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d-Refinable (dual) pseudo-splines and their regularities

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In this paper, we introduce the concept of d-dilation (dual) pseudo-splines in $L^2(\mathbb{R})$ with dilation $d \geq 2$, characterize the linear independence of the integer shifts, the subdivision schemes of polynomial reproduction and Hölder regularity for (dual) pseudo-splines with general dilation. We present a new method to determine the Hölder regularity of refinable functions with general dilation $d \geq 2$. Furthermore, we compare the regularities between pseudo-splines and dual ones.

Keywords: (dual) Pseudo-splines; linear independence; polynomial reproduction; Hölder regularity.

AMS Subject Classification: 42C15, 94A12

1. Introduction

It is well known that pseudo-splines, which neatly fill the gap between B-splines and orthogonal Daubechies splines (or the interpolatory Dubuc–Deslauriers functions), not only enrich the theory of refinable functions, but also allow to meet various

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demands for balancing the approximation order, the support size and the regularity in applications. Binary pseudo-splines of type I were introduced in Refs. 5 and 18 in order to construct tight framelets of required approximation order. Dong and Shen[9] introduced Binary pseudo-splines of type II to construct symmetric or anti-symmetric tight framelets with desired approximation orders. Dyn[10] *et al.* studied dual subdivision schemes and introduced binary dual pseudo-splines, which is used to construct symmetric orthonormal complex wavelets with arbitrarily linear-phase moments and sum rules in Ref. 14.

The properties such as the regularity, stability, convergence and linear independence were derived for the binary (dual) pseudo-splines; see Refs. 6, 8, 9, 17, 19 and 20 for example. Moreover, the binary (dual) pseudo-splines were used to construct wavelet frames (real, complex, symmetric) and shearlets; see Refs. 7, 16, 20, 22, 23 and 24 for example. Zhuang[24] further introduced the d-refinable pseudo-splines of order $(m, 2n-1)$ with $2n-1 < m, n, m \in \mathbb{N}_0$. He obtained the linear independence of integer shifts of pseudo-splines and provided an algorithm for the construction of tight wavelet frames with some interesting properties.

Motivated by Refs. 9, 10 and 24, we introduce the general d-refinable (dual) pseudo-splines and discuss their properties such as the independence of integer shifts, polynomial reproduction and regularities in this paper. Let us first introduce some notations and definitions. The symbol d denotes a dilation factor with $d \in \mathbb{Z}$ and $d > 1$. We define $\sigma_d(z) = \left(\frac{1+z+\cdots+z^{d-1}}{d}\right)$ as the d-dilation smoothing factor or the mask symbol of d-refinable B-splines. Bi[1] *et al.* have already used this factor for estimating the Sobolev exponent of d-dilation scaling functions. Conti and Hormann[3] treat this factor as the main reason for polynomial generation of subdivision schemes, which is also derived from refining symbols and have some relationship with Strang–Fix conditions. A function $\phi \in L^2(\mathbb{R})$ is called *d-refinable* if

$$\phi = \sum_{k\in\mathbb{Z}} a(k)\phi(d\cdot - k), \tag{1.1}$$

where a is a (finitely supported) sequence on $\mathbb{Z}$ and satisfies $\sum_{k\in\mathbb{Z}} a(k) = d$. We say a the refinement mask of ϕ. Considering the *Fourier transform* of ϕ,

$$\hat{\phi}(\xi) = \int_{\mathbb{R}} \phi(t)e^{-i\xi t}dt, \quad \xi \in \mathbb{R}.$$

The refinement equation (1.1) can be written as

$$\hat{\phi}(\xi) = \widehat{a}\left(\frac{\xi}{d}\right)\hat{\phi}\left(\frac{\xi}{d}\right),$$

where $\widehat{a}(\xi) = 1/d\sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}, \xi \in \mathbb{R}$, is the refinable symbol of sequence a.

