ON THE COHOMOLOGY RINGS OF SMALL CATEGORIES

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ABSTRACT. Let $\mathcal{C}$ be a small category. Its cohomology ring $H^*(\mathcal{C})$ is defined as the cohomology ring of the topological realization of its nerve. We produce a family of small categories, $\{\mathcal{C}_n\}_{n>1}$ where $n$ is the number of (non-isomorphic) objects, such that $H^*(\mathcal{C}_n)$ is infinitely generated for each $n > 1$. When $\mathcal{C}$ is a finite EI-category, we conjecture that the quotient of $H^*(\mathcal{C})$ by nilpotents is finitely generated. At present, whether the conjecture is true or false is not known. This paper attempts to establish some tools which may be useful in proving the conjecture. Especially, we study the relationship between the cohomology ring of a category and those of its subcategories and extensions.

KEYWORDS. Cohomology ring, finite generation, restriction, extension, generalized LHS spectral sequence.

1. INTRODUCTION

Let $\mathcal{C}$ be a small category and $\mathcal{A}b$ the category of abelian groups. We denote by $\mathcal{C}$-mod the abelian category of all covariant functors from $\mathcal{C}$ to $\mathcal{A}b$. The $n$th cohomology group of $\mathcal{C}$ with coefficients in a functor $F \in \mathcal{C}$-mod, $H^n(\mathcal{C}; F)$, can be defined as the $n$th higher inverse limit $\varprojlim F$. If $A$ is an abelian group and $A$ is the corresponding constant functor which sends every object to $A$ and every morphism to $\text{Id}_A$, then $H^n(\mathcal{C}; A) \cong H^n(|\mathcal{C}|, A)$, where $|\mathcal{C}|$ is the topological realization of $\mathcal{N}\mathcal{C}$—the nerve of $\mathcal{C}$. We are particularly interested in the case where $A$ is a commutative ring with identity, because then $H^*(\mathcal{C}; A) \cong H^*(|\mathcal{C}|, A)$ will become a graded commutative ring, which is called the cohomology ring of $\mathcal{C}$. To this end, we will focus on cohomology of $\mathcal{C}$ with coefficients in a functor $F : \mathcal{C} \to R$-mod for a commutative ring with identity $R$. Since $H^*(\mathcal{C}; R) \cong H^*(|\mathcal{C}|, R)$ is a generalization of the cohomology ring of a group, it is naturally called the cohomology ring of $\mathcal{C}$.

For any small category $\mathcal{C}$, one can define an associative algebra, called category algebra $RC$ of $\mathcal{C}$ (see [34]). It’s a natural generation of the group algebra of a group and the incidence algebra of a poset. If $\text{Ob}\mathcal{C}$ is finite, Mitchell [23] showed $RC$-mod $\simeq (R$-mod)$^\mathcal{C}$. Since all categories here in this paper are assumed to have finite object sets, without any confusion we can denote by $RC$-mod the category of all...
covariant functors from $\mathcal{C}$ to $R$-mod. Upon this equivalence of categories, $H^n(\mathcal{C}; F)$ can be identified with $\text{Ext}^n_{R\mathcal{C}}(R, F) := \text{Ext}^n_{R\mathcal{C}\text{-mod}}(R, F)$. In 2.1, we will give more information about cohomology of small categories and, more generally, Ext and Tor over category algebras.

When a finite group $G$ is regarded as a category with a single object, usually written as $\hat{G}$, Evens [9] and Venkov [31] proved that $H^*(\hat{G}; R) \cong H^*(G; R)$ is finitely generated when $R$ is Noetherian. This is one of the fundamental theorems in group cohomology theory, and is the start point of the theory of varieties for modules [5]. Other categories whose cohomology rings are finitely generated include finite posets and centric linking systems (when they exist) in the theory of $p$-local finite groups [6].

Motivated by these known results, one wants to know whether the finite generation property is true in greater generality. Since many categories considered in group representation and cohomology theory are the so-called EI-categories and are finite, it is natural to focus on this special class of finite categories. By definition, a category is finite if $\text{Mor} \mathcal{C}$ is a finite set and is EI if every endomorphism is an isomorphism. Given an EI-category, it’s easy to see that the isomorphism classes of objects in it possesses a natural poset structure. Hence a finite EI-category is an amalgam of a finite poset and several finite groups. More precisely, Słomińska [29] showed that $|\mathcal{C}|$ is homotopy equivalent to a homotopy colimit of some functor from a finite poset to the category of finite groupoids. Since cohomology rings of finite groups and finite posets are finitely generated, one wish to generalize the finite generation property of cohomology rings to finite EI-categories. The finite generation property of a cohomology ring would be useful in computing higher limits over a category. For example, in the theory of $p$-local finite groups [6], in order to know the existence and uniqueness of a centric linking system for a given fusion system $\mathcal{F}$ over a $p$-group $S$, one needs to calculate the higher limits of the central functor $\mathcal{Z}$ over a full subcategory $\mathcal{F}^c \subset \mathcal{F}$, which sends a subgroup of $S$ to its center. A characterization of $H^*(\mathcal{F}^c; \mathbb{Z})$, where $\mathbb{Z}$ is a ring of integers, would be useful in computing $\lim_{\mathcal{F}^c} \mathcal{Z}$ because the ring acts on it. The finite generation property of cohomology rings with coefficients in a field would also lead us to a theory of varieties for modules over category algebras. Since the category algebra $RC$ for a finite EI-category is not a Hopf algebra in general, the finite generation property doesn’t follow from the result of Friedlander-Suslin [10].

It turned out that the cohomology ring of a finite EI-category is not finitely generated in general. A family of finite EI-categories with infinitely generated cohomology rings are constructed and their cohomology rings are explicitly calculated. However, we conjecture that the cohomology ring of a finite EI-category modulo nilpotents is finitely generated, and the present paper is an attempt to find ways to prove the conjecture.

This paper is organized as follows. In Section 2, after a brief review of the cohomology theory of small categories, we give a family of finite categories whose cohomology rings are not finitely generated. Then in Section 3, we prove that for a given category there exist certain subcategories such that the cohomology ring of the category modulo some nilpotents is embedded into the cohomology ring of such a subcategory.
modu some nilpotents. This generalizes the well-known fact in group cohomology.

**Theorem A** Let $\mathcal{C}$ be a finite EI-category and $\mathcal{D}$ a subcategory. Suppose $p$ is a prime. If $\text{Ob} \mathcal{D} = \{x \in \text{Ob} \mathcal{C} : p | |\text{Aut}_\mathcal{C}(x)||\}$ and $\text{Aut}_\mathcal{D}(x)$ contains a Sylow $p$-subgroup of $\text{Aut}_\mathcal{C}(x)$, then the map induced by the restriction $H^*(\mathcal{C}; \mathbb{F}_p)/\mathcal{N}_\mathcal{C} \to H^*(\mathcal{D}; \mathbb{F}_p)/\mathcal{N}_\mathcal{D}$ is injective, where $\mathcal{N}_\mathcal{C}$ and $\mathcal{N}_\mathcal{D}$ are the ideals of nilpotents in $H^*(\mathcal{C}; \mathbb{F}_p)$ and $H^*(\mathcal{D}; \mathbb{F}_p)$, respectively.

We note that for any finite EI-category $\mathcal{C}$, a subcategory $\mathcal{D}$ satisfying the conditions in Theorem A always exists. At last in Section 4, we consider the extensions of small categories and deduce a generalized LHS spectral sequence. We introduce the concepts of opposite extensions and subextensions. Based on the spectral sequence, we can find connections between the cohomology ring of a category and those of its extensions.

**Theorem B** Suppose there is an extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$. If $\iota_\mathcal{D} : \mathcal{D} \hookrightarrow \mathcal{C}$ is the inclusion such that the undercategory $\iota_\mathcal{D}^{-1}[y]$ is contractible for every $y \in \text{Ob} \mathcal{C}$, then $H^*(\mathcal{E}; F) \cong H^*(\mathcal{E}_\mathcal{D}; F)$ for any contra-variant functor $F : \mathcal{E} \to R\text{-mod}$. Here $\mathcal{E}_\mathcal{D} \subset \mathcal{E}$ is the subextension corresponding to $\mathcal{D}$.

Suppose there is an opposite extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$. If $\iota_\mathcal{D} : \mathcal{D} \hookrightarrow \mathcal{C}$ is the inclusion such that overcategory $\iota_\mathcal{D}^{-1}[y]$ is contractible for every $y \in \text{Ob} \mathcal{C}$, then $H^*(\mathcal{E}; F) \cong H^*(\mathcal{E}_\mathcal{D}; F)$ for any covariant functor $F : \mathcal{E} \to R\text{-mod}$. Here $\mathcal{E}_\mathcal{D} \subset \mathcal{E}$ is the opposite subextension corresponding to $\mathcal{D}$.

**Conventions** We use curly letters, such as $\mathcal{C}$, to denote small categories in this paper. Symbols like $\mathcal{D}$ and $\mathcal{E}$ are used to denote subcategories and extensions of $\mathcal{C}$, respectively. When $G$ is a group, we use $\hat{G}$ to denote the corresponding category with a single object. For instance, let $x$ be an object of $\mathcal{C}$. We frequently refer to its automorphism group $\text{Aut}_\mathcal{C}(x)$ via the corresponding subcategory $\text{Aut}_\mathcal{C}(x) \subset \mathcal{C}$. For each $x \in \text{Ob} \mathcal{C}$, we denote by $[x] \subset \text{Ob} \mathcal{C}$ the set of all objects isomorphic to $x$ in $\mathcal{C}$. We also use $\hat{[x]} \subset \mathcal{C}$ to denote the groupoid which consists of all objects in $[x]$ and all isomorphisms among them. The symbol $\mathcal{A}$ is reserved for the subcategory of $\mathcal{C}$ which is the disjoint union of all these groupoids $\hat{[x]}$, $[x] \subset \text{Ob} \mathcal{C}$.

For each category $\mathcal{C}$, we use the corresponding blackboard bold letter $\mathbb{C}$ to denote the naturally constructed chain complex from its nerve. For simplicity, when $R$ is understood, sometimes we omit $R$ or $\mathcal{R}$ in the cohomology ring of a category $\mathcal{C}$ and will write it as $H^*(\mathcal{C})$ instead of $H^*(|\mathcal{C}|, R)$ or $H^*(\mathcal{C}; R)$. A category $\mathcal{C}$ is said to be contractible if $|\mathcal{C}|$ is.

We mainly focus on finite EI-categories in this paper. A category $\mathcal{C}$ is finite if $\text{Mor}(\mathcal{C})$ is finite. A category is EI if every endomorphism is an isomorphism. Given an EI-category $\mathcal{C}$, there is a natural poset structure defined on the set of isomorphism classes of objects in $\mathcal{C}$. For any $[x], [y] \subset \text{Ob} \mathcal{C}$, we define $[x] \leq [y]$ if $\text{Hom}_\mathcal{C}(x, y) \neq \emptyset$. The resulting poset is written as $|\mathcal{C}|$, and there is a canonical functor $\mathcal{C} \to |\mathcal{C}|$. 

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Based on the existence of the partial order on the set of isomorphism classes of objects, for any \( x \in \text{Ob} \mathcal{C} \), we can define a full subcategory \( \mathcal{C}_{\leq x} \subset \mathcal{C} \) such that \( \text{Ob} \mathcal{C}_{\leq x} = \{ y \in \text{Ob} \mathcal{C} \mid \text{Hom}_\mathcal{C}(y, x) \neq \emptyset \} \). We can similarly define full subcategories such as \( \mathcal{C}_{< x}, \mathcal{C}_{\geq x} \) and \( \mathcal{C}_{> x} \).

