The Hyperspace of the Regions Below of Continuous Maps from the Converging Sequence*

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Abstract: Let $S = \{1, \frac{1}{2}, \frac{1}{2^2}, \cdots, \frac{1}{2^n} = 0\}$ and $I = [0, 1]$ be the unit interval. We use $\downarrow \text{USC}(S)$ and $\downarrow \text{C}(S)$ to denote the families of the regions below of all upper semi-continuous maps and of the regions below of all continuous maps from $S$ to $I$ and $\downarrow \text{C}_0(S) = \{f \in \text{C}(S) : f(0) = 0\}$. $\downarrow \text{USC}(S)$ endowed with the Vietoris topology is a topological space. A pair of topological spaces $(X,Y)$ means that $X$ is a topological space and $Y$ is its subspace. Two pairs of topological spaces $(X,Y)$ and $(A,B)$ is called pair-homeomorphic ($\approx$) if there exists a homeomorphism $h : X \rightarrow A$ from $X$ onto $A$ such that $h(Y) = B$. It is proved that, $(\downarrow \text{USC}(S), \downarrow \text{C}_0(S)) \approx (Q, s)$ and $(\downarrow \text{USC}(S), \downarrow \text{C}(S) \setminus \downarrow \text{C}_0(S)) \approx (Q, c_0)$, where $Q = [-1, 1]^{\omega}$ is the Hilbert cube and $s = (-1, 1)^{\omega}$, $c_0 = \{(x_n) \in Q : \lim_{n \rightarrow \infty} x_n = 0\}$. But we do not know what $(\downarrow \text{USC}(S), \downarrow \text{C}(S))$ is.

Key words: regions below, upper semi-continuity, the Hilbert cube, pseudo-interior, strongly universal, the converging sequence


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1 Introduction

For a metric space $(X,d)$ and a subset $L$ of the set $R$ of all real numbers, we consider the sets $C(X,L)$ and $\text{USC}(X,L)$ which consist of all continuous maps and all upper semi-continuous maps from $X$ to $L$, respectively. The Kadec-Anderson Theorem states that when $X$ is a compactum and $L = R$ the space $C(X,L)$ endowed with the uniform convergence topology is homeomorphic to $(\approx) R^{\omega} \approx l^2$ (see [1], [2]). The so-called Dobrowolski-Marciszewski-Mogilski Theorem (see [3], Theorem 6.12.15 of [4]) asserts that if $X$ is nondiscrete, countable and $L = R$ or $L = I = [0,1]$ then the space $C_p(X,L)$, the set $C(X,L)$ topologized with the topology of pointwise convergence, is homeomorphic to the space $c_0 = \{(x_n) \in [-1,1]^\omega : \lim_{n \rightarrow \infty} x_n = 0\}$.
\[ \lim_{n \to \infty} x_n = 0 \]. In [5],[6], a map \( f \in \text{USC}(X, I) \) is carried by a bijection to its region below \( \downarrow f = \{(x, \lambda) \in X \times I : \lambda \leq f(x)\} \) in the product space \( X \times I \), and we studied the space \( |\text{USC}(X, I) = \{f : f \in \text{USC}(X, I)\} \) and its subspace \( |\text{C}(X, I) = \{f : f \in \text{C}(X, I)\} \), where they are topologized as the subspaces of the hyperspace \( \text{Cld}(X \times I) \) which denotes the set of all nonempty closed subsets in \( X \times I \) endowed with the Vietoris topology (see Section 2). The family \( |\text{C}(X, I) \) with this topology is usually different from the above two spaces \( C_0(X, I) \) and \( C_p(X, I) \) though there are bijections among them. In [7], it was proved that if \( X \) is an infinite locally connected compact metric space then there exists a homeomorphism \( h : \text{USC}(X, I) \to Q \) such that \( h(|\text{C}(X, I)) = c_0 \), that is, \( (|\text{USC}(X, I), \text{C}(X, I)) \approx (Q, c_0) \), where \( Q = [-1, 1]^\omega \) is the Hilbert cube. But when \( X \) is an infinite non-locally-connected compact metric space, \( \text{USC}(X, I) \) and especially \( |\text{C}(X, I) \) become complicated. In [7], we pointed out that it is not necessary that \( |\text{C}(X, I) \approx c_0 \).

In the present paper, we consider the simplest case, that is, \( X = S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{\infty} = 0\} \) is a convergent sequence with its limit. Even in this case, we do not know what \(|\text{C}(S, I) \) is. We discuss the topological space \( |\text{USC}(S) = \text{USC}(S, I) \) and its some natural subspaces \( |C_0(S) = \{f : f \in C(S), f(0) = 0\} \) and \( |C_{>0}(S) = C(S) \setminus C_0(S) \), where \( C(S) = C(S, I) \) and \( |\text{C}(S) = |\text{C}(S, I) \). We prove the following theorems.