For a refinable function ϕ, its integer shifts are linearly independent if and only if there exists a compactly supported dual refinable function ϕ^d for ϕ (see Ref. 15). The integer shifts of $f \in L^2(\mathbb{R})$ are linearly independent if

$$\sum_{j\in\mathbb{Z}} c(j)f(\cdot - j) = 0 \quad \text{implies } c(j) = 0, \quad \forall j \in \mathbb{Z}.$$

We say that the integer shifts of $f \in L^2(\mathbb{R})$ are stable if there exits $0 < A \leq B < \infty$ such that for any sequence $c \in \ell^2(\mathbb{Z})$

$$A\|c\|_{\ell^2} \leq \left\| \sum_{j\in\mathbb{Z}} c(j)f(\cdot - j) \right\|_{L^2} \leq B\|c\|_{\ell^2}. \tag{1.2}$$

The upper bound of (1.2) always exists for any compactly supported function $f \in L^2(\mathbb{R})$, and the existence of the lower bound is equivalent to

$$(\widehat{f}(\xi + 2\pi k))_{k\in\mathbb{Z}} \neq \mathbf{0}, \quad \forall\, \xi \in \mathbb{R}, \tag{1.3}$$

where $\mathbf{0}$ is the zero sequence in $\ell^2(\mathbb{Z})$. It is also known that the integer shifts of a compactly supported function $f \in L^2(\mathbb{R})$ are linearly independent if and only if

$$(\widehat{f}(\xi + 2\pi k))_{k\in\mathbb{Z}} \neq \mathbf{0}, \quad \forall\, \xi \in \mathbb{C}. \tag{1.4}$$

Consequently, the linear independence of shifts of compactly supported function $f \in L^2(\mathbb{R})$ induces the stability of shifts of f.

The pseudo-splines and dual pseudo-splines are defined based on the following two polynomials:

$$P_{m,n}(y) := \sum_{j=0}^{n} \left[\sum_{j_1+\cdots+j_{d-1}=j} \prod_{k=1}^{d-1} \binom{m-1+j_k}{j_k} \sin\left(\frac{k\pi}{d}\right)^{-2j_k} \right] y^j, \tag{1.5}$$

and

$$\widetilde{P}_{m,n}(y) := \sum_{j=0}^{n} \left[\sum_{j_1+\cdots+j_{d-1}=j} \prod_{k=1}^{d-1} \binom{m-\frac{1}{2}+j_k}{j_k} \sin\left(\frac{k\pi}{d}\right)^{-2j_k} \right] y^j, \tag{1.6}$$

respectively, where $n < m \in \mathbb{N}$. By Riesz Lemma, there exists polynomial $b_{m,n}$ satisfying $|b_{m,n}(z)|^2 = P_{m,n}(\sin^2(\xi/2))$. Now, we give the basic result of this paper about the general (dual) pseudo-splines as follows.

Theorem 1.1. *Let d be a dilation factor. For fixed $n, m \in \mathbb{N}$ with $n < m$, let $P_{m,n}$ and $\widetilde{P}_{m,n}$ be defined as in* (1.5) *and* (1.6), *respectively. Define*

$${}_I\widehat{a}_{m,n}(z) := \sigma_d^m(z) b_{m,n},\ {}_{II}\widehat{a}_{m,n}(z) := |{}_I\widehat{a}_{m,n}(z)|^2,\ \widehat{\widetilde{a}_{m,n}}(z) := z^{-(d-1)/2}|\sigma_d(z)|^{2m}\widetilde{P}_{m,n},$$

and

$$\widehat{{}_I\phi_{m,n}}(\xi) := \prod_{j=1}^{\infty} {}_I\widehat{a}_{m,n}(d^{-j}\xi),\ \widehat{{}_{II}\phi_{m,n}}(\xi) := \prod_{j=1}^{\infty} {}_{II}\widehat{a}_{m,n}(d^{-j}\xi),\ \widehat{\widetilde{\phi}_{m,n}}(\xi) := \prod_{j=1}^{\infty} \widehat{\widetilde{a}_{m,n}}(d^{-j}\xi). \tag{1.7}$$

Then the d-refinable functions ${}_I\phi$, ${}_{II}\phi$ and $\widetilde{\phi}$ are compactly supported scaling function in $L^2(\mathbb{R})$.