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## 2. Infinitely generated cohomology rings

We construct an infinite family of finite categories \( \mathcal{C}_n \) \((n \geq 2)\), where \( n \) represents the number of isomorphism classes of objects, and then prove their cohomology rings are not finitely generated.

### 2.1. Cohomology of small categories.

This section intends to review some standard materials that we need before computing the cohomology rings of categories \( \mathcal{C}_n \) constructed in 2.3. In the introduction we mentioned that for any small category \( \mathcal{C} \) one can define a category algebra \( R \mathcal{C} \), which is a free \( R \)-module with basis the set of morphisms in \( \mathcal{C} \) (see [34]). When \( \text{Ob} \mathcal{C} \) is finite, Mitchell [23] showed the category of left \( R \mathcal{C} \)-modules is equivalent to the category of covariant functors from \( \mathcal{C} \) to \( R \)-mod, i.e. \( R \mathcal{C} \)-mod \( \simeq (R \text{-mod})^\mathcal{C} \). The equivalence is given as follows. If \( F \in (R \text{-mod})^\mathcal{C} \) is a functor, then we define an \( R \mathcal{C} \)-module to be the \( R \)-module \( M_F = \bigoplus_{x \in \text{Ob} \mathcal{C}} F(x) \) equipped with natural actions by morphisms in \( \mathcal{C} \). Conversely if \( M \in R \mathcal{C} \)-mod, then we define a functor \( F_M \in (R \text{-mod})^\mathcal{C} \) such that \( F_M(x) = 1_x M \) for each \( x \in \text{Ob} \mathcal{C} \). It’s easy to verify that these two assignments are functors which are inverse to each other. Similarly, the category of right \( R \mathcal{C} \)-modules (same as \( R \mathcal{C}^{\text{op}} \text{-mod} \)) is equivalent to the category of contra-covariant functors from \( \mathcal{C} \) to \( R \)-mod.

These equivalences allow us to consider \( \text{Ext}_{R \mathcal{C}}^*(F_1, F_2) \) and \( \text{Tor}_{R \mathcal{C}}^*(M_1, M_2) \), where \( F_1, F_2 \) and \( M_2 \) are covariant functors and \( M_1 \) is a contra-variant functor (for further information, see tom Dieck [7], Lück [18], Webb [32] or Xu [34]). For the reader’s convenience, we point out that the tensor product of a contra-variant functor \( M \) and a covariant functor \( N \) as an \( R \)-module is the following \( M \otimes_{R \mathcal{C}} N \): \( \sum_{x \in \text{Ob} \mathcal{C}} M(x) \otimes_R N(x)/ \sim \), where the relation is given by \( m_x M(\alpha) \otimes_R N(\alpha)n_y = m_x \otimes_R N(\alpha)n_y \) for any \( m_x \in M(x), n_y \in N(y) \) and \( \alpha \in \text{Hom}_\mathcal{C}(y, x) \).

In order to compute \( H^*(\mathcal{C}; F) := \text{Ext}_{R \mathcal{C}}^*(R, F) \) and \( H_*(\mathcal{C}; F) := \text{Tor}_{R \mathcal{C}}^*(R, F) \), we need to construct a projective resolution of \( R \). Note that in Ext, both \( R \) and \( F \) are either covariant functors or contra-variant functors, while in Tor \( R \) is a contra-variant functor and \( F \) is a covariant functor. Since the constructions of the following well-known projective resolution of \( R \) as a left or a right \( R \mathcal{C} \)-module are similar, we’ll just describe it in the former situation. This canonical projective resolution (see Grodal [12]) can be defined using the so-called overcategories (see MacLane [24]) associated with the identity functor \( \text{Id}_\mathcal{C} : \mathcal{C} \to \mathcal{C} \). Recall that a functor \( \mu : \mathcal{D} \to \mathcal{C} \) can be used to produce an overcategory \( \mu \downarrow_x \) for each \( x \in \text{Ob} \mathcal{C} \). This overcategory consists of objects of the form \((y, \alpha)\), where \( y \in \text{Ob} \mathcal{D} \) and \( \alpha \in \text{Hom}_\mathcal{C}(\mu(y), x) \). A morphism
between two objects \((y, \alpha)\) and \((z, \beta)\) is given by some \(\gamma \in \text{Hom}_P(y, z)\) which makes the following diagram commutative:

\[
\begin{array}{ccc}
\mu(y) & \xrightarrow{\alpha} & x \\
\downarrow & & \downarrow \\
\mu(z) & \xleftarrow{\beta} & \mu(\gamma)
\end{array}
\]

Dually, one can define undercategories \(\mu \downarrow^x\) for each object \(x \in \text{Ob} C\). Now fix
an object \(x \in \text{Ob} C\), \(\text{Id}_C \downarrow^x\) is always contractible because it has a terminal object
\((x, 1_x)\), by Quillen's Theorem A. For each integer \(n \geq 0\) we can define a functor \(\text{Id}_C \downarrow^x_n : C \to \text{R-mod}\) sending \(x\) to \(\text{Id}_C \downarrow^x_n\) - the free \(\text{R}\)-module spanned over the set of
\(n\)-chains of morphisms in \(\text{Id}_C \downarrow^x\). There is a natural way to assemble these functors
into a sequence of functors, \(\text{Id}_C \downarrow^x_n : \text{R} \to 0\), which evaluated at each \(x \in \text{Ob} C\) is a
complex for computing the reduced homology of \(|\text{Id}_C \downarrow^x|\). Since every \(|\text{Id}_C \downarrow^x|\) is
contractible, the sequence is exact. In order to show it's a projective resolution of \(\text{R}\),
we need to prove each \(\text{Id}_C \downarrow^x_n\) is a projective object in \(\text{R}C\)-mod. This can be seen by
rewriting the sequence in a slightly different form: \(P_n : C \to \text{R}\) - mod is defined as follows (see Oliver [25]). The new form of this resolution is less
conceptual and is easier to use in practice. For any \(x \in \text{Ob} C\), \(P_n(x)\) is the free abelian
group with a basis the set of all sequences \([x_0 \to x_1 \to \cdots \to x_n]\) of morphisms
in \(\text{Mor}(C)\) ending in \(x\). For any morphism \(f \in \text{Hom}_C(x, y)\), \(P_n(f)\) is defined by its
action on base elements: \([x_0 \to \cdots \to x_n] \mapsto [x_0 \to \cdots \to x_n f\phi y]\). The boundary
map \(\sigma = \{\sigma_x\} : P_n \to P_{n-1}\) is given by setting
\[
\sigma_x([x_0 \to x_1 \to \cdots \to x_n]) = \sum_{i=0}^{n} (-1)^i [x_0 \to \cdots \to \hat{x}_i \to \cdots \to x_n \to x],
\]
on base elements. Since
\[
\text{Hom}_{\text{RC}}(P_n, F) \cong \prod_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(C)} F(x_n),
\]
where \(N(C)\) is the nerve of \(C\), we know \(\text{Hom}_{\text{RC}}(P_n, F)\) is exact and hence \(P_n\) is projective for any \(n \geq 0\). Thus \(H^*(C; F)\) are the homology groups of the co-chain complex
\(0 \to \text{Hom}_{\text{RC}}(P_*, F)\), or equivalently \(0 \to \text{Hom}_{\text{RC}}(\text{Id}_C \downarrow^x_n, F)\). When \(F = \text{A}\) is a constant
functor, one can easily see \(H^*(C; \text{A}) \cong H^*(|C|, \text{A})\), because \(0 \to \text{Hom}_{\text{RC}}(P_*, \text{A})\) can be identified with a co-chain complex used to compute \(H^*(|C|, \text{A})\).

2.2. **Two Torsion Groups.** Using \(\text{Tor}\), one can define and compute the homology
groups of \(C\) with coefficients in a functor \(F: H_*(C; F) \cong \lim_{\text{R}C}^{\ast} F \cong \text{Tor}_*^{\text{RC}}(\text{R}, F)\), where
\(\text{R}\) is considered as a contra-variant functor. As in cohomology, similarly we have
\(H_*(C; \text{A}) \cong H_*(|C|, \text{A})\) for any constant functor \(\text{A}\). One can use the same projective
resolution we described above to prove \(\text{Tor}_*^{\text{RC}}(\text{R}, \text{A}) \cong H_*(|C|, \text{A})\). Fortunately in the
rest of this paper, we only need to calculate torsions involving constant functors and atomic functors, so there is no need to recall the general theory of Ext and Tor.

A functor is atomic if it takes non-zero values at only one isomorphism class of objects in \( C \). For example, we can define an atomic bi-functor \( S_{x,R} : C \to R\text{-mod} \) such that \( S_{x,R}(y) = R \) if \( y \cong x \) and \( S_{x,R}(y) = 0 \) otherwise. It’s not hard to see using the projective resolution \( \mathcal{P}'_* \cong \text{Id}_C \lceil \cdot \rceil \) (see for example [11] Appendix II 3.2) of the contra-variant functor \( R^\alpha \); given by undercategories, one can show that \( H_n(C; S_{x,R}) \cong \text{Tor}^R_\alpha (R, S_{x,R}) \) can be calculated as homology groups of the chain complex given by

\[
\left\{ \bigoplus_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(C)} S_{x,R}(x_0) \right\} = \left\{ \bigoplus_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(C), \ x_0 \cong x} R \right\},
\]

where chain maps are induced by the face maps for \( N(C) \). We will encounter torsions like \( \text{Tor}^R_\alpha (S_{x,R}, R) \) as well in the next subsection. This time they are the homology groups of the chain complex given by

\[
\left\{ \bigoplus_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(C)} S_{x,R}(x_n) \right\} = \left\{ \bigoplus_{[x_0 \to x_1 \to \cdots \to x_n] \in N_n(C), \ x_n \cong x} R \right\},
\]

where chain maps are induced by the face maps for \( N(C) \).

2.3. Categories with infinitely generated cohomology rings. For each integer \( n > 1 \) we construct a category \( C_n \) which has exactly \( n \) objects and then examine the ring structure of \( H^*(C_n) := H^*(C_n; \mathbb{F}_2) \cong H^*(|C_n|, \mathbb{F}_2) \), where \( \mathbb{F}_2 \) is a field of characteristic 2. Let \( C_n, n > 1 \), be the following category

\[
\xymatrix{ *+<0mm,0mm>{} \ar[rr]^{1_{x_1}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_2}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_3}} & & \cdots \ar[rr]^{1_{x_n}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_1}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_2}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_3}} & & \cdots \ar[rr]^{1_{x_n}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_1}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_2}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_3}} & & \cdots \ar[rr]^{1_{x_n}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_1}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_2}} & & *+<0mm,0mm>{} \ar[rr]^{1_{x_3}} & & \cdots \ar[rr]^{1_{x_n}} & & *+<0mm,0mm>{} \}
\]

where \( \alpha_i \cdot g_i = \alpha_i = g_{i+1} \cdot \alpha_i \) and \( g_i^2 = 1_{x_i} \) for all \( i \). We’ll first calculate the homology of \( C_n \). Then the structure of \( H^*(C; \mathbb{F}_2) \) can be easily seen through the Universal Coefficient Theorem.

According to 2.2, one can just take the (normalized) chain complex associated to the nerve of \( C_n \) to compute its homology groups: \( \cdots \to C_k \to \cdots \to C_1 \to C_0 \to 0 \), where \( C_0 = \bigoplus_i \mathbb{F}_2 x_i \) and \( C_k, k \geq 1, \) is an \( \mathbb{F}_2\)-vectorspace spanned by chains of \( k \) consecutive morphisms in \( C_n \). For example in \( C_2 \) of \( C_2 \), the basis is \( \{(g_1, g_1), (g_1, \alpha_1), (\alpha_1, g_2), (g_2, g_2)\} \).