**Theorem 1.1** Let \( S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{\infty} = 0\} \). Then \( \downarrow |\text{USC}(S), |C_0(S)) \approx (Q, s) \), where \( s = (-1, 1)^\omega \).

**Theorem 1.2** Let \( S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{\infty} = 0\} \). Then \( \downarrow |\text{USC}(S), |C_{>0}(S)) \approx (Q, c_0) \).

We look forward to know what \(|\text{C}(S) \) and the pair \(|\text{USC}(S), |\text{C}(S)) \) are. We find out that this problem is a little complicated. By the following Lemma 3.2, we know that \(|\text{C}(S) \) is an AR and it was proved in [5],[6] that \( |\text{C}(S) \) is an \( F_{\sigma}\)-set in \( |\text{USC}(S) \) but not a \( G_{\delta}\)-set. Now we have

**Lemma 1.1** \(|\text{C}(S)\) is homeomorphic to neither \( c_0 \) nor \( s \).

**Proof.** By Theorem 3 of [6], \(|\text{C}(S)\) is not a \( G_{\delta}\)-set in \(|\text{USC}(S)\). It follows that \(|\text{C}(S)\) is not topologically complete but \( s \) is, and by Theorem 2 of [6], \(|\text{C}(S)\) has the Baire property while \( c_0 \) has not. We are done.

Hence, we put the following problems:

**Problem 1.1** What are the spaces \(|\text{C}(S)\) and \((|\text{USC}(S), |\text{C}(S))\)?

**Problem 1.2** Whether or not \((|\text{USC}(X)) \approx Q \) for any infinite compactum \( X \)?

## 2 Preliminaries

All spaces under discussion here are assumed to be separable and metrizable.

In this section we recall some necessary fundamental concepts and facts. For more information on them, please refer to [7] and [4].
We first recall that for a metric space $X$, the hyperspace $\text{Cld}(X)$ is the set consisting of all non-empty closed subsets in $X$ endowed with the Vietoris topology which is generated by $\{U^-, U^+ : U \subset X \text{ is open }\}$ as a subbase, where
\[
U^- = \{A \in \text{Cld}(X) \mid A \cap U \neq \emptyset\} \quad \text{and} \quad U^+ = \{A \in \text{Cld}(X) \mid A \subset U\}.
\]
It is well-known that $\text{Cld}(X)$ with this topology is metrizable if and only if $X$ is compact and metrizable (see Theorem 1.3.4 of [8]). For a compact metric space $X = (X,d)$, the Vietoris topology of $\text{Cld}(X)$ is induced by the Hausdorff metric $d_H$ defined as follows:
\[
d_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}.
\]
A closed subset $A$ of a space $X$ is said to be a $Z$-set if for any continuous map $\varepsilon : X \to (0, \infty)$ there is a continuous map $f : X \to X \setminus A$ with $d(f(x), x) < \varepsilon(x)$ for $x \in X$. If $X$ is compact then $\varepsilon$ map can be replaced by an arbitrary positive real number $\varepsilon$. A $Z_\sigma$-set is a countable union of $Z$-sets. The families of all $Z$-sets and all $Z_\sigma$-sets of a space $X$ are denoted by $\mathcal{Z}(X)$ and $\mathcal{Z}_\sigma(X)$, respectively. A $Z$-embedding is an embedding with a $Z$-set image. We say that a subspace $Y$ of $X$ is homotopy dense in $X$ provided there exists a homotopy $h : X \times I \to X$ such that $h_0 = \text{id}_X$ and $h_t(X) \subset Y$ for any $t > 0$.

Let $\mathcal{M}_0$ denote the class of compact metric spaces, and for a topological class $\mathcal{C}$ of spaces, let $(\mathcal{M}_0, \mathcal{C})$ denote the class of the pairs $(Z, C)$ such that $Z \in \mathcal{M}_0$, $C \in \mathcal{C}$ and $C$ is a subspace of $Z$.

**Definition 2.1** We say that a pair $(X,Y)$ in $(\mathcal{M}_0, \mathcal{C})$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal provided for any $(A,C) \in (\mathcal{M}_0, \mathcal{C})$, any $\varepsilon > 0$ and any continuous map $f : A \to X$, any closed subset $K$ of $A$ such that $f|_K : K \to X$ is a $Z$-embedding, there exists a $Z$-embedding $g : A \to X$ such that $g|_K = f|_K$, $g^{-1}(Y) \setminus K = C \setminus K$ and $d(g(a), f(a)) < \varepsilon$ for any $a \in A$.