Remark 1.2. Since the refinable masks ${}_I\widehat{a}_{m,n}$, ${}_{II}\widehat{a}_{m,n}$ and $\widehat{\widetilde{a}_{m,n}}$ are Laurent polynomials and equal to 1 at point $z = 1$, then ${}_I\phi$, ${}_{II}\phi$ and $\widetilde{\phi}$ are compactly supported distributions. As a result, to obtain Theorem 1.1, we need only to show that (dual) pseudo-splines are in $L^2(\mathbb{R})$. One important method is to calculate the regularity. One may refer to Refs. 6 and 9. This method is also used for binary refinable (dual) pseudo-splines. However, we use the Hölder regularity of the subdivision schemes. Note that this regularity is different from the one used for binary refinable (dual) pseudo-splines. For details, see in Sec. 4. Then following Propositions. 5.3.1 and 5.3.2 in Ref. 4 and the standard processing for construction of orthonormal wavelets, one can obtain that all of ${}_I\phi$, ${}_{II}\phi$ and $\widetilde{\phi}$ are scaling functions.

The structure of this paper is as follows. In Sec. 2, we obtain some important properties of polynomials $P_{m,n}(y)$ and $\widetilde{P}_{m,n}(y)$ and prove the linear independence of (dual) pseudo-splines defined by (1.7). In Sec. 3, we concentrate on the polynomial reproduction and generation of subdivision schemes associated with the mask of (dual) pseudo-splines. Moreover, we prove that the subdivision schemes with symbol $\widehat{a}_{m,n}(z)(\widehat{\widetilde{a}_{m,n}}(z))$ can reproduce polynomial up to degree $\min\{2m-1, 2n+1\}(\min\{2m, 2n+2\})$. In Sec. 4, the regularity of symmetric subdivision schemes with general dilation will be discussed. In the end, different regularity exponents for schemes of (dual) pseudo-splines will be compared.

2. *d*-Refinable (Dual) Pseudo-Splines and the Linear Independence

In this section, we shall give the definition of d-refinable pseudo-splines and d-refinable dual pseudo-splines. Throughout this paper, we denote $z = e^{-i\xi}$. Since $(1-z)(1+z+\cdots+z^{d-1}) = 1-z^d$, then

$$\frac{1}{d^2}|1-z|^2|1+z+\cdots+z^{d-1}|^2 = |1-z^d|^2 = |z^{-d/2}-z^{d/2}|^2.$$

As a result, we have

$$\begin{aligned}\frac{1}{d^2}|1+z+\cdots+z^{d-1}|^2 &= \frac{|z^{-d/2}-z^{d/2}|^2}{|z^{-1/2}-z^{1/2}|^2}\\ &= \frac{\sin^2\left(\dfrac{d\xi}{2}\right)}{\sin^2\left(\dfrac{\xi}{2}\right)} \triangleq h\left(\sin^2\left(\frac{\xi}{2}\right)\right),\end{aligned}$$

where $h(y)$ is some polynomial functions. It is not difficult to find that the zeros of $h(y)$ are $\sin^2(k\pi/d), k = 1, \ldots, d-1$. By $h(0) = 1$, we have

$$h(y) = \prod_{k=1}^{d-1}\left(1 - \frac{y}{\sin^2\left(\dfrac{k\pi}{d}\right)}\right), \quad y \in \mathbb{R}.$$