In order to proceed, we need some auxiliary constructions. Suppose \( C \) is a finite EI-category. For each \( x \in \text{Ob} C \), we’ll be interested in two full subcategories of \( C \) \( C_{<x} \subseteq C_{\leq x} \) and the associated complexes \( C_{<x} \subseteq C_{\leq x} \) and the corresponding quotient complex, written as \( \{ (\cdot, x) \}_k \) \( \text{Ob} C \to 0 \), which contains the linear combinations of chains ending at \( x \). If \( x \leq y \in \text{Ob} C \), then we also define \( C_{[x,y]} \) to be a full subcategory of \( C \) consisting of objects \( z \) such that both \( \text{Hom}_C(x, z) \) and \( \text{Hom}_C(z, y) \) are not empty. There are naturally defined full subcategories of \( C_{[x,y]}, C_{(x,y)}, C_{[x,y]} \) and \( C_{(x,y)} \). The
Proof. Suppose \( C \) is the normalized chain complex associated with the nerve of \( C \). Let \( \sigma \) be any chain in any \( C_i \). A significant property of \( \delta_i \) in this particular case is that all the non-zero summands of \( \delta_i(\sigma) \) contain the same isomorphisms as those in \( \sigma \) itself. For this reason, we can classify chains of morphisms in \( C \) by the (ordered) lists of isomorphisms in them.

We say two chains in \( C_i \) are of the same type if, after removing non-isomorphisms in them, the resulting ordered lists (possibly empty) of isomorphisms are the same. We’ll denote the type of a chain by \( T = \{t_1, t_2, \ldots, t_n\} \), where \( t_i \) is a non-negative integer and represents the number of \( g_i \) in the chain, for every \( i \). The length of \( T \) is \( |T| = t_1 + \cdots + t_n \). Due to our observation on \( \delta_i \), chains of morphisms of the same type \( T \) form a subcomplex of \( C \), denoted by \( C^T \), which is always finite because there are finitely many non-isomorphisms. Furthermore \( C_\ast = \bigoplus_T C_\ast^T \), where \( T \) runs over the (infinite) set of all possible types. In particular, the type \( \{0, \cdots, 0\} \) chains are chains of non-isomorphisms and if we demand \( C_0 \) to be of type \( \{0, \cdots, 0\} \) then they form a natural subcomplex of \( C \), which is the order complex for the underlying poset [\( C_n \)] of \( C_n \). In general, when a type \( T \) is given, all chains of type \( T \) have lengths greater than or equal to \( |T| \) (that is, \( C_\ast^T \) begins at degree \( |T| \)).

Now we start to calculate \( H_\ast(C_n) \). In order to compute the homology groups of \( C_\ast \), we just have to know how to do it for the subcomplex formed by chains of each type. Fix a type \( T \), it is not hard to see that \( C_\ast^T \) is isomorphic to the product of finitely many chain complexes of the forms \( [x_1, y], \langle x, y \rangle \) and \( \langle x, x_n \rangle \) coming from the underlying poset \( [C_n] \) of \( C_n \) (since isomorphisms have been excluded and there is only one morphism between any two non-isomorphic objects.). Hence one can use the Künneth Theorem to compute the homology of each \( C_\ast^T \) if the homology groups of the three explicitly constructed quotient complexes are known.

Next we re-interpret the homology groups of the following chain complexes \( [x, y], \langle x, y \rangle \) and \( \langle x, y \rangle \) for objects \( x, y \) belonging to a finite poset. Note that when \( x = y \), these complexes are trivial.

Proposition 2.3.1. Let \( C_n(n \geq 2) \) be the category defined above. Then

\[
H^i(C_n; \mathbb{F}_2) = \begin{cases} 
\mathbb{F}_2, & \text{if } i = 0; \\
0, & \text{if } 1 \leq i \leq 2n - 2; \\
\mathbb{F}_2^{m(i,n)}, & \text{if } i \geq 2n - 1.
\end{cases}
\]

Here \( m(i,n) \in \mathbb{Z} \) is the number of chains of \( i \) consecutive morphisms in which every non-identity morphism in \( C_n \) appears at least once.
Lemma 2.3.2. Let $\mathcal{C}$ be a finite poset. For any two objects $x < y$, the homology groups of $[x, y)$, $(x, y)$ and $(x, y]$ are isomorphic to $\text{Tor}^R_{*}(S_{y,R}, R)$, $\text{Tor}^R_{*}(S_{y,R}, S_{x,R})$ and $\text{Tor}^R_{*}(R, S_{x,R})$, respectively, where $\mathcal{D}$ is the subposet $\mathcal{C}$ consisting of all objects $z$ such that $x < z < y$.

The groups $\text{Tor}^R_{i}(R, S_{y,R})$ and $\text{Tor}^R_{i}(S_{x,R}, R)$ vanish unless $i = 0$, when they are equal to $R$. If there exists $x < z < y$ then $\text{Tor}^R_{i}(S_{y,R}, S_{x,R}) = 0$. Otherwise there is only one non-zero torsion $\text{Tor}^R_{i}(S_{y,R}, S_{x,R}) = R$.

Assuming the lemma is proved, we use it to calculate the $i$-th homology of $C_*$ when $i \geq 1$

\[
H_i(C_*) = \oplus T H_i(C_*^T).
\]

By Künneth Theorem, we have

\[
H_i(C_*^T) = \oplus_{i_1 + \cdots + i_k = i} H_{i_1}([\cdot, \cdot]) \otimes \cdots \otimes H_{i_k}(\langle \cdot, \cdot \rangle)
\]

where the dots represent appropriate objects in $\mathcal{C}_n$.

The second equality is true because $H_{i_1}([\cdot, \cdot])$ and $H_{i_k}(\langle \cdot, \cdot \rangle)$ are non-zero if and only if $i_1 = i_k = 0$, when they equal $R$ by the previous lemma. The same lemma also implies that $H_{i_2}(\langle \cdot, \cdot \rangle) \otimes \cdots \otimes H_{i_{k-1}}(\langle \cdot, \cdot \rangle)$ is non-zero if and only if $i_2 = \cdots = i_{k-1} = 1$ and each $\langle \cdot \rangle$ is some $\langle x, x_{x+1} \rangle$, or equivalently every isomorphism in $\text{Mor}(\mathcal{C})$ occurs at least once in $T$. When it’s non-zero, the tensor product must be $R$. Thus $H_i(C_*) = \oplus T H_i(C_*^T) = R^{m(i,n)}$, where $m(i,n)$ is the number of $i$-chains of all possible types in which every morphism in $\mathcal{C}_n$ appears at least once.

Proof of the lemma. The homology groups of $[x, y)$ are exactly $\text{Tor}^R_{*}(S_{y,R}, R)$ by definition of $[x, y)$ and discussion on $\text{Tor}^R_{*}(S_{y,R}, R)$ in 2.2. All positive degree torsions are zero because $R = R \text{Hom}(x, ?)$ is a representable functor. Similarly, we can prove $H_*([x, y)) \cong \text{Tor}^R_{*}(R, S_{x,R})$ and all positive degree torsions are zero because $R = R \text{Hom}([?, x]$ is a contra-variant representable functor. As for $(x, y)$, we note that $\text{Tor}^R_{*}(S_{y,R}, S_{x,R})$ can be computed through the long exact sequence

\[
\cdots \rightarrow \text{Tor}^R_{*}(S_{y,R}, R_{<z}) \rightarrow \text{Tor}^R_{*}(S_{y,R}, R) \rightarrow \text{Tor}^R_{*}(S_{y,R}, S_{x,R}) \rightarrow \cdots
\]

coming from the following short exact sequence of functors: $0 \rightarrow R_{<z} \rightarrow R \rightarrow S_{x,R} \rightarrow 0$. Here $R_{<z}$ is the maximal subfunctor of $R$ (note that if there is no $z$ such that $x < z < y$ then $R_{<z} \cong S_{y,R}$). Since $\text{Tor}^R_{*}(S_{y,R}, R) \cong H_*(\langle x, y \rangle)$, $\text{Tor}^R_{*}(S_{y,R}, R_{>z}) \cong H_*(\langle x, y \rangle)$ and $0 \rightarrow (x, y) \rightarrow [x, y] \rightarrow [x, y] \rightarrow 0$ also induces a long exact sequence, a Five Lemma argument shows that $\text{Tor}^R_{*}(S_{y,R}, S_{x,R}) \cong H_*(\langle x, y \rangle)$.

Finally from the long exact sequence for $\text{Tor}^R_{*}$ and known values for $\text{Tor}^R_{*}(S_{y,R}, R_{>z})$ and $\text{Tor}^R_{*}(S_{y,R}, R)$, we conclude that if there exists $z$ such that $x < z < y$ then $\text{Tor}^R_{*}(S_{y,R}, S_{x,R}) = 0$. Otherwise there is only one non-zero torsion $\text{Tor}^R_{i}(S_{y,R}, S_{x,R}) = R$.

Corollary 2.3.3. For each $n > 1$, the cohomology ring $H^*(\mathcal{C}_n; \mathbb{F}_2)$ is not finitely generated. \qed
Proof. Over the field $\mathbb{F}_2$ we always have $H^\ast(C_n) \cong \text{Hom}_{\mathbb{F}_2}(H_n(C_n), \mathbb{F}_2)$ in a natural way as vector spaces by the Universal Coefficient Theorem. It implies as a vector space $H^\ast(C_n)$ is infinite dimensional. It also means, for any $k \geq 0$, $H_k(C_n)$ is spanned by functions in $\text{Hom}_{\mathbb{F}_2}(C_k, \mathbb{F}_2)$ which are dual to the base elements for $H_k(C_n)$. From the description of $H_\ast(C_n)$ we know what the generators of $H^\ast(C_n)$ are and thus we can compute the cup product of any two of these functions, which is always zero. Hence the statement is proved. □

Note that $C_2$ is the smallest (in terms of number of objects and/or morphisms) non-contractible category that is neither a group nor a poset.

3. The cohomology ring modulo nilpotents

Although the cohomology ring of a finite EI-category is not finitely generated, based on known evidences it could be true that the cohomology ring modulo the ideal of all nilpotent elements is finitely generated, as happened to the above examples where in $H^\ast(C_n)$ all positive degree elements are nilpotent. We make the following conjecture.

**Conjecture** Given a finite EI-category $C$ and a Noetherian ring $R$, the cohomology ring $H^\ast(C; R)$ modulo nilpotents is finitely generated.

We would like to point out that in general there exist non-nilpotent elements of positive degree in the cohomology rings of EI-categories so the quotient by nilpotents won’t always be the base ring and our conjecture won’t be trivial. An example is the following category $C$:

![Diagram](https://example.com/diagram.png)

where $g^2 = k^2 = 1_x, gk = kg, \alpha g = \alpha, \alpha k = \beta, \beta k = \beta, \beta g = \beta k = \alpha, h^2 = 1_y$ and $h \beta = \alpha$. Using Quillen’s Theorem A we can show $C$ and $\hat{\text{Aut}}_C(y)$ are homotopy equivalent. Therefore the cohomology ring of $C$ has non-nilpotent elements of positive degree.