**Definition 2.2** We say that a subset $Y$ is a $C$-absorber in $X$ if
(a) $Y \in C$;
(b) $Y$ is contained in a $Z_\sigma$-set of $X$;
(c) $(X,Y)$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal.

**Lemma 2.1** Theorem 8.2 of [9],[10] If $X$ and $Y$ are $C$-absorbers in a $Q$-manifold $M$, then $(M, X) \approx (M, Y)$.

In this paper we are concerned with the class $\mathcal{F}_\sigma$ of absolute $F_\sigma$ spaces and the collection $\mathcal{F}_\sigma$ of $\sigma$-compact spaces, respectively. It was proved, $c_0 = \{x \in Q : \lim_{n \to \infty} x_n = 0\}$ is a $\mathcal{F}_\sigma$-absorber in $Q$ (see [11]) and $B(Q) = Q \setminus s$ is an $\mathcal{F}_\sigma$-absorber in $Q$ (see Theorem 6.5.8 of [7]).

Let $\phi : A \to B$ be a map. If $A \subset \text{USC}(X)$ and/or $B \subset \text{USC}(Y)$ for spaces $X$ and $Y$, We may define a corresponding map $|\phi| : \downarrow A \to \downarrow B$ or $|\phi| : A \to \downarrow B$ or $\downarrow \phi : \downarrow A \to B$ by $|\phi|(\downarrow f) = |\downarrow (\phi(f))|$ or $\downarrow \phi(f) = |\downarrow (\phi(f))|$ or $\downarrow \phi(f) = \phi(\downarrow f)$, respectively.
3 Proof of $|\text{USC}(S)| \approx Q$

We start with recalling some facts and concepts needed in this section. A space $X$ is called an absolute retract (abbreviated AR) provided that for any space $Y$ containing $X$ as a closed subspace there exists a continuous map $r : Y \to X$ such that $r|_X = id_X$. We say that a space $X$ has the disjoint-cells property provided that for every natural number $n$, every continuous function $f : I^n \times \{0, 1\} \to X$ can be approximated (arbitrarily closely) by a continuous map sending $I^n \times \{0\}$ and $I^n \times \{1\}$ to disjoint sets.

**Lemma 3.1** (The Toruńczyk’s Characterization Theorem) [12], Corollary 7.8.4 of [7]. A space $X$ is homeomorphic to the Hilbert cube $Q$ if and only if it is a compact AR with the disjoint-cells property.

Now we use it to prove

**Proposition 3.1** $|\text{USC}(S)| \approx Q$.

For $n = 0, 1, 2, \cdots, \infty$, let $a_n = \frac{1}{n}$ and $S = \{a_0, a_1, a_2, \cdots, a_\infty = 0\}$.

**Lemma 3.2** $|C(S)|$ is an AR.

**Proof.** In [7], it was proved that $|C(I)|$ is homeomorphic to $c_0$ which is a known AR. Hence, it suffices to prove that $|C(S)|$ is homeomorphic to a retract of $|C(I)|$.

**Claim 1.** There exists a closed embedding $\downarrow \varphi : |\text{USC}(S)| \to |\text{USC}(I)|$ carrying $\downarrow C(S)$ into $|C(I)|$ as a closed set of $|C(I)|$.

For any $n \in \mathbb{N}$, choose $c_n, b_n \in (a_{n+1}, a_n)$ with $c_n < b_n$. For any $f \in \text{USC}(S)$, define the break line function $\varphi(f) : I \to I$ through points $(a_n, f(a_n))$, $(b_n, \min\{f(a_n), f(a_{n+1})\})$ and $(c_n, \min\{f(a_n), f(a_{n+1})\})$ for any $n \in \mathbb{N}$, and $(0,0)$. It is not hard to check that the function $\downarrow \varphi : |\text{USC}(S)| \to |\text{USC}(I)|$ is a closed embedding and so is $|\varphi|_{C(S)} : |C(S)| \to |C(I)|$.