Note that for fixed integer $N > 0$,

$$(1-x)^{-N} = \sum_{k=0}^{\infty} \binom{N-1+k}{k} x^k, \quad |x| < 1.$$

One obtains

$$h(y)^{-m} = \prod_{k=1}^{d-1} \left(\sum_{j_k=0}^{\infty} \binom{m+j_k-1}{j_k} \frac{y^{j_k}}{\sin^{2j_k}\left(\frac{k\pi}{d}\right)} \right) = \sum_{j=0}^{\infty} c_{m,j} y^j, \quad |y| < \sin^2\left(\frac{\pi}{d}\right) \tag{2.1}$$

with

$$c_{m,j} = \sum_{j_1+\cdots+j_{d-1}=j} \prod_{k=1}^{d-1} \binom{m-1+j_k}{j_k} \sin\left(\frac{k\pi}{d}\right)^{-2j_k}, \quad j \in \mathbb{N}. \tag{2.2}$$

Let $P_{m,n}$ be the nth-degree Taylor polynomial of $h(y)^{-m}$ at $x = 0$, that is,

$$P_{m,n}(y) = \sum_{j=0}^{n} c_{m,j} y^j$$

with $0 \le n \le m-1$. Similarly, define the nth-degree polynomial $\widetilde{P}_{m,n}$ from $h(y)^{-1/2-m}$ with

$$h(y)^{1/2-m} = \sum_{j=0}^{\infty} \tilde{c}_{m,j} y^j, \quad |y| < \sin^2\left(\frac{\pi}{d}\right)$$

and

$$\tilde{c}_{m,j} = \sum_{j_1+\cdots+j_{d-1}=j} \prod_{k=1}^{d-1} \binom{m-\frac{1}{2}+j_k}{j_k} \sin\left(\frac{k\pi}{d}\right)^{-2j_k}, \quad j \in \mathbb{N},$$

where the binomials are defined as $\binom{m-1/2+j}{j} = \frac{(m-1/2+j)(m-3/2+j)\cdots(m+1/2)}{j!}$. Note that most of the properties of ${}_I\phi$ are similar to those of ${}_{II}\phi$, we only discuss the pseudo-spline of type II in this paper. For simplicity, we redefine the general d-refinable mask of the pseudo-spline (of type II) $\phi_{m,n}$ with $0 \le n \le m-1$ as

$$\hat{a}_{m,n}(\xi) = \left| \frac{1+\cdots+z^{d-1}}{d} \right|^{2m} P_{m,n}\left(\sin^2\left(\frac{\xi}{2}\right)\right) \tag{2.3}$$

and the general d-refinable mask of the dual pseudo-spline $\tilde{\phi}_{m,n}$ with $0 \le n \le m-1$ as

$$\widehat{\tilde{a}_{m,n}}(\xi) = \frac{1+\cdots+z^{d-1}}{dz^{(d-1)/2}} \left| \frac{1+\cdots+z^{d-1}}{d} \right|^{2m} \widetilde{P}_{m,n}\left(\sin^2\left(\frac{\xi}{2}\right)\right), \tag{2.4}$$

respectively.

Next, we discuss the stability and linear independence of shifts for the general (dual) pseudo-splines. The linear independence of shifts were discussed in Ref. 24 for a special class of d-refinable pseudo-splines of type II, which is ${}_{II}\phi_{m,2k-1}$ of integer k satisfied $0 \leq 2k-1 \leq m-1$. So we just show the linear independence of shifts of $\tilde{\phi}_{m,n}$.

It is known that of Ref. 8 Proposition 2.3 for a polynomial $P(z) = c_0 + c_1 z + \cdots + c_n z^n$ with real coefficients satisfying $c_n > c_{n-1} > \cdots > c_0 > 0$, all the zeros of $P(z)$ contain in the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. We also need the following lemma.

Lemma 2.1. *Let $f \in L^2(\mathbb{R})$ be a compactly supported d-refinable function associated with a low-pass filter a. Then, the shifts of f are linearly independent if and only if*

(1) *the shifts of f are stable*;
(2) *the symbol a of f satisfies*: $\{a(ze^{i2\pi j/d}) : j = 0, \ldots, d-1\} \neq 0$ *for all* $z \in \mathbb{C}$.