**Remark 3.0.4.** Let $\tau : C \to V$ be a functor from $C$ to a poset $V$. Slomińska [29] showed there exists “a decomposition of $|C|$ over $V$” in the sense that there is a homotopy equivalence

$$\text{hocolim}_{\text{sd}V}|\tilde{\tau}| \simeq |C|,$$

where $\text{sd}V$ is the subdivision of $V$ and $\tilde{\tau}$ is a functor from $\text{sd}V$ to the category of groupoids, induced by $\tau$. The Bousfield-Kan cohomology spectral sequence for homotopy colimits gives the following

$$H^i(\text{sd}V; H^j(\tilde{\tau})) \Rightarrow H^{i+j}(C).$$

However, it seems that a more elementary method is the better tool than the above spectral sequence for us to analyze the ring structure of $H^\ast(C)$. 
3.1. Comparing the cohomology of a category with those of its subcategories. Let \( \mathcal{C} \) be a small category and \( \mathcal{D} \) a subcategory. Then the inclusion \( i : \mathcal{D} \to \mathcal{C} \) naturally induces the restriction \( H^*(\mathcal{C}) \to H^*(\mathcal{D}) \). We want to compare the cohomology rings of \( \mathcal{C} \) and its various subcategories. In [6], [15], [12] and [34] people studied the case where \( \mathcal{D} \subset \mathcal{C} \) is a full subcategory which has less objects, and showed under certain assumptions one can have \( H^*(\mathcal{C}) \cong H^*(\mathcal{D}) \). Here we investigate subcategories \( \mathcal{D} \subset \mathcal{C} \) with the same set of objects but with less morphisms.

Let's consider the subcategory \( \mathcal{A} \subset \mathcal{C} \) consisting of all objects and exactly all isomorphisms in \( \mathcal{C} \). Then the chain complexes associated with the nerves of these two categories give a short exact sequence of complexes

\[
0 \to \mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{\pi} \mathcal{C}/\mathcal{A} \to 0,
\]

where \( i \) is the inclusion and \( \pi \) is the natural surjection. Note that for any abelian group \( A \), \( H^*(\mathcal{C}; A) = H^*(\mathcal{C}, A) \) and \( H^*(\mathcal{A}; A) = H^*(\mathcal{A}, A) \cong \bigoplus_{x \in \text{Ob}\mathcal{C}} H^*(\text{Aut}_\mathcal{C}(x), A) \), where \( [x] \) is the isomorphism class of an object \( x \in \text{Ob}\mathcal{C} \). Suppose \( \mathcal{D} \subset \mathcal{C} \) is a subcategory such that \( \text{Ob} \mathcal{D} = \text{Ob} \mathcal{C} \) and \( \mathcal{D} \) is the chain complex associated with the nerves of \( \mathcal{D} \). Let \( \mathcal{D} \cap \mathcal{A} \subset \mathcal{A} \) be the obvious subcategory consisting of objects and morphisms in both \( \mathcal{D} \) and \( \mathcal{A} \). Then we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & \mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{\pi} \mathcal{C}/\mathcal{A} \to 0 \\
& \searrow \ 
\downarrow \iota & \downarrow \iota & \downarrow \kappa \\
0 & \to & \mathcal{A}_\mathcal{D} \xrightarrow{i} \mathcal{D} \xrightarrow{\pi} \mathcal{D}/\mathcal{A}_\mathcal{D} \to 0,
\end{array}
\]

where \( \mathcal{A}_\mathcal{D} \) is \( \mathcal{A} \cap \mathcal{D} \), the chain complex of \( \mathcal{D} \cap \mathcal{A} \). Let \( A \) be an abelian group. This diagram gives rise to an infinite commutative diagram with a typical segment as follows

\[
\begin{array}{cccc}
H^{n-1}(A, A) & \to & H^n(\mathcal{C}/\mathcal{A}, A) & \to & H^n(\mathcal{C}, A) \xrightarrow{\iota^*} H^n(A, A) \xrightarrow{\iota^*} H^{n+1}(\mathcal{C}/\mathcal{A}, A) \\
\downarrow \iota^* & & \downarrow \kappa^* & & \downarrow \iota^* \\
H^{n-1}(\mathcal{A}_\mathcal{D}, A) & \to & H^n(\mathcal{D}/\mathcal{A}_\mathcal{D}, A) & \to & H^n(\mathcal{D}, A) \xrightarrow{\iota^*} H^n(\mathcal{A}_\mathcal{D}, A) \xrightarrow{\iota^*} H^{n+1}(\mathcal{D}/\mathcal{A}_\mathcal{D}, A),
\end{array}
\]

which induces a commutative diagram for each \( n \)

\[
\begin{array}{ccc}
H^n(\mathcal{C}; A)/I^n_\mathcal{C} & \xrightarrow{i'} & H^n(\mathcal{A}; A) \\
\downarrow \iota & & \downarrow \iota^* \\
H^n(\mathcal{D}; A)/I^n_\mathcal{D} & \xrightarrow{i'} & H^n(\mathcal{A}_\mathcal{D}; A),
\end{array}
\]

where \( i' \) is induced by \( \iota^* : H^n(\mathcal{C}; A) \to H^n(\mathcal{D}; A) \), \( I^n_\mathcal{C} = \pi^* H^n(\mathcal{C}/\mathcal{A}, A) \) and \( I^n_\mathcal{D} = \pi^* H^n(\mathcal{D}/\mathcal{A}_\mathcal{D}, A) \) satisfying \( i'(I^n_\mathcal{D}) = \pi^* \kappa^*(H^n(\mathcal{C}/\mathcal{A}, A)) \subset \pi^*(H^n(\mathcal{D}/\mathcal{A}_\mathcal{D}, A)) = I^n_\mathcal{D} \). By the exactness of the long exact sequences, both \( i' \) are injective.

Remark 3.1.1. When \( \mathcal{C} \) is a finite group, \( I^n_\mathcal{C} \) and \( I^n_\mathcal{D} \) vanish and the map \( i' \) coincides with the usual restriction map.
We show, when $A = R$ is a ring, $I_C = \oplus I^n_C$ and $I_D = \oplus I^n_D$ are nilpotent ideals in $H^*(C)$ and $H^*(D)$, respectively. Thus we have a commutative diagram of rings, not just of groups, with injective horizontal homomorphisms.

**Proposition 3.1.2.** Let $C$ be a finite category. The image of $H^*(C/\mathbb{A})$, denoted by $I_C = \oplus_{n\geq 1} I^n_C$, is an ideal of $H^*(C)$ consisting of nilpotents, and $i^*(H^*(C)) = i'(H^*(C)/I_C) \cong H^*(C)/I_C$ is a sub-algebra of $H^*(A)$.

**Proof.** The first observation comes from the long exact sequence induced by

$$0 \rightarrow A \xrightarrow{i} C \xrightarrow{\pi} C/\mathbb{A} \rightarrow 0,$$

$$\cdots \rightarrow H^n(C/\mathbb{A}) \rightarrow H^n(C) \xrightarrow{\pi^*} H^n(C) \xrightarrow{i^*} H^n(A) \rightarrow H^{n+1}(C/\mathbb{A}) \rightarrow \cdots.$$ 

It’s a canonical result that $i^*$ induces a ring homomorphism, still denoted by $i^*$. The image $I_C = \pi^*(H^*(C/\mathbb{A})) \subseteq \oplus_{n\geq 1} H^n(C)$ because $H^0(C/\mathbb{A}) = 0$. It implies $i^*(H^*(C))$ contains the identity in $H^*(A)$. Since $I_C$ is the kernel of the ring homomorphism $i^*$, it’s an ideal in $H^*(C)$. Furthermore elements of $I_C$ are nilpotent because the category $C$ is finite. If $k$ is the maximum length of chains of non-isomorphisms in $C$, then for any $f \in I_C$, we must have $f^{k+1} = 0$ by direct calculations using the definition of cup product.

Unfortunately a sub-algebra of a finitely generated algebra doesn’t have to be finitely generated so our conjecture is not proved yet.

In the commutative diagram we considered, we can artificially add an upwards map, which is the transfer, pictured as follows

$$\xymatrix{ H^*(C)/I_C \ar[r]^{\text{ind.}} \ar[d]_{?} & H^*(A) \ar[d]^{i^*} \\
H^*(D)/I_D \ar[r]_{\text{ind.}} & H^*(A_D). }$$

Although it’s unclear whether or not the transfer restricts to a well-defined map $H^*(D)/I_D \rightarrow H^*(C)/I_C$, the composite $\text{tr} \circ \alpha^* : H^*(A) \rightarrow H^*(A)$ is given by

$$\alpha = (\alpha_x)_{[x] \in \text{Ob} C} \mapsto \text{tr} \circ \alpha^*(\alpha) = (n_x \alpha_x)_{[x] \in \text{Ob} C},$$

by a standard result in group cohomology theory, where $\alpha = (\alpha_x)_{[x] \in \text{Ob} C}$ is an element of $H^*(A) = \oplus_{[x] \in \text{Ob} C} H^*(\text{Aut}_C(x))$ such that for each $[x] \subset \text{Ob} C \alpha_x \in H^*(\text{Aut}_C(x))$ and $n_x := |\text{Aut}_C(x) : \text{Aut}_D(x)|$.

**Corollary 3.1.3.** Let $n_C$ be the least common multiplier of the integers $\{|\text{Aut}_C(x)|\}$, where $x$ runs over the set of isomorphism classes of objects whose automorphism groups have order not invertible in $R$. Then, for any $i > 0$ and $\alpha \in H^i(C)/I_C$, $n_C \alpha = 0$.

**Remark 3.1.4.** If $A$ is an abelian group and $\underline{A}$ is the corresponding constant functor, usually $H^*(C; \underline{A})$ doesn’t have a ring structure. Hence the preceding proposition makes
no sense in this case. However, the restriction $H^\ast(C; A) \to H^\ast(A; A)$ is interesting in its own right.

Let $G$ be a finite group, $k$ an algebraic closed field of characteristic $p$ and $F$ a fusion system of a block of the group algebra $kG$, see Linckelmann [22]. Let $k^x$ be the constant functor from $F$ to $Ab$, sending every object in $F$ to the multiplicative group $k^\times$ of $k$. People are interested in the cohomology groups of the full subcategory $F^c \subset F$, consisting of $F$-centric subgroups, and its quotient category (orbit category of $F^c$) $\bar{F}^c$. In particular, one wants to understand the restrictions $H^i(F^c; \mathbb{Z}) \to H^i(A_{\bar{F}^c}; \mathbb{Z})$ and $H^2(F^c; k^x) \to H^2(A_{\bar{F}^c}; k^x)$, because they may provide important information for studying Alperin’s Conjecture in modular representation theory of finite groups, see Linckelmann [19] Section 4.1. There is a commutative diagram

$$
\begin{align*}
H^2(F^c; k^x) & \xrightarrow{\pi^*} H^2(A_{\bar{F}^c}; k^x) \\
\pi^* \downarrow & & \downarrow \pi^* \\
H^2(F^c; k^x) & \xrightarrow{\iota^*} H^2(A_{\bar{F}^c}; k^x)
\end{align*}
$$

where the maps are induced by the functors $\pi : F^c \to \bar{F}^c$, $\iota : A_{\bar{F}^c} \to F^c$ and $\iota : A_{\bar{F}^c} \to F^c$.

3.2. Restriction to subcategories with less isomorphisms. We assume $\text{Ob} D = \text{Ob} C$ and $D$ contains all non-isomorphisms in $\text{Mor}(C)$. We show there exist subcategories $D$ of $C$ such that $D$ shares the same objects and non-isomorphisms with $C$, while for every $x \in \text{Ob} C = \text{Ob} D$ the automorphism group $\text{Aut}_D(x)$ is a designated subgroup (up to isomorphism) of $\text{Aut}_C(x)$.