**Claim 2.** $|\varphi|_{|\text{USC}(S)|}$ is a retract of $|C(I)|$.

For a continuous function $g : I \to I$, define a function $r(g) : S \to I$ as follows:

$$r(g)(a_1) = \sup\{g(x) : x \in (c_1, a_1)\},$$

$$r(g)(a_n) = \sup\{g(x) : x \in (c_n, b_{n-1})\}, \quad n = 2, 3, \cdots$$

and

$$r(g)(a_\infty) = g(a_\infty) = g(0).$$

Trivially, $r(g)$ is continuous. Thus we define a map $\downarrow r : C(I) \to C(S)$. To complete the proof of the Claim, it suffices to show that $\downarrow r : \downarrow C(I) \to \downarrow C(S)$ is continuous and $\downarrow r \circ \downarrow \varphi = id_{\downarrow C(S)}$. It is not hard to verify the latter. The remainder is to check the continuity of $\downarrow r$. Let $g, g_k \in C(I)$ with $\lim_{k \to \infty} g_k = g$. For any $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $a_N < \frac{\varepsilon}{2}$. By Lemma 14 of [6], the map $g \mapsto r(g)(a_n)$ is continuous from $|C(I)|$ to $I$ for any $n \in \mathbb{N}$. Thus we may pick $K \in \mathbb{N}$ such that for any $k > K$,

$$d_H(g_k, g) < \varepsilon = \min\{\frac{\varepsilon}{2}, \frac{b_{N-1} - b_N}{2}\};$$

and

$$\left| r(g_k)(a_n) - r(g)(a_n) \right| < \varepsilon,$$
for $n \leq N$. Then

$$d_H(\downarrow r(g_k) \cap (\{a_0, \ldots, a_N\} \times I), \downarrow r(g) \cap (\{a_0, \ldots, a_N\} \times I)) < \varepsilon.$$  

Moreover, for any $n > N$ and $k > K$, there exists $(x_1, y_1) \in \downarrow g_k$ with $x_1 \in (c_n, b_{n-1})$ and $|y_1 - r(g_k)(a_n)| < \frac{\varepsilon}{2}$. It follows from $d_H(\downarrow g_k, \downarrow g) < \delta$ that there exists $(x_2, y_2) \in \downarrow g$ such that $d((x_2, y_2), (x_1, y_1)) < \delta$. Then $x_2 \in [0, b_N]$, and hence there exists $n' \geq N$ such that $(a_{n'}, y_2) \in \downarrow r(g)$. Trivially, $d((a_{n'}, y_2), (a_n, y)) < \varepsilon$. Thus

$$\downarrow r(g_k) \cap (\{a_{n+1}, a_{N+2}, \ldots, a_\infty\} \times I) \subset B_{d_H}(\downarrow r(g), \varepsilon)$$

for $k > N$. Similarly, for $k > N$, we have

$$\downarrow r(g) \cap (\{a_{n+1}, a_{N+2}, \ldots, a_\infty\} \times I) \subset B_{d_H}(\downarrow r(g_k), \varepsilon).$$

Hence $d_H(\downarrow r(g), \downarrow r(g_k)) < \varepsilon$ when $k > K$.

From Claim 1, Claim 2 and the fact that $\|C(I)\|$ is an AR, we obtain Lemma 3.2.

**Lemma 3.3** $\|C(S)\|$ is homotopy dense in $\|USC(S)\|$.

**Proof.** We first embed $\|USC(S)\|$ into $\|USC(I)\|$ as a closed subspace by a method more direct than the one used in the above lemma. For any $f \in USC(S)$, define a function $\varphi(f) : \mathbb{I} \rightarrow \mathbb{I}$ by

$$\varphi(f)(x) = \left\{ \begin{array}{ll} \frac{a_n - x}{a_n - a_{n+1}} f(a_{n+1}) + \frac{x - a_{n+1}}{a_n - a_{n+1}} f(a_n), & x \in [a_{n+1}, a_n] \text{ for some } n, \\ f(0), & x = 0. \end{array} \right.$$  

Then it is easily seen that $\|\varphi : \|USC(S)\| \rightarrow \|USC(I)\|$ is a closed embedding. Define a map $M : \|USC(S)\| \times (0, 1] \rightarrow \mathbb{I}$ by

$$M(\langle f, t \rangle) = \sup\{\varphi(f)(x) : 0 \leq x \leq t\}$$

for any $\langle f \rangle \in \|USC(S)\|$ and $t \in (0, 1]$. Then $M$ is continuous (see Lemma 14 and its proof of [6]). Define $H : USC(S) \times \mathbb{I} \rightarrow USC(S)$ as follows:

$$H(f, t)(a_n) = \left\{ \begin{array}{ll} \varphi(f)(a_n) = f(a_n), & a_n \geq t; \\ \varphi(f)(t) \frac{a_n - t}{a_n} + M(\langle f, t \rangle) \frac{t - a_n}{t}, & a_n < t. \end{array} \right.$$  

