogeneous on sets. In addition, because \(f_\lambda\) is a translation, \(f_\lambda\) preserves the Choquet capacity of sets. Particularly, for any compact set \(K \subseteq [0, 1)\), we have \(T(f_\lambda^{-1}(K)) = T(K), K \in \mathcal{K}\). Therefore, along with the extension of Choquet capacities from compact sets to open sets and then to arbitrary sets (see [26, 60]), any rotation \(f_\lambda\) preserves the Choquet capacity of any capacitable set.
Further, if \( \lambda \) is an irrational number, then the two-sided orbit of every point \( x \in [0, 1) \), i.e., \( \text{orb}(x) = \{ x + \lambda z \ (mod\ 1) \mid z \in \mathbb{Z} \} \), is dense in \([0, 1)\) and is \( f_\lambda \) invariant. Every capacitable invariant set, closed or not, holds capacity 1 except the empty set, and therefore the capacity system \(([0, 1), B, T, f_\lambda)\) is ergodic.

To investigate the ergodicity of the induced hyperspace measure system \((\mathcal{F}([0, 1)), B, P, 2^{f_\lambda})\), it requires more studies of the relation between \(2^{f_\lambda}\)-invariant sets and \(f_\lambda\)-invariant sets, as well as the relation between \(P\) and \(T\). This will be further investigated in a separate paper.

Choquet capacities are non-additive functions of special type: while these functions are non-additive, they correspond to the hyperspace probability measures which are additive. The examples constructed have demonstrated and restored the fundamental role of Choquet capacities in probability theory, engage the investigation of non-additive probability measures, are useful in investigating dynamical properties and ergodic theory of capacity systems, and hence enrich the dynamical theory and ergodic theory. The methods employed are powerful for constructing various types of capacity systems, including capacity systems on the torus. It should be emphasized that properties of capacity systems explored in Example 4.4 and Example 4.5 are remarkably different from the properties of the usual additive measures.

References


Methods for Constructing Choquet-Capacity Preserving and Ergodic Systems: Examples