**Lemma 3.2.1.** Let $C$ be a finite EI-category. For each $[y] \subset \text{Ob} C$ we fix a representative $y \in [y]$ and choose a subgroup $H_x \subset \text{Aut}_C(x)$. Then there exists a subcategory $D$ such that $D$ contains all objects and non-isomorphisms in $C$ and such that $\text{Aut}_D(z)$ is isomorphic to $H_z$ if $z \cong x$.

Suppose for different choices of sets of representatives $x'$ and corresponding subgroups $H_{x'} \subset \text{Aut}_C(x')$, $x' \in [y] \subset \text{Ob} C$, there is a $D'$ with the same properties as $D$. Then $D' \cong D$, if $H_{x'} = gH_{x'}g^{-1}$ for some $g \in \text{Is}_C(x', x)$ whenever $[x] = [x']$.

**Proof.** Since we want to keep all objects and all non-isomorphisms, the key to find such a subcategory is to find a set of isomorphisms in $C$ such that they can form $\text{Mor}(D)$ along with the non-isomorphisms. In fact, we need to construct for each isomorphism class $[x] \subset \text{Ob} C$ a subgroupoid of $[x]$ whose skeleton is isomorphic to the chosen $\bar{H}_x$.

We proceed in two steps. First of all, we want to find a unique isomorphism $\alpha_{ij} \in \text{Hom}_C(x_i, x_j)$ for every pair of objects in $[x] = \{x_1, \cdots, x_i, \cdots, x_j, \cdots\}$ such that this set of isomorphisms is closed under composition, and such that $\alpha_{ii} = 1_{x_i}$ and $\alpha_{ij} = \alpha_{ji}^{-1}$. Let’s do this by induction on the number of objects in $[x]$. When $[x]$ only has one or two objects, the construction is trivial. When there are three objects in $[x]$, say $x_1, x_2, x_3$, we can choose arbitrary $\alpha_{12} \in \text{Hom}_C(x_1, x_2)$, $\alpha_{13} \in \text{Hom}_C(x_1, x_3)$
and then define \( \alpha_{23} = \alpha_{13}^{-1} \alpha_{12}^{-1} \). Suppose we have constructed such sets for all isomorphism classes with less than \( n \) objects. For an isomorphism class of objects in \( \mathcal{C} \) with \( n \) objects, we fix a set of compatible isomorphisms for the subgroupoid consisting of any \( n - 1 \) objects, say \( x_1, \cdots, x_{n-1} \). Then any choice of an isomorphism \( \alpha_{1,n} \in \text{Hom}_\mathcal{C}(x_1, x_n) \) can be used to get a compatible set of isomorphisms for \([x]\), by composing \( 1_{x_n}, \alpha_{1,n} \) and its inverse with existing isomorphisms among \( x_1, \cdots, x_{n-1} \).

Second of all, without loss of generality, for a fixed object \( x_1 = x \in [x] \), we let \( H_{x_1} = H_x \). Then for each \( x_i \in [x] \), we define \( H_{x_i} \) to be \( \alpha_{1i}H_{x_1}\alpha_{1i}^{-1} \). These groups of automorphisms, along with the compatible set of isomorphisms \( \{\alpha_{ij}\} \) define a subgroupoid in \([x]\) whose skeleton is isomorphic to \( \hat{H}_x \).

Let \( x \) run over the set of isomorphism classes of objects in \( \mathcal{C} \). Then we get a collection of isomorphisms from subgroupoids constructed above. These isomorphisms, together with all non-isomorphisms in \( \mathcal{C} \), form the morphism set of a subcategory \( \mathcal{D} \) satisfying our conditions.

Now let’s prove the second half of the lemma. We want to show with the given property there exists a \( g_y \in \text{Aut}_\mathcal{C}(y) \) for each \( y \in \text{Ob} \mathcal{C} \) such that \( g_y \text{Aut}_\mathcal{D}(y) = \text{Aut}_\mathcal{D}(y) \). If this is true, then we can define a functor \( \tau : \mathcal{D}' \to \mathcal{D} \) by setting \( \tau(y) = y \) for each object \( y \), and \( \tau(\alpha) = g_z^{-1}\alpha g_y \) if \( \alpha \in \text{Hom}_{\mathcal{D}'}(y, z) \). It is easy to verify that \( \tau \) is an isomorphism of categories.

Let \( \{x_i'\} \) be a different set of representatives of objects and \( \{H_{x_i}\} \) the corresponding chosen subgroups. We just proved there exists a \( \mathcal{D}' \) satisfying conditions in the first part of our statement. Without loss of generality we can assume \( \{x_i'\} = \{x_i\} \) and then use new symbols \( \{H_{x_i}'\} \), instead of \( \{H_{x_i}\} \), for the set of subgroups different from \( \{H_{x_i}\} \). Suppose \( H_x = g_x H_{x_i}' g_x^{-1} \) for some \( g_x \in \text{Aut}_\mathcal{C}(x) \), any chosen \( x \). We need to find such element for every object in \( \mathcal{C} \), not just the given representatives \( \{x_i\} \). Let’s take a compatible set of isomorphisms \( \{\alpha_{ij}'\} \) for each \( [x] \subset C \) and another compatible set \( \{\beta_{ij}'\} \) in \( \mathcal{D}' \). Then for any \( x_i, x_j \in [x] \), we must have \( \text{Aut}_{\mathcal{D}}(x_j) = \alpha_{ij}' \text{Aut}_{\mathcal{D}}(x_i) \alpha_{ji}'^{-1} \) and \( \text{Aut}_{\mathcal{D}'}(x_j) = \beta_{ij}' \text{Aut}_{\mathcal{D}'}(x_i) \beta_{ji}'^{-1} \). Assume \( x = x_1 \). Since \( \text{Aut}_{\mathcal{D}}(x) = H_x = g_x H_{x_i}' g_x^{-1} = g_x \text{Aut}_{\mathcal{D}'}(x) g_x^{-1} \), for each \( x_i \in [x] \) we can find \( g_{x_i} = \alpha_{1i}' g_x \beta_{1i}' \in \text{Aut}_\mathcal{C}(x_i) \) satisfying \( g_{x_i} \text{Aut}_{\mathcal{D}'}(x_i) g_{x_i}^{-1} = \text{Aut}_{\mathcal{D}}(x_i) \). Hence we’re done. \( \square \)

In particular, for a fixed prime \( p \) we can choose \( \text{Aut}_\mathcal{D}(x) \) to be a Sylow \( p \)-subgroup of \( \text{Aut}_\mathcal{C}(x) \), where \( x \) runs over the set of isomorphism classes in \( \text{Ob} \mathcal{C} \). We’ll use \( \mathcal{C}_p \) to denote each of such subcategories, because they are isomorphic to each other.

**Corollary 3.2.2.** Suppose \( p \) is a prime and \( \mathcal{C} \) is a finite EI-category. Then there exists a unique subcategory \( \mathcal{C}_p \) up to isomorphism such that \( \text{Ob} \mathcal{C}_p = \text{Ob} \mathcal{C} \), \( \mathcal{C}_p \) contains all non-isomorphisms in \( \mathcal{C} \) and for each object \( x \) \( \text{Aut}_{\mathcal{C}_p}(x) \) is a Sylow \( p \)-subgroup of \( \text{Aut}_\mathcal{C}(x) \).

One may call each \( \mathcal{C}_p \) a Sylow \( p \)-subcategory of \( \mathcal{C} \) since if \( \mathcal{D} \) is a subcategory such that it shares all objects and all non-isomorphisms with \( \mathcal{C} \) and such that \( \text{Aut}_\mathcal{D}(x) \) is a \( p \)-subgroup of \( \text{Aut}_\mathcal{C}(x) \), for each \( x \in \text{Ob} \mathcal{C} \), then \( \mathcal{D} \) is contained in a \( \mathcal{C}_p \). To some extent, \( \mathcal{C}_p \) plays the role of a Sylow \( p \)-subgroup of a finite group. If we consider the
chain complexes of categories $\mathcal{C}, \mathcal{A}, \mathcal{C}_p$ and $\mathcal{A}_p = \mathcal{C}_p \cap \mathcal{A}$, we’ll have

\[
\begin{array}{c}
0 \to \mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{\pi} \mathcal{C}/\mathcal{A} \to 0 \\
\downarrow{\text{Id}} \quad \downarrow{i} \quad \downarrow{\kappa} \\
0 \to \mathcal{A}_p \xrightarrow{i} \mathcal{C}_p \xrightarrow{\pi} \mathcal{C}_p/\mathcal{A}_p \to 0,
\end{array}
\]

and hence

\[
\begin{array}{c}
\mathbb{H}^*(\mathcal{C})/I_{\mathcal{C}} \xrightarrow{i'} \mathbb{H}^*(\mathcal{A}) \\
\downarrow{i'} \\
\mathbb{H}^*(\mathcal{C}_p)/I_{\mathcal{C}_p} \xrightarrow{i^*} \mathbb{H}^*(\mathcal{A}_p).
\end{array}
\]

The next result follows from the standard fact that the restriction $i^*: \mathbb{H}^*(\mathcal{A}; \mathbb{F}_p) \to \mathbb{H}^*(\mathcal{A}_p; \mathbb{F}_p)$ is injective.

**Proposition 3.2.3.** Let $\mathcal{C}$ be a finite EI-category and $p$ a prime. Then the map

\[
i': \mathbb{H}^*(\mathcal{C}; \mathbb{F}_p)/I_{\mathcal{C}} \to \mathbb{H}^*(\mathcal{C}_p; \mathbb{F}_p)/I_{\mathcal{C}_p}
\]

is an injection, where $I_{\mathcal{C}} = \pi^* \mathbb{H}^*(\mathcal{C}/\mathcal{A})$ and $I_{\mathcal{C}_p} = \pi^* \mathbb{H}^*(\mathcal{C}_p/\mathcal{A}_p)$ are two nilpotent ideals.

At present, it’s not clear that whether or not $i'$ is an isomorphism if the other restriction $i^*: \mathbb{H}^*(\mathcal{A}; \mathbb{F}_p) \to \mathbb{H}^*(\mathcal{A}_p; \mathbb{F}_p)$ is an isomorphism.

3.3. **Restriction to subcategories with less non-isomorphisms.** We assume $\text{Ob} \mathcal{D} = \text{Ob} \mathcal{C}$ and $\mathcal{D}$ and $\mathcal{C}$ have the same set of isomorphisms. Then the commutative diagram of short exact sequences reads as follows

\[
\begin{array}{c}
0 \to \mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{\pi} \mathcal{C}/\mathcal{A} \to 0 \\
\downarrow{\text{Id}} \quad \downarrow{i} \quad \downarrow{\kappa} \\
0 \to \mathcal{A}_p \xrightarrow{i} \mathcal{D} \xrightarrow{\pi} \mathcal{D}/\mathcal{A}_p \to 0.
\end{array}
\]

It induces a commutative diagram

\[
\begin{array}{c}
\mathbb{H}^*(\mathcal{C})/I_{\mathcal{C}} \xrightarrow{i'} \mathbb{H}^*(\mathcal{A}) \\
\downarrow{i'} \\
\mathbb{H}^*(\mathcal{D})/I_{\mathcal{D}} \xrightarrow{i^*} \mathbb{H}^*(\mathcal{A}_p),
\end{array}
\]

where $i'$ is induced by $i^*$, $i'$ is induced by $i^*: \mathbb{H}^*(\mathcal{C}) \to \mathbb{H}^*(\mathcal{D})$, $I_{\mathcal{C}} = \pi^* \mathbb{H}^*(\mathcal{C}/\mathcal{A})$ and $I_{\mathcal{D}} = \pi^* \mathbb{H}^*(\mathcal{D}/\mathcal{A}_p)$ are two nilpotent ideals satisfying $i^*(I_{\mathcal{C}}) = \pi^* \kappa^*(\mathbb{H}^*(\mathcal{C}/\mathcal{A})) \subset \pi^*(\mathbb{H}^*(\mathcal{D}/\mathcal{A}_p))$.