Obviously, $\downarrow H(\|USC \times (0, 1]\|) \subset \|C(S)\|$ and $\downarrow H(\downarrow f, t) = \|f \|$ for any $\downarrow f \in \|USC(S)\|$. We next prove that $\downarrow H$ is continuous. Consider a sequence $(\langle f_n, t_n \rangle)_n$ converging to $(\langle f, t \rangle)$. We shall show that $\downarrow H(\langle f_n, t_n \rangle)_n$ converges to $\downarrow H(\langle f, t \rangle)$. This follows from the following facts:

$$\lim_{n \rightarrow \infty} H(f_n, t_n)(a_m) = M(\langle f_n, t_n \rangle), \quad \lim_{n \rightarrow \infty} H(f, t)(a_m) = M(\langle f, t \rangle), \quad \lim_{n \rightarrow \infty} M(\langle f_n, t_n \rangle)_n = M(\langle f, t \rangle) \text{ and the sequence } (H(f_n, t_n))_n \text{ of maps pointwise converges to } H(f, t).$$

Consequently, the homotopy $\|H\|$ is as required.

**Lemma 3.4** $\|USC(S)\|$ is an AR.

**Proof.** It was proved in [13] that, if $Y$ is a homotopy dense subspace in $X$ then $Y$ is an AR if and only if $X$ is. It follows directly from Lemmas 3.2 and 3.3 that $\|USC(S)\|$ is an AR.

**Remark** The authors want to say that, in order to prove Proposition 3.1, the fact that $\|USC(S)\|$ is an AR is necessary. The reason we make a detour which lead us via the proof of $\|C(S)\|$ being an AR is that we also can embed $\|USC(S)\|$ into $\|USC(I)\|$ as a closed subspace while no natural retraction just as in the proof of Lemma 3.2 can be found.
Define a map $M : \downarrow \text{USC}(S) \times S \setminus \{0\} \to I$ by, for any $\downarrow f \in \downarrow \text{USC}(S)$ and $n \in \mathbb{N}$, $M(\downarrow f, \frac{1}{2^n}) = \max\{f(c)(\frac{1}{2^n}) : k \geq n\}$. Then $M$ is continuous. This map will be used in the following Lemma 3.5 and Lemma 4.3.

**Lemma 3.5** $\downarrow \text{USC}(S)$ has the disjoint-cells property.

**Proof.** For two given continuous functions $f$ and $g : I^k \to \downarrow \text{USC}(S)$, choose $N \in \mathbb{N}$ such that $\frac{1}{2N} < \epsilon$. Define $f'$ and $g' : I^k \to \downarrow \text{USC}(S)$ as follows,

$$f'(q)(\frac{1}{2^n}) = \begin{cases} f(q)(\frac{1}{2^n}), & i \neq N, i \neq N + 1, \\ \max\{\epsilon, f(q)(\frac{1}{2^n}), f(q)(\frac{1}{2^{n+1}})\}, & i = N, \\ 0, & i = N + 1, \\ \max\{\epsilon, f(q)(\frac{1}{2^n}), f(q)(\frac{1}{2^{n+1}})\}, & i = N + 1, \\ 0, & i = N. \end{cases}$$

$$g'(q)(\frac{1}{2^n}) = \begin{cases} g(q)(\frac{1}{2^n}), & i \neq N, i \neq N + 1, \\ \max\{\epsilon, g(q)(\frac{1}{2^n}), g(q)(\frac{1}{2^{n+1}})\}, & i = N + 1, \\ 0, & i = N. \end{cases}$$

Then it is clear that $d_{\text{USC}}(f, f') < \epsilon$ and $d_{\text{USC}}(g, g') < \epsilon$, and moreover, $f'(I^k) \cap g'(I^k) = \emptyset$.

Now we give the proof the main result of this section.

The Proof of Proposition 3.1. It follows directly from Lemmas 3.1, 3.4 and 3.5.

### 4 Proofs of Theorems 1.1 and 1.2

We first recall some concepts related to capsets and present an interesting characterization of capsets which will be applied in the proof of Theorem 1.1.

Let $M^Q$ be a copy of the Hilbert cube $Q$. An element $A \in \mathcal{Z}_\sigma(M^Q)$ is called a capset provided that $A$ can be written as the union of an increasing sequence $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ of $Z$-sets in $M^Q$ such that, for any $\epsilon > 0$, $n \in \mathbb{N}$, and $Z \in Z(M^Q)$ there exist $m > n$ and a homeomorphism $h : M^Q \to M^Q$ with the properties $d(h, 1) < \epsilon, h|_{A_n} = id_{A_n}$, and $h(Z) \subset A_n$.