Theorem 2.2. *The shifts of any dual pseudo-spline are linearly independent.*

Proof. It is plain to get the stable pseudo-splines by (1.3) and the fact

$$\widehat{\tilde{\phi}_{m,n}}(\xi) \geq \widehat{B}_{2m}(\xi)$$

for all $\xi \in \mathbb{R}$, where B_{2m} denotes the d-refinable B-spline of order $2m$. We rewrite the Laurent polynomial $\widetilde{P}_{m,n}$ as

$$\widetilde{P}_{m,n} = \sum_{j=0}^{n} 2^{-j} \tilde{c}_{m,j} \left(2\sin^2\left(\frac{\xi}{2}\right)\right)^j = \sum_{j=0}^{n} 2^{-j} c_{m,j} (1 - \cos\xi)^j. \tag{2.5}$$

Obviously, $\xi = 2\pi j/d$ is a zero of $\widehat{\tilde{a}_{m,n}}(\xi)$ with $j = 1, \ldots, d-1$. Noting that $\widehat{\tilde{a}_{m,n}}(0) = 1$, it is sufficient to show that $\{\widetilde{P}_{m,n}(ze^{i2\pi j/d}) : j = 0, \ldots, d-1\} \neq 0$ for all $z \in \mathbb{C}$. Suppose that $\widetilde{P}_{m,n} = \sum_{j=0}^{n} 2^{-j} c_{m,j} (1 - \mathrm{Re}(z))^j$ has zeros at $z_j = \rho e^{i\theta_j}$ with $j = 0, \ldots, d-1$, where $\theta_j = \theta + 2\pi j/d$ and $\mathrm{Re}(z)$ denotes the real part of z. Note that

$$\max_{0 \leq j \leq d-1} |1 - \mathrm{Re}(z_j)| = \max_{0 \leq j \leq d-1} (1 - \rho\cos(\theta_j)) \geq 1,$$

and

$$2^{-j+1} \tilde{c}_{m,j-1} < 2^{-j} \tilde{c}_{m,j}, \quad \text{for } j = 1, \ldots, n-1.$$

Similar to the proof of [Lemma 3 of Ref. 24], one obtains a contradiction to the fact that all roots of the polynomial $\sum_{j=0}^{n-1} 2^{-j} \tilde{c}_{m,j} z^j$ are contained inside the unit disk $\{z \in \mathbb{C} : |Z| < 1\}$. ☐

In the rest of this section, we give some interesting propositions.

Proposition 2.3. *For fixed integers d and m, define*

$$E_{m,h,n} := \sum_{j_1+\cdots+j_{d-1}=n} \binom{m+j_h}{j_h} \sin^{-2j_h}\left(\frac{h\pi}{d}\right) \times \prod_{k=1,k\neq h}^{d-1} \binom{m-1+j_k}{j_k} \sin^{-2j_k}\left(\frac{k\pi}{d}\right) \tag{2.6}$$

and

$$F_{m,n} = \sum_{j_1+\cdots+j_{d-1}=n} \prod_{k=1}^{d-1} \binom{m-1+j_k}{j_k} \sin^{-2j_k}\left(\frac{k\pi}{d}\right)$$

with the integers h and n satisfying $1 \leq h \leq d-1$ and $0 \leq n \leq m-1$. Then

$$E_{m,h,n} = \sum_{k=0}^{n} \sin^{-2(n-k)}\left(\frac{h\pi}{d}\right) F_{m,k}. \tag{2.7}$$