**Proposition 3.3.1.** Let $\mathcal{C}$ be a finite EI-category. Assume there exists a subcategory $\mathcal{D} \subset \mathcal{C}$ such that $\text{Ob} \mathcal{D} = \text{Ob} \mathcal{C}$ and $\text{Aut}_{p}(x) = \text{Aut}_{\mathcal{C}}(x)$ for each $x \in \text{Ob} \mathcal{C}$. Then $i': \mathbb{H}^*(\mathcal{C})/I_{\mathcal{C}} \to \mathbb{H}^*(\mathcal{D})/I_{\mathcal{D}}$ is injective for any base ring $R$. 

Proof. It follows directly from Proposition 3.1.2 and the commutative diagram above it. \(\square\)

As a simple example, one can verify that if the orders of all automorphism groups in \(\mathcal{C}\) are invertible in the base ring \(R\), then \(H^*(\mathcal{C})/I_{\mathcal{C}} = R\). If for some \(x \in \text{Ob}\mathcal{C}\), \(|\text{Aut}_\mathcal{C}(x)|^{-1} \in R\), then the isomorphism class \([x]\) can be “dropped” due to the following result. Let \(x \in \text{Ob}\mathcal{C}\). We define \(\mathcal{C}_x\) to be the full subcategory of \(\mathcal{C}\) consisting of all objects but those isomorphic to \(x\).

**Corollary 3.3.2.** Let \(\mathcal{C}\) be a finite EI-category. Suppose \(x \in \text{Ob}\mathcal{C}\) such that \(|\text{Aut}_\mathcal{C}(x)|\) is invertible in \(R\). Then \(H^*(\mathcal{C})/\mathcal{N}_\mathcal{C} \rightarrow H^*(\mathcal{C}_x)/\mathcal{N}_{\mathcal{C}_x}\) is injective.

**Proof.** The subcategory \(\mathcal{D} = \mathcal{C}_x \cup [x]\) (a disjoint union) obviously satisfies the condition in our previous proposition. Thus we have an injective map \(i' : H^*(\mathcal{C})/I_{\mathcal{C}} \rightarrow H^*(\mathcal{D})/I_{\mathcal{D}}.\) Since \(H^*(\mathcal{D}) = H^*(\mathcal{C}_x) \oplus H^*([x])\) and \(H^*([x])\) vanishes at any positive degree, \(pr \circ i' : H^*(\mathcal{C})/I_{\mathcal{C}} \rightarrow H^*(\mathcal{C}_x)/I_{\mathcal{D}}\) is an injective homomorphism, where \(pr\) is the natural projection \(H^*(\mathcal{D})/I_{\mathcal{D}} \rightarrow H^*(\mathcal{C}_x)/I_{\mathcal{D}}.\) Hence the statement follows. \(\square\)

Thus to some extent, when considering mod \(p\) cohomology one can focus on categories whose automorphism groups of objects are \(p\)-groups. Combining Propositions 3.2.3 and 3.3.1 and Corollary 3.3.2, we have the following main result.

**Theorem 3.3.3.** Let \(\mathcal{C}\) be a finite EI-category and \(\mathcal{D}\) a subcategory. Suppose \(p\) is a prime. If \(\text{Ob}\mathcal{D} = \{x \in \text{Ob}\mathcal{C} : p \mid |\text{Aut}_\mathcal{C}(x)|\}\) and \(\text{Aut}_\mathcal{D}(x)\) contains a Sylow \(p\)-subgroup of \(\text{Aut}_\mathcal{C}(x)\), then the map induced by the restriction \(H^*(\mathcal{C};\mathbb{F}_p)/\mathcal{N}_\mathcal{C} \rightarrow H^*(\mathcal{D};\mathbb{F}_p)/\mathcal{N}_\mathcal{D}\) is injective, where \(\mathcal{N}_\mathcal{C}\) and \(\mathcal{N}_\mathcal{D}\) are the ideals of nilpotents in \(H^*(\mathcal{C};\mathbb{F}_p)\) and \(H^*(\mathcal{D};\mathbb{F}_p)\), respectively.

A non trivial example can be found in the theory of \(p\)-local finite groups, see [2], [6] and [17] for a complete description of results stated in the next example.

**Example 3.3.4.** Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local finite group [6] with finite covering data (see [17]) \((\theta, \Gamma, H)\), where \(\Gamma\) is a \(p\)-group or a \(p'\)-group, \(\theta : S \rightarrow \Gamma\) is a group homomorphism and \(H \subset \Gamma\) is a subgroup. Define \(S_H = \theta^{-1}(H)\). Then there exists a fusion system \(\mathcal{F}_H\) over \(S_H\) ([2] Proposition 3.8) and a \(p\)-local finite group \((S_H, \mathcal{F}_H, \mathcal{L}_H)\) ([2] Theorem 3.9) such that \(|\mathcal{L}_H|\) is a covering space of \(|\mathcal{L}|\). Let \(\Gamma = H = 1\). Then \(S_1 = S\) and \(\mathcal{L}_1 \subset \mathcal{L}\) satisfies the condition that, for any \(P \in \text{Ob}\mathcal{L} = \text{Ob}\mathcal{L}_1, \text{Aut}_{\mathcal{L}_1}(P)\) contains a Sylow \(p\)-subgroup of \(\text{Aut}_{\mathcal{L}}(P)\) ([2] Proposition 3.8 c). By the preceding theorem, we get an injection \(H^*(\mathcal{L})/\mathcal{N}_{\mathcal{L}} \rightarrow H^*(\mathcal{L}_1)/\mathcal{N}_{\mathcal{L}_1}\).

On the other hand, Levi-Ragnarsson [17] showed that in this case one can construct a transfer map for \(p\)-local finite groups which composes with the restriction is the identity: \(H^*(\mathcal{L}) \rightarrow H^*(\mathcal{L}_1) \rightarrow H^*(\mathcal{L}).\) Especially, it implies \(\text{Res} : H^*(\mathcal{L}) \rightarrow H^*(\mathcal{L}_1)\) is injective hence another proof of the injectivity of the induced map \(H^*(\mathcal{L})/\mathcal{N}_{\mathcal{L}} \rightarrow H^*(\mathcal{L}_1)/\mathcal{N}_{\mathcal{L}_1}\).

We conclude this section with a couple of final remarks. Let \(\mathcal{C}\) be a small category. One can define the subdivision \(S(\mathcal{C})\) of \(\mathcal{C}\), which is homotopy equivalent to \(\mathcal{C}\).
(see Linckelmann [20]). The subdivision is a category with subobjects, so especially all morphisms are monomorphic. Thus when studying cohomology rings of small categories one can just focus on categories whose morphisms are monomorphic. We comment that in the theory of \( p \)-local finite groups [6], as well as in local representation theory [30], many categories have the property that either all morphisms are monomorphic or all of them are epimorphic.

A possible approach to proving our conjecture is to embed each finite EI-category \( C \) into a category \( \bar{C} \) that has a finitely generated cohomology ring, and then show \( H^*(C)/N_C \) is finitely generated over \( H^*(\bar{C})/N_{\bar{C}} \). According to above remark, one can assume all the morphisms in a finite EI-category are monomorphic. It seems that one can begin with finite concrete EI-categories whose morphisms are monomorphic. Under the assumption, there is a faithful functor from such a category to the category \( mSets \) of finite sets with monomorphisms. It would be useful to solve our problem if one could classify some or even all finite subcategories of \( mSets \) with finitely generated cohomology rings.

4. Extensions of categories and the generalized LHS spectral sequences

In last section we considered certain subcategories of a category \( C \) and the relationship between the cohomology rings of those subcategories and of \( C \). In this section, we do the “converse” and consider the cohomology rings of extensions of \( C \), which have the same sets of objects but have “bigger” morphism sets. At the beginning we go over some basic knowledge about extensions of categories. The reader is referred to Hoff [13] or Webb [32] for more information about extensions of categories.

Given a functor \( \mu : \mathcal{E} \to \mathcal{C} \), Gabriel and Zisman [11] showed there exists a spectral sequence converging to the homology of \( \mathcal{E} \). When the category \( \mathcal{E} \) is an extension of \( \mathcal{C} \), we deduce a spectral sequence, based on Gabriel and Zisman’s, as a generalization of the LHS spectral sequence for group extensions. It will be used to compare cohomology rings of \( \mathcal{E} \) and \( \mathcal{C} \).

4.1. Extensions of categories. An extension \( \mathcal{E} \) of a category \( \mathcal{C} \) via a category \( \mathcal{K} \), in the sense of Hoff [13], is a sequence of functors

\[
\mathcal{K} \xleftarrow{\iota} \mathcal{E} \xrightarrow{\pi} \mathcal{C},
\]

which has the following properties:

1. \( \text{Ob}\mathcal{K} = \text{Ob}\mathcal{E} = \text{Ob}\mathcal{C} \), \( \iota \) is injective and \( \pi \) is surjective on morphisms;
2. if \( \pi(\alpha) = \pi(\beta) \), for two morphisms \( \alpha, \beta \in \text{Mor}(\mathcal{E}) \), if and only if there is a unique \( g \in \text{Mor}(\mathcal{K}) \) such that \( \beta = \iota(g)\alpha \);
3. if \( \alpha\iota(h) \) exists for \( \alpha \in \text{Mor}(\mathcal{E}) \) and \( h \in \text{Mor}(\mathcal{K}) \), then there exists a unique \( h' \in \text{Mor}(\mathcal{K}) \) such that \( \iota(h')\alpha = \alpha\iota(h) \);
4. for any \( \alpha \in \text{Hom}_\mathcal{C}(x, y) \), \( \mathcal{K}(y) \) acts regularly on \( \pi^{-1}(\alpha) \).

Note that (3) and (4) can be deduced from (1) and (2) which are normally used to define an extension of a category. It’s known to Hoff [13] that \( \mathcal{K} \) is a disjoint union of the groups \( \pi^{-1}(1_x) \) for all \( 1_x \in \text{Mor}(\mathcal{C}) \) (regarded as categories), and can be identified
with a functor $\mathcal{K} : \mathcal{E} \to \text{Groups}$. Usually from the context, one can easily see when we take $\mathcal{K}$ to be a category and when it is regarded as a functor.

An extension is \textit{split} if it admits a functor $s : \mathcal{C} \to \mathcal{E}$ such that $\pi \circ s = 1_{\mathcal{C}}$. In this case, $\mathcal{E}$ is a Grothendieck construction [32].

For future reference, we define an \textit{opposite extension} $\mathcal{E}$ of $\mathcal{C}$ via $\mathcal{K}$ to be a sequence of functors $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ such that the following sequence is an extension of $\mathcal{C}^{\text{op}}$

$$\mathcal{K}^{\text{op}} \to \mathcal{E}^{\text{op}} \to \mathcal{C}^{\text{op}}.$$ 

When it won’t cause any confusion, we’ll just say $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is an opposite extension of $\mathcal{C}$.