Let $B = (B_i)_i$ be a tower of subsets of $M^Q$, that is, $B_i \subset B_{i+1} \subset M^Q$ for every $i$. We say that $B$ has the deformation property if there is a homotopy $H : M^Q \times I \to M^Q$ such that $H_0 = id_{M^Q}$, and moreover, for each $t \in (0, 1]$ there is an $i \in \mathbb{N}$ such that $H(M^Q \times [t, 1]) \subset B_i$.

**Lemma 4.1** Theorem 5.4.9 of [4] An element $A \in \mathcal{Z}_\sigma(M^Q)$ is a capset if and only if $(M^Q, A) \approx (Q, B(Q))$.

**Lemma 4.2**[14], Corollary 5.4.11 of [4] An element $B \in \mathcal{Z}_\sigma(Q)$ is a capset if and only if $B$ can be written as the union of a tower $(B_n)_n$ of $Z$-sets in $Q$ such that

(a) each $B_n$ is homeomorphic to $Q$;

(b) each $B_n$ is a $Z$-set in $B_{n+1}$;

(c) $(B_n)_n$ has the deformation property.

To show that $(\downarrow \text{USC}(S), \downarrow C_0(S)) \approx (Q, s)$, by Proposition 3.1 and Lemma 4.1, it suffices to prove that $\downarrow \text{USC}(S) \setminus \downarrow C_0(S)$ is a capset in $\downarrow \text{USC}(S)$. For any $n \in \mathbb{N}$, put $F_n = \{\downarrow f \in \downarrow \text{USC}(S) : f(0) \geq \frac{1}{n}\}$. Then $F = \bigcup_{n=1}^{\infty} F_n = \downarrow \text{USC}(S) \setminus \downarrow C_0(S)$. We need to check that
the tower \((F_n)_n\) satisfies the conditions (a),(b) and (c) in the above lemma. Then the proof of Theorem 1.1 is completed.

**Lemma 4.3** \(F_n \in Z(\downarrow \text{USC}(S)) \cap Z(F_{n+1})\) and \(\downarrow \text{C}_{>0}(S) \cup \bigcup_{n=1}^{\infty} F_n\).

**Proof.** Let \(\varepsilon > 0\) be given and \(n_0\) fixed. Choose \(N_0 \in \mathbb{N}\) such that \(\frac{1}{2n_0} < \varepsilon\) and define \(M : \downarrow \text{USC}(S) \times S \setminus \{0\} \to I\) by, for any \(f \in \downarrow \text{USC}(S)\), and \(n \in \mathbb{N}\),

\[
M(f, \frac{1}{2n}) = \max\{f(c)(\frac{1}{2^k}) : k \geq n\}.
\]

Then \(M\) is continuous. Define a continuous map \(\phi : \downarrow \text{USC}(S) \to \downarrow \text{USC}(S)\) as follows: for any \(f \in \text{USC}(S)\),

\[
\phi(f)(\frac{1}{2m}) = \begin{cases} f(\frac{1}{2m}), & m < N_0, \\ M(f, \frac{1}{2m0}), & m = N_0, \\ 0, & \text{others}. \end{cases}
\]

Then \(d(\phi, \text{id}_{\downarrow \text{USC}(S)}) < \varepsilon\). Thus \(F_{n_0} \in Z(\downarrow \text{USC}(S))\). Replacing 0 in the above formula by \(\frac{1}{n_0+1}\) we may show \(F_{n_0} \in Z(F_{n_0+1})\). \(\downarrow \text{C}_{>0}(S) \cup \bigcup_{n=1}^{\infty} F_n\) is trivial.

**Lemma 4.4** \(F_n\) is homeomorphic to \(Q\) for any \(n \in \mathbb{N}\).

**Proof.** The function \(\downarrow r : \downarrow \text{USC}(S) \to F_n\) defined by

\[
r(f)(x) = \begin{cases} f(x), & x \neq 0; \\ \max\{f(0), \frac{1}{2}\}, & x = 0. \end{cases}
\]

is a retraction from \(\downarrow \text{USC}(S)\) to \(F_n\). Hence \(F_n\) is an AR. By the same method as the one used in Lemma 3.5 we may verify that \(F_n\) has the disjoint-cells property. Hence, by Lemma 3.1 we conclude that \(F_n \approx Q\).

**Lemma 4.5** \(\{F_n : n \in \mathbb{N}\}\) is a tower with the deformation property in \(\downarrow \text{USC}(S)\).

**Proof.** For any \(f \in \downarrow \text{USC}(S)\) and for any \(t \in I\), let \(H(\downarrow f, t) = [f \cup (S \times [0, t])\). Then it is easy to see that \(H\) is as required.