Proof. For convenience, define

$$\mathbb{J}_{\{k\neq h\}} := \sum_{k=1,k\neq h}^{d-1} j_k \quad \text{and} \quad G_{h,n} := \prod_{k=1,k\neq h}^{d-1} \binom{m-1+j_k}{j_k} \sin^{-2j_k}\left(\frac{k\pi}{d}\right)$$

with $j_1+\cdots+j_{d-1}=n$. By induction, suppose that (2.7) holds for $n-1$. Next, we establish (2.7) is true for n. By direct calculation, one obtains

$$\begin{aligned}
E_{m,h,n} &= \sum_{j_h=0}^{n} \sum_{\mathbb{J}_{\{k\neq h\}}=n-j_h} \binom{m+j_h}{j_h} \sin^{-2j_h}\left(\frac{h\pi}{d}\right) G_{h,n} \\
&= \sum_{j_h=1}^{n} \sum_{\mathbb{J}_{\{k\neq h\}}=n-j_h} \left(\binom{m+j_h-1}{j_h} + \binom{m+j_h-1}{j_h-1}\right) \sin^{-2j_h}\left(\frac{h\pi}{d}\right) G_{h,n} \\
&\quad + \sum_{\mathbb{J}_{\{k\neq h\}}=n} G_{h,n} \\
&= F_{m,n} + \sin^{-2}\left(\frac{h\pi}{d}\right) \sum_{j_h=1}^{n} \sum_{\mathbb{J}_{\{k\neq h\}}=n-j_h} \binom{m+j_h-1}{j_h-1} \sin^{-2(j_h-1)}\left(\frac{h\pi}{d}\right) G_{h,n} \\
&= F_{m,n} + \sin^{-2}\left(\frac{h\pi}{d}\right) E_{m,h,n-1} = \sum_{k=0}^{n} \sin^{-2(n-k)}\left(\frac{h\pi}{d}\right) F_{m,k}.
\end{aligned} \tag{2.8}$$

Thus, we obtain the required results. □

Proposition 2.4. *Let $E_{m,h,n}$ and $F_{m,n}$ be defined as in Proposition* 2.3. *Define*

$$M_{d,m} := \frac{n-1}{m}\sin^{-2}\left(\frac{\pi}{d}\right) + \sum_{h=1}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right). \tag{2.9}$$

Then

$$\frac{n}{m}F_{m,n} = \sum_{h=1}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right)E_{m,h,n-1} \quad \textit{and} \quad \frac{n}{m}F_{m,n} \le M_{d,m}F_{m,n-1}.$$

Furthermore,

$$M_{d,m} \le \frac{2m+n-1}{m}\sin^{-2}\left(\frac{\pi}{d}\right) + \frac{d}{\pi}\int_{\frac{3\pi}{2d}}^{\frac{(2d-3)\pi}{2d}} \sin^{-2} x dx.$$

Proof. By directly calculating, we have

$$\begin{aligned}\sum_{h=1}^{d-1} E_{m,h,n} &= \sum_{h=1}^{d-1}\sum_{j_1+\cdots+j_{d-1}=n}\frac{m+j_h}{m}\prod_{k=1}^{d-1}\binom{m-1+j_k}{j_k}\sin^{-2j_k}\left(\frac{k\pi}{d}\right)\\ &= \sum_{j_1+\cdots+j_{d-1}=n}\sum_{h=1}^{d-1}\frac{m+j_h}{m}\prod_{k=1}^{d-1}\binom{m-1+j_k}{j_k}\sin^{-2j_k}\left(\frac{k\pi}{d}\right)\\ &= \frac{m(d-1)+n}{m}F_{m,n}.\end{aligned}$$

This together with (2.8) deduces

$$\frac{n}{m}F_{m,n} = \sum_{h=1}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right)E_{m,h,n-1}$$

since

$$\begin{aligned}\frac{n}{m}F_{m,n} = &\sum_{j_1+\cdots+j_{d-1}=n-1}\sum_{h=1}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right)\frac{m+j_h}{m}\\ &\times\prod_{k=1}^{d-1}\binom{m-1+j_k}{j_k}\sin^{-2j_k}\left(\frac{k\pi}{d}\right).\end{aligned} \tag{2.10}$$

Note that

$$\sum_{h=1}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right)\frac{m+j_h}{m} \le \frac{n-1}{m}\sin^{-2}\left(\frac{\pi}{d}\right) + \sum_{h=1}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right).$$