\textbf{Example 4.1.1.}  
(1) Let $G$ be a finite group and $p$ a prime dividing the order of $G$. A collection $\mathcal{C}$ of $p$-subgroups of $G$ is a set of $p$-subgroups which is closed under conjugations in $G$. The transporter category $\text{Tr}_C(G)$ is an extension of the orbit category $\mathcal{O}_C(G)$

$$\mathcal{K}_s \to \text{Tr}_C(G) \to \mathcal{O}_C(G),$$

where $\mathcal{K}_s(H) = H \subset \text{Aut}_{\text{Tr}_C(G)}(H) = N_G(H)$;

(2) Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. Then $\mathcal{Z} \to \mathcal{L} \to \mathcal{F} \to \mathcal{C}$ is an opposite extension of $\mathcal{F}^{\text{c}}$, where $\mathcal{F}^{\text{c}} \subset \mathcal{F}$ is a full subcategory consisting of all $F$-centric subgroups and $\mathcal{Z}(P) = Z(P)$ is the so-called central functor [?].

4.2. \textbf{Generalized Lyndon-Hochschild-Serre spectral sequence.} Gabriel and Zisman ([11] Appendix II Theorem 3.6) gave a homology spectral sequence for a functor between two categories $\pi : \mathcal{E} \to \mathcal{C}$

$$E^2_{ij} = H_i(\mathcal{C}; H_j(\pi \downarrow y; F)) \Rightarrow E^i_{i+j}(\mathcal{E}; F),$$

where $F \in R\mathcal{E}$-mod is a functor from $\mathcal{E}$ to $R$-mod.

Since we are more interested in cohomology, we also write out the cohomology version of their spectral sequence

$$E^2_{ij} = H^i(\mathcal{C}^{\text{op}}; H^j(\pi \downarrow y; F)) \Rightarrow H^{i+j}(\mathcal{E}^{\text{op}}; F),$$

where $F \in R\mathcal{E}^{\text{op}}$-mod, or a \textit{contra-variant} functor $\mathcal{E} \to R$-mod. When $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is an opposite extension, we prove the cohomology groups of the overcategory with coefficients in $F$, $H^*(\pi \downarrow y; F)$, can be identified with $H^*(\mathcal{K}(y), F(y))$ by a formula of Jackowski-Slomińska ([15] Proposition 5.4), which says if a functor $\mu : \mathcal{C}_1 \to \mathcal{C}_2$ satisfies the condition that every undercategory $\mu \downarrow y$, $y \in \text{Ob}\mathcal{C}_2$, is contractible, then $H^*(\mathcal{C}_2; F) \cong H^*(\mathcal{C}_1; F \circ \mu)$ for any contra-variant functor $F : \mathcal{C}_2 \to R$-mod. (The contractibility of undercategories guarantees the left Kan extension (the left adjoint), see [14], of $\text{Res}_\mu$, the restriction along $\mu$, is exact. Hence it sends a certain projective resolution of the contra-variant functor $R \in \mathcal{C}_1$-mod to a projective resolution of the contra-variant functor $R \in \mathcal{C}_2$-mod. Similarly one can get a Jackowski-Slomińska formula for the cases when $F$ is a covariant functor and for homology, and we leave the formulations to the reader.)

Note that if there is a covariant functor $\pi : \mathcal{E} \to \mathcal{C}$ then any functor from $\mathcal{C}$ to $R$-mod induces a functor from $\mathcal{E}$ to $R$-mod (restriction along $\pi$).
Lemma 4.2.1. Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension. Then there exists a natural functor $\iota : \widehat{\mathcal{K}(y)} \to \pi_{\downarrow y}$ such that every undercategory associated with it is contractible. Hence

(1) $H_i(\pi_{\downarrow y}; F) \cong H_i(\mathcal{K}(?), F(?))$ as functors in $\text{RC-mod}$ for any $F \in \text{RC-mod}$; and

(2) $H^*(\pi_{\downarrow y}; F) \cong H^*(\mathcal{K}(?), F(?))$ as functors in $\text{RC}^{\text{op}}$-mod for any $F \in \text{RC}^{\text{op}}$-mod.

Proof. The category $\pi_{\downarrow y}$ has objects of the form $(x, \alpha)$, where $x \in \text{Ob} \mathcal{K} = \text{Ob} \mathcal{C}$ and $\alpha \in \text{Hom}_\mathcal{C}(x, y)$. From the definition of $\pi_{\downarrow y}$, it’s easy to see the maximal objects are $(y, g)$, $g \in \text{Aut}_\mathcal{C}(y)$, which are isomorphic to each other and have automorphism groups isomorphic to $\mathcal{K}(y)$.

Next we take the full subcategory of $\pi_{\downarrow y}$, consisting of all maximal objects. This full subcategory is denoted by $[(y, 1_y)]$ and its skeleton is isomorphic to the group $\widehat{\mathcal{K}(y)}$. Using Quillen’s Theorem A, we show the undercategories associated with $\iota : [(y, 1_y)] \hookrightarrow \pi_{\downarrow y}$ are contractible, and thus we can apply Jackowski-Słomińska’s result we just mentioned to get the isomorphism of cohomology groups.

Fix an object $(x, \alpha) \in \pi_{\downarrow y}$. The undercategory $(\pi_{\downarrow y})^{(x, \alpha)}$ has objects of the form $(\beta, (y, g))$, where $\beta : (x, \alpha) \to (y, g)$ is an morphism in $\pi_{\downarrow y}$ satisfying $g\pi(\beta) = \alpha$. Since $\pi(\beta) = g^{-1}\alpha$, we can deduce that $(\beta, (y, g)) \cong (\beta', (y, g'))$ for any $(y, g')$ and $\beta' : (x, \alpha) \to (y, g')$, and that $((\beta, (y, g)) \in (\pi_{\downarrow y})^{(x, \alpha)}$ has a trivial automorphism group. These imply $((\pi_{\downarrow y})^{(x, \alpha)}$ is equivalent to a point, and hence is contractible.

The isomorphism of homology and cohomology groups follows from the result of Jackowski and Słomińska ([15] Proposition 5.4) we quoted above. It’s not hard to see the isomorphisms give rise to the desired isomorphisms of functors. \hfill \Box

Combining the above lemma and the spectral sequences of Gabriel-Zisman, one can write out the following spectral sequences for category extensions. For simplicity we use $H^*(\mathcal{K}; F)$ etc, instead of $H^*(\mathcal{K}(?); F(?))$ etc, for the functors in Lemma 4.2.1.

Proposition 4.2.2. Given a functor $F : \mathcal{C} \to \text{R-mod}$, there are two spectral sequences associated with an extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ as follows:

(1) a homology spectral sequence

$$E^2_{ij} = H_i(\mathcal{C}; H_j(\mathcal{K}; F)) \Rightarrow H_{i+j}(\mathcal{E}; F);$$

and

(2) a cohomology spectral sequence

$$E^2_{ij} = H^i(\mathcal{C}^{\text{op}}; H^j(\mathcal{K}^{\text{op}}; F)) \Rightarrow H^{i+j}(\mathcal{E}^{\text{op}}; F).$$

Note that $H^i(\mathcal{K}^{\text{op}}; F) \cong H^i(\mathcal{K}; F)$.

Remark 4.2.3. From these two spectral sequence, one can obtain two five term exact sequences

$$H_2(\mathcal{E}; F) \to H_2(\mathcal{C}; F) \to H_0(\mathcal{C}; H_1(\mathcal{K}; F)) \to H_1(\mathcal{E}; F) \to H_1(\mathcal{C}; F) \to 0,$$

and

$$0 \to H^1(\mathcal{C}^{\text{op}}; F) \to H^1(\mathcal{E}^{\text{op}}; F) \to H^0(\mathcal{C}^{\text{op}}; H^1(\mathcal{K}^{\text{op}}; F)) \to H^2(\mathcal{C}^{\text{op}}; F) \to H^2(\mathcal{E}^{\text{op}}; F),$$
where $F : \mathcal{C} \to \text{R-mod}$ is a functor. When $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is a group extension then these two exact sequences are the usual five term sequence in group homology and cohomology.

Webb [32] has produced the same five term exact sequences using other techniques.

In general the finite generation of cohomology rings of both $\mathcal{K}$ and $\mathcal{C}$ doesn’t guarantee the cohomology ring of $\mathcal{E}$ has the same property. One of the examples, $\mathcal{C}_2$, we used to demonstrate that the cohomology rings of EI-categories are not finitely generated, is an extension of a contractible category:

![Diagram]

However, when $\mathcal{K}$ is cohomologically trivial, the cohomology rings of $\mathcal{E}$ and $\mathcal{C}$ are isomorphic.

**Corollary 4.2.4.** Suppose $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is an extension, and $|\mathcal{K}(x)|$ is invertible in $\text{R}$ for every object $x$. Then for any $F \in \text{R}\mathcal{C}$-mod

$$H_\ast(\mathcal{E}; F) \cong H_\ast(\mathcal{C}; H_0(\mathcal{K}; F)) \cong H_\ast(\mathcal{C}; \lim_{\mathcal{K}} F).$$

$$H^\ast(\mathcal{E}^{\text{op}}; F) \cong H^\ast(\mathcal{C}^{\text{op}}; H^0(\mathcal{K}^{\text{op}}; F)) \cong H^\ast(\mathcal{C}^{\text{op}}; \lim_{\mathcal{K}^{\text{op}}} F).$$

Since $\lim_{\mathcal{K}^{\text{op}}} \text{R} \cong \text{R}$ in $\text{R}\mathcal{C}^{\text{op}}$-mod, we have $H^\ast(\mathcal{C}; \text{R}) \cong H^\ast(\mathcal{C}^{\text{op}}; \text{R}) \cong H^\ast(\mathcal{E}^{\text{op}}; \text{R}) \cong H^\ast(\mathcal{E}; \text{R})$ as algebras.

**Proof.** Under the assumption, the $E_2$ (resp. $E^2$) page of the cohomology (resp. homology) spectral sequence collapses to the vertical (resp. horizontal) axis. \qed

### 4.3. Sub-extensions and reduction.

Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension. The generalized LHS spectral sequences establish connections between the cohomology and homology of $\mathcal{E}$ and $\mathcal{D}$. Since there is a natural correspondence between the subcategories of $\mathcal{C}$ and those of $\mathcal{E}$, one would like to exploit further connections between the homological properties of $\mathcal{C}$ and $\mathcal{E}$. Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$ and $\mathcal{E}_D$ its “preimage” in $\mathcal{E}$. We show the undercategories (or overcategories) associated with the inclusions are equivalent, when $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is an extension (or an opposite extension) of $\mathcal{C}$.

**Definition 4.3.1.** Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension and $\mathcal{D} \subset \mathcal{C}$ a subcategory. The subextension of $\mathcal{D}$ in $\mathcal{E}$ via $\mathcal{K}|_{\mathcal{D}}$, named $\mathcal{E}_D$, is a subcategory of $\mathcal{E}$ whose object set is the same as $\mathcal{D}$ and whose morphism set consists of morphisms in $\mathcal{E}$ which are preimages of morphisms in $\mathcal{D}$. 

![Diagram]
If \(D\) is a full subcategory of \(C\) then \(\mathcal{E}_D\) is a full subcategory of \(\mathcal{E}\). Given an extension \(\mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{C}\), \(\text{Aut}_{\mathcal{K}}(x) \rightarrow \text{Aut}_{\mathcal{E}}(x) \rightarrow \text{Aut}_{\mathcal{C}}(x)\) is a subextension for any \(x \in \text{Ob}\mathcal{C}\).