**The Proof of Theorem 1.1.** It follows from the above lemmata.

Now we turn to proving Theorem 1.2.

Let \(\mathcal{F}_{\sigma\delta}\) be the family of all absolute \(F_{\sigma\delta}\)-sets. Then \(c_0\) is an \(\mathcal{F}_{\sigma\delta}\)-absorber in \(Q\). To show that \((\downarrow \text{USC}(S), \downarrow \text{C}_{>0}(S))\) is homeomorphic to \((Q, c_0)\), it suffices to verify that \(\downarrow \text{C}_{>0}(S)\) is an \(\mathcal{F}_{\sigma\delta}\)-absorber in \(\downarrow \text{USC}(S)\). We notice that \(\downarrow \text{C}_{>0}(S)\) is an absolute \(F_{\sigma\delta}\)-set since \(\downarrow \text{C}_{>0}(S)\) is a \(F_\alpha\)-subset of \(\downarrow \text{USC}(S)\) which is an absolute \(F_{\sigma\delta}\)-set (see [5] and [15]). Moreover, by Lemma 4.3, \(\downarrow \text{C}(S)\) is contained in a \(Z_\sigma\)-set of \(\downarrow \text{USC}(S)\). The remainder is to show that the pair \((\downarrow \text{USC}(S), \downarrow \text{C}_{>0}(S))\) is strongly \((\mathcal{M}_0, \mathcal{F}_{\sigma\delta})\)-universal. At first, we give the following lemma in [6].

**Lemma 4.6** \([6]\) Let \(Q_u = [0, 1]^\omega\) and \(c_1 = \{(x_n) \in Q_u : \lim_{n \to \infty} x_n = 1\}\). Then the pair \((Q_u, c_1)\) is strongly \((\mathcal{M}_0, \mathcal{F}_{\sigma\delta})\) -universal.

**Lemma 4.7** The pair \((\downarrow \text{USC}(S), \downarrow \text{C}_{>0}(S))\) is strongly \((\mathcal{M}_0, \mathcal{F}_{\sigma\delta})\) -universal.
Proof. For any \((C, A) \in (\mathcal{M}_0, \mathcal{F}_\sigma)\), any continuous map \(|F| : C \to |\text{USC}(S)|\), and any closed subset \(K\) of \(C\) such that \(|F|_K : K \to |\text{USC}(S)|\) is a \(Z\)-embedding. By Lemma 1.1 of [16], we may, without loss of generality assume that \(|F|(C \setminus K) \cap |F|(K) = \emptyset\).

We first define \(M : C \times S \setminus \{0\} \to I\) by, for any \(c \in C\), and \(n \leq \infty, M(c, \frac{1}{2^n}) = \max\{F(c)(\frac{1}{2^n}) : k \geq n\}\). Then \(M\) is continuous. By Lemma 4.6, there exists \(g : C \to Q_u\) such that \(g\) is an embedding and \(g^{-1}(c_1) = A\). Now for any \(\varepsilon \in (0, 1), \) put \(\delta : C \to [0, 1)\) by \(\delta(c) = \min\{\varepsilon, \frac{1}{2\delta}(|F|(c), |F|(K))\}\). Then \(\delta\) is continuous and \(\delta(c) = 0\) if and only if \(c \in K\).

For \(c \in C_k = \{c \in C : \frac{1}{2^k} \leq \delta(c) \leq \frac{1}{2^{k+1}}\}, k = 1, 2, \ldots\), define \(H_k(c) \in \text{USC}(S)\) as follows: for any \(n \leq \infty,
\begin{align*}
H_k(c)(\frac{1}{2^n}) &= \begin{cases} 
F(c)(\frac{1}{2^n}), & n \leq 2k, \\
M(c, \frac{1}{2^{k+1}})(1 - t) + F(c)(\frac{1}{2^k})t, & n = 2k + 1, \\
F(c)(\frac{1}{2^k})t, & n = 2k + 2, \\
M(c, \frac{1}{2^{k+1}}), & n = 2k + 3, \\
0, & n = 2k + 4, \\
\delta(c)(1 - t) + M(c, \frac{1}{2^{k+1}})t, & n = 2k + 5, \\
\delta(c)g(c)(1)(1 - t), & n = 2k + 6, \\
\delta(c)[(1 - t)g(c)(m) + tg(c)(m)], & n > 2k + 6, n \text{ is even}, \\
\delta(c), & \text{others},
\end{cases}
\end{align*}
\)
where \(t = 2 - 2^k\delta(c)\), and \(m = 2^{-1}n - k - 3\).