We obtain that

$$\frac{n}{m}F_{m,n} \le M_{d,m}F_{m,n-1}.$$

It is well known that for convex function $f(x)$ defined on $[a,b]$, the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx$$

holds. Thus,

$$\sin^{-2}\left(\frac{h\pi}{d}\right) \le \frac{d}{\pi}\int_{\frac{(2h-1)\pi}{2d}}^{\frac{(2h+1)\pi}{2d}} \sin^{-2} x dx$$

for $1 \le h \le d-2$, $0 \le k \le n-1$ and $d \ge 3$, since function $\sin^{-2} x$ is convex on $x \in (0, 2\pi)$. Therefore, we obtain

$$M_{d,m} \le \frac{2m+n-1}{m}\sin^{-2}\left(\frac{\pi}{d}\right) + \frac{d}{\pi}\int_{\frac{3\pi}{2d}}^{\frac{(2d-3)\pi}{2d}} \sin^{-2(n-k)}(x)dx$$

and the proof is complete. □

For $\widetilde{P}_{m,n}(y)$ with $y = \sin^2(\xi/2)$, we have the following.

Proposition 2.5. *For fixed integers d and m, define*

$$\widetilde{E}_{m,h,n} := \sum_{j_1+\cdots+j_{d-1}=n} \binom{m-\frac{1}{2}+j_h}{j_h} \sin^{-2j_h}\left(\frac{h\pi}{d}\right) \times \prod_{k=1,k\neq h}^{d-1} \binom{m-\frac{1}{2}+j_k}{j_k} \sin^{-2j_k}\left(\frac{k\pi}{d}\right)$$

and

$$\widetilde{F}_{m,n} = \sum_{j_1+\cdots+j_{d-1}=n} \prod_{k=1}^{d-1} \binom{m-\frac{1}{2}-j_h}{j_h} \sin^{-2j_k}\left(\frac{k\pi}{d}\right)$$

with the integers h and n satisfying $1 \le h \le d-1$ and $0 \le n \le m-1$. Then

$$\widetilde{E}_{m,h,n} = \sum_{k=0}^{n} \sin^{-2(n-k)}\left(\frac{h\pi}{d}\right)\widetilde{F}_{m,k}. \tag{2.11}$$

Moreover,

$$\frac{n}{m}\widetilde{F}_{m,n} = \sum_{h=1}^{d-1} \sin^{-2}\left(\frac{h\pi}{d}\right)\widetilde{E}_{m,h,n-1} \quad \textit{and} \quad \frac{n}{m}\widetilde{F}_{m,n} \le M_{d,m}\widetilde{F}_{m,n-1}, \tag{2.12}$$

where $M_{d,m}$ is defined as in (2.9).

3. Polynomial Reproduction for the *d*-Refinable (Dual) Pseudo-Splines

In this section, we shall concentrate on the polynomial reproduction property of subdivision schemes of d-refinable (dual) pseudo-splines. Polynomial reproduction of subdivision schemes is different from polynomial generation and their relations are systematically discussed in Ref. 3. A convergent subdivision scheme generates the space of polynomial of degree up to r, if and only if its mask satisfies sum rules of order $r+1$, or its symbol is divided by polynomial $\sigma_d(z)^{r+1}$. This is the so-called polynomial generation property, which is just a necessary condition of the associated refinable function having approximation order $d+1$. A sufficient condition for having approximation order $d+1$ is the property of polynomial reproduction, reproducing in the limit of subdivision scheme the same polynomial up to degree of d from which the data is sampled. These properties have been expanded to multivariate scalar subdivision schemes in Ref. 2. The subdivision scheme is the recursive algorithm based on subdivision operator, S_a,

$$f_{\ell+1,k} = S_a(f_{\ell,k}) := \sum_{j\in\mathbb{Z}} a_{k-dj} f_{\ell,j}, \quad k \in \mathbb{Z} \tag{3.1}$$