**Proposition 4.3.2.** Let \(\mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{C}\) a sequence of functors and \(D\) a full subcategory of \(\mathcal{C}\) with the inclusion \(\iota_D : D \rightarrow \mathcal{C}\). Then

1. if \(\mathcal{E}\) is an extension of \(\mathcal{C}\), \(\mathcal{K} \downarrow_D \rightarrow \mathcal{E}_D \rightarrow \mathcal{D}\) is the subextension and \(\iota_{\mathcal{E}_D} : \mathcal{E}_D \rightarrow \mathcal{E}\) is the inclusion, then for any \(y \in \text{Ob}\mathcal{C} = \text{Ob}\mathcal{E}\), the undercategory \(\mathcal{E}_D \downarrow y\) is isomorphic to a subcategory of the undercategory \(\iota_{\mathcal{E}_D} \downarrow y\), which is equivalent to \(\iota_{\mathcal{E}_D} \downarrow y\);

2. if \(\mathcal{E}\) is an opposite extension of \(\mathcal{C}\), \(\mathcal{K}_D \rightarrow \mathcal{E}_D \rightarrow \mathcal{D}\) is the (opposite) subextension and \(\iota_{\mathcal{E}_D} : \mathcal{E}_D \rightarrow \mathcal{E}\) is the inclusion, then for any \(y \in \text{Ob}\mathcal{C} = \text{Ob}\mathcal{E}\), the overcategory \(\mathcal{E}_D \downarrow y\) is isomorphic to a subcategory of the overcategory \(\iota_{\mathcal{E}_D} \downarrow y\), which is equivalent to \(\iota_{\mathcal{E}_D} \downarrow y\).

**Proof.** We’ll prove (2). In \(\iota_{\mathcal{E}_D} \downarrow y\), any two objects \((x, \alpha)\) and \((x, \beta)\) are isomorphic if and only if \(\pi(\alpha) = \pi(\beta)\). Let \(\iota_{\mathcal{E}_D} \downarrow y \subset \iota_{\mathcal{E}_D} \downarrow y\) be the full subcategory consisting of one object from each isomorphism class of objects described above. Then \(\iota_{\mathcal{E}_D} \downarrow y\) and \(\iota_{\mathcal{E}_D} \downarrow y\) are equivalent. We prove the former is isomorphic to \(\iota_{\mathcal{D}} \downarrow y\).

There is a natural bijection between objects sets of these two categories \((x, \alpha) \rightarrow (x, \pi(\alpha))\) (\(\pi\) is surjective on morphisms). We show there is a bijection between the morphism sets and the bijections extend to a functor which gives an isomorphism between two categories. Any \((x, \alpha) \xrightarrow{\gamma}(z, \beta)\) in \(\text{Mor}(\iota_{\mathcal{E}_D} \downarrow y)\) gives rise to a morphism \((x, \pi(\alpha)) \xrightarrow{\pi(\gamma)}(z, \pi(\beta))\) in \(\iota_{\mathcal{D}} \downarrow y\). On the other hand, a morphism \((x, \pi(\alpha)) \xrightarrow{\pi(\gamma)}(z, \pi(\beta))\) in \(\iota_{\mathcal{D}} \downarrow y\) implies \(\pi(\beta)\pi(\gamma) = \pi(\alpha)\), which means there exists a unique \(g \in \mathcal{K}(x)\) such that \(\beta\gamma = \alpha g\). Thus we have a uniquely defined morphism \((x, \alpha) \xrightarrow{\gamma^{-1}}(x, \alpha g)\xrightarrow{\pi}(z, \beta) = (x, \alpha) \xrightarrow{g^{-1}}(z, \beta)\) in \(\text{Mor}(\iota_{\mathcal{E}_D} \downarrow y)\). Note that a different \(\gamma'\) such that \(\pi(\gamma') = \pi(\gamma)\) gives the same morphism \((x, \alpha) \xrightarrow{\gamma'^{-1}}(z, \beta)\), so the map from \(\text{Mor}(\iota_{\mathcal{D}} \downarrow y)\) to \(\text{Mor}(\iota_{\mathcal{E}_D} \downarrow y)\) is well-defined. It’s straightforward to check these two assignments on morphisms are mutually inverse to each other.

In order to show the bijections on objects and morphisms defining an isomorphism between categories, we need to verify they preserve composition and identity. We’ll just prove the former and leave the proof of preserving identity to the reader. Suppose \((x, \alpha) \rightarrow (z, \beta) \rightarrow (w, \gamma)\) is a composite of two morphisms in \(\iota_{\mathcal{E}_D} \downarrow y\). Then our map naturally sends it to a composite of morphisms \((x, \pi(\alpha)) \rightarrow (z, \pi(\beta)) \rightarrow (w, \pi(\gamma))\). Conversely, if \((x, \pi(\alpha)) \xrightarrow{\pi(\alpha)}(z, \pi(\beta)) \xrightarrow{\pi(\gamma)}(w, \pi(\gamma)) = (x, \pi(\alpha)) \xrightarrow{\pi(\gamma)}(w, \pi(\gamma))\) is the composite of two morphisms in \(\iota_{\mathcal{D}} \downarrow y\), then we need to show the two morphisms \((x, \alpha) \xrightarrow{\gamma^{-1}}(z, \beta) \xrightarrow{h^{-1}}(w, \gamma) = (x, \alpha) \xrightarrow{h^{-1}g^{-1}}(w, \gamma)\) and \((x, \alpha) \xrightarrow{v^{-1}}(w, \gamma)\) are equal, where \(g, h, t\) are isomorphisms, described in the preceding paragraph. Since \(\pi(vh^{-1}ug^{-1}) = \pi(vt^{-1})\), there is a unique isomorphism \(s\) satisfying \(vh^{-1}ug^{-1} = vt^{-1}s\). But then we have \(\alpha = \gamma v^{-1} = \gamma vh^{-1}ug^{-1} = \gamma vt^{-1}s\), and this forces \(s = 1\) because \(\mathcal{K}^{op}\) acts freely on morphisms in \(\text{Mor}(\mathcal{E}^{op})\). Hence we get \(vh^{-1}ug^{-1} = vt^{-1}\). \(\square\)
The following corollary is a natural outcome of the proposition. A space $X$ is said to be $R$-acyclic, if the reduced homology groups $\tilde{H}^r(X, R)$ vanish.

**Corollary 4.3.3.** Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be a sequence of functors and $\mathcal{D} \subset \mathcal{C}$ a full subcategory with the inclusion $\iota_\mathcal{D}: \mathcal{D} \to \mathcal{C}$. Then

1. if $\mathcal{E}$ is an extension of $\mathcal{C}$, then $\iota_\mathcal{D} \downarrow y$ is contractible (or $R$-acyclic or connected) if and only if $\iota_{\mathcal{E}_\mathcal{D}} \downarrow y$ is;
2. if $\mathcal{E}$ is an opposite extension of $\mathcal{C}$, then $\iota_\mathcal{D} \downarrow y$ is contractible (or $R$-acyclic or connected) if and only if $\iota_{\mathcal{E}_\mathcal{D}} \downarrow y$ is.

**Example 4.3.4.** Let $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ be an extension with a unique maximal object $x$ such that $\text{Aut}_\mathcal{C}(x)$ acts freely and transitively on $\text{Hom}_\mathcal{C}(y, x)$ for any $y \in \text{Ob}\mathcal{C}$. Then it’s easy to check that $\iota: \text{Aut}_\mathcal{C}(x) \hookrightarrow \mathcal{C}$ induces a homotopy equivalence since all undercategories associated to it are contractible. Hence we know $\text{Aut}_\mathcal{E}(x) \hookrightarrow \mathcal{E}$ is a homotopy equivalence as well.

Since any category can be regarded as a trivial extension of itself, the following result is a generalization of Jackowski-Słomińska’s formula we mentioned in the previous subsection.

**Corollary 4.3.5.** Suppose there is an extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$. If $\iota_\mathcal{D}: \mathcal{D} \hookrightarrow \mathcal{C}$ is an inclusion such that $\iota_\mathcal{D} \downarrow y$ is contractible for every $y \in \text{Ob}\mathcal{C}$, then $H^s(\mathcal{E}; F) \cong H^s(\mathcal{E}_\mathcal{D}; F)$ for any contra-variant functor $F: \mathcal{E} \to R\text{-mod}$, and $H_*\mathcal{E}(\mathcal{E}; F) \cong H_*\mathcal{E}(\mathcal{E}_\mathcal{D}; F)$ for any covariant functor $F: \mathcal{E} \to R\text{-mod}$. Here $\mathcal{E}_\mathcal{D}$ is the subextension corresponding to $\mathcal{D}$.

Suppose there is an opposite extension $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$. If $\iota_\mathcal{D}: \mathcal{D} \hookrightarrow \mathcal{C}$ is an inclusion such that $\iota_\mathcal{D} \downarrow y$ is contractible for every $y \in \text{Ob}\mathcal{C}$, then $H^s(\mathcal{E}; F) \cong H^s(\mathcal{E}_\mathcal{D}; F)$ for any covariant functor $F: \mathcal{E} \to R\text{-mod}$. Here $\mathcal{E}_\mathcal{D}$ is the opposite subextension corresponding to $\mathcal{D}$.

**Proof.** We prove the statements for cohomology. Since $\iota_\mathcal{D} \downarrow y$ is contractible for every $y \in \text{Ob}\mathcal{C}$, $\iota_{\mathcal{E}_\mathcal{D}} \downarrow y$ is contractible for every $y \in \text{Ob}\mathcal{E}$ as well by Proposition 4.3.2. That means if we take the dual version $P'_r$ of the projective resolution $P_r$, described in 2.1, of the contra-variant functor $R: \mathcal{E}_\mathcal{D} \to R\text{-mod}$, the left Kan extension of $P'_r \to R \to 0$ is still a projective resolution $K(P'_r) \to K(R) \cong R \to 0$ of the contra-variant functor $R: \mathcal{E} \to R\text{-mod}$. Thus our results follows from the isomorphism of complexes of $R$-modules $\text{Hom}_{\mathcal{E}_\mathcal{D}}(P'_r, F \circ \iota_{\mathcal{E}_\mathcal{D}}) \cong \text{Hom}_{\mathcal{E}}(K(P'_r), F)$.

When we have an opposite extension, using the same proposition we get $\iota_{\mathcal{E}_\mathcal{D}} \downarrow y$ is contractible for every $y \in \text{Ob}\mathcal{E}$. If $P_r \to R \to 0$ is the projective resolutions of $R$ as a covariant functor, then the left Kan extension of it, $K(P_r) \to K(R) \cong R \to 0$, is a projective resolution of $R$ as a covariant functor from $\mathcal{E}$ to $R\text{-mod}$. Hence we can obtain a similar isomorphism of complex using the adjunction of $K$ and $\text{Res}_{\iota_{\mathcal{E}_\mathcal{D}}}$, the restriction along $\iota_{\mathcal{E}_\mathcal{D}}$. \qed

As an example when $\mathcal{K} \to \mathcal{E} \to \mathcal{C}$ is an extension (or an opposite extension) and $\mathcal{C}$ has a unique maximal (or minimal) object $x$ and $\text{Aut}_\mathcal{C}(x)$ acts regularly on
\[ \text{Hom}_C(y, x) \ (\text{or } \text{Hom}_C(x, y)) \] for any \( y \in \text{Ob} \mathcal{C} \), we have \( \text{H}^*(\mathcal{C}; F) \cong \text{H}^*(\text{Aut}_\mathcal{C}(x), F(x)) \)

hence \( \text{H}^*(\mathcal{E}; F) \cong \text{H}^*(\text{Aut}_\mathcal{E}(x), F(x)) \) for any contra-variant (or covariant) functor \( F \).

\textbf{References}


ON THE COHOMOLOGY RINGS OF SMALL CATEGORIES