If \(c \in C_k \cap C_{k-1}\) for some \(k \in \mathbb{N}\) then \(\delta(c) = \frac{1}{2^k}\) and an easy check shows that \(H_k(c) = H_{k-1}(c)\). Thus we may define \(|H| : C \to |\text{USC}(S)|\) by \(|H(c)| = |H_k(c)|\) for \(c \in C_k\) and \(|H(c)| = \downarrow F(c)\) if \(c \in K\). We shall prove that \(\downarrow H\) is as required. Obviously, \(\lim_{m \to \infty} H(c)(\frac{1}{2^m})\) exists if and only if \(\lim\, g(c)(m) = 1\), if and only if \(c \in A\). Hence \(H^{-1}(|\text{USC}(S)| \setminus K) = C \setminus K\).

For any \(c \in C \setminus K\), there exists \(n_0 \in \mathbb{N}\) such that \(H(c)(\frac{1}{2^n}) = 0\). It follows that \(|H|(C)\) is a \(Z\)-set in \(|\text{USC}(S)|\) (see Lemma 5 of [6]). Notice that \(H(c)(\frac{1}{2^n}) = F(c)(\frac{1}{2^n})\) for any \(\frac{1}{2^n} \geq \delta(c)\) and there exists \(n_0 \in \mathbb{N}\) such that
\[
H(c)(\frac{1}{2^n}) = \max\{F(c)(\frac{1}{2^n}) : \frac{1}{2^n} \leq \delta(c)\}.
\]
Thus \(B_{d_H}(|H(c)|, \delta(c)) \supset |F(c)|\). Moreover,
\[
\max\{|H(c)(\frac{1}{2^n}) : \frac{1}{2^n} \leq \delta(c)\} \leq \max\{\max\{F(c)(\frac{1}{2^n}) : \frac{1}{2^n} \leq \delta(c)\}, \delta(c)\}.
\]
It follows that \(B_{d_H}(|F(c)|, \delta(c)) \supset |H(c)|\). Hence
\[
d_H(|H(c)|, |F(c)|) \leq \delta(c) \leq \varepsilon \quad \text{for any } c \in C.
\]
To complete the proof it suffices to check the following two facts:

Fact 1. \(|H| : C \to |\text{USC}(S)|\) is continuous.

It follows from the continuity of \(F, M, g\) and \(\delta\) that \(|H| : C_k \to \text{USC}(S)\) is continuous where \(\text{USC}(S)\) is topologized by the pointwise convergence. Moreover, \(|H(c)(\frac{1}{2^n})| \leq H(c)(0) = \delta(c)\) for any \(n > 2k + 6\). Thus \(|H| : C_k \to \text{USC}(S)\) is continuous for any \(k \in \mathbb{N}\) since \(\delta\) is continuous. It follows from \(d_H(|F(c)|, |H(c)|) < \delta(c)\) that \(|H|\) is continuous on \(K\).

Thus \(|H : C \to |\text{USC}(S)|\) is continuous.

Fact 2. \(|H| : C \to |C_{>0}(S)|\) is an injection.

Let \(c_1, c_2 \in C\) and \(|H(c_1)| = |H(c_2)|\). We need to show that \(c_1 = c_2\). Since \(|H|_K = |F|_K\) is an injection and \(d_H(|H(c)|, |F(c)|) \leq \delta(c) \leq \frac{1}{2\delta}(|F(c)|, |F(K)|)\), we only consider the case
that $c_1, c_2 \in C \backslash K$. By the assumption, we have

$$\delta(c_1) = \lim_{m \to \infty} H(c_1)(\frac{1}{2^m}) = \lim_{m \to \infty} H(c_2)(\frac{1}{2^m}) = \delta(c_2),$$

and hence there exists $k \in \mathbb{N}$ such that $c_1, c_2 \in C_k$. Then

$$\delta(c_1) g(c_1)(1)(2 - 2^k \delta(c_1)) = \delta(c_2) g(c_2)(1)(2 - 2^k \delta(c_2)).$$

Since $\delta(c_1) = \delta(c_2) \neq 0$ we have $g(c_1)(1) = g(c_2)(1)$, and by a similar argument, $g(c_1)(m) = g(c_2)(m)$ for any $m \in \mathbb{N}$ and consequently, $g(c_1) = g(c_2)$. Since $g$ is an embedding we have $c_1 = c_2$. The proof is completed.

The proof of Theorem 1.2. By Lemma 2.1, Lemma 4.3 and Lemma 4.7, we are done.

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References