with finitely supported mask $a = (a_k)_{k\in\mathbb{Z}} \in \mathbb{R}$ acting on the initial data $f_0 = \{f_{0,k}, k \in \mathbb{Z}\}$. Define piecewise linear function f_ℓ by attaching the data $f_{\ell,k}$ to the parameter values t_k^ℓ with $t_{k+1}^\ell - t_k^\ell = d^{-\ell}$ for $k \in \mathbb{Z}, \ell \in \mathbb{N}$ and requiring that the piecewise linear functions with values $f_{\ell,k}(t_k^\ell)$. The scheme is convergent if it has pointwise limit $f = \lim_{\ell\to\infty} f_\ell$. We will always choose the initial data $f_0(t_k^0)$ to be δ_k, which denotes Dirac sequence.

While the parameter values $t_k^\ell, k \in \mathbb{Z}$ do not affect most properties of the subdivision schemes, e.g., its convergence, smoothness or its support size, it is crucial for polynomial reproduction of degree $d \geq 0$. The associated parametrization is determined by a suitable parametric shift $\tau \in \mathbb{R}$,

$$t_k^\ell = t_0^\ell + \frac{k}{d^\ell}, \quad t_0^\ell = t_0^{\ell-1} - \frac{\tau}{d^\ell}, \quad t_0^{(0)} = 0, \quad k \in \mathbb{Z}, \quad 0 \leq \ell \in \mathbb{N}. \tag{3.2}$$

The following lemma derives from [Theorem 4.3, Proposition 4.5 of Ref. 3] which provides a slightly simpler way to determine the degree of polynomial reproduction for some given subdivision scheme with general dilation $d \in \mathbb{N}$.

Lemma 3.1. *A convergent subdivision scheme S_a with subdivision symbol $a(z)$ reproduces polynomials of degree r with respect to the parametrization in* (3.2) *if and only if $a(z)$ satisfies sum ruler of order $r+1$ and $b(z) = a(z^d)z^{-d\tau}$ satisfies*

$$b(1) = 1, \quad \text{and} \quad b^{(k)}(1) = 0, \quad k = 1, \ldots, d \tag{3.3}$$

with $b^{(k)}$ denotes the k derivative of $b(z)$, which in turn is equivalent to require that $b(z) = (1-z)^{r+1}c(z) + 1$ for some $c(z)$.

With this lemma in hand, we can get the polynomial reproduction properties of subdivision schemes with symbols $\hat{a}_{m,n}(z)$ and $\widehat{\tilde{a}_{m,n}}(z)$.

Theorem 3.2. *The primal subdivision schemes with symbol $\hat{a}_{m,n}(z)$ reproduce polynomial up to degree* $\min(2m-1, 2n+1)$. *The dual subdivision schemes with symbol $\widehat{\tilde{a}_{m,n}}(z)$ reproduce polynomial up to degree* $\min(2m, 2n+1)$.

Proof. It follows directly from (2.3) that $\hat{a}_{m,n}(z)$ is divisible by $(1 + z + \cdots + z^{d-1})^{2m}$, but the rest $P_n(z)$ is not, hence the scheme generates polynomial of degree at most $2m-1$. Let $\delta(z) = -\frac{(1-z)^2}{4z} = \sin^2(\xi/2)$ with $z = e^{-i\xi}$. Note that $P_n(\delta(z))$ be the nth-degree Taylor polynomial of $h(\delta(z))^{-m}$, $P_n(\delta(z)) = h(\delta(z))^{-m} - \sum_{j=n+1}^{\infty} c_{m,n}\delta^j(z)$ with $c_{m,n}$ defined as in (2.2). We have

$$\hat{a}_{m,n}(z) = h(\delta(z))^m P_n(\delta(z)) = 1 + h(\delta(z))^m \sum_{j=n+1}^{\infty} c_{m,n}\delta^j(z).$$

For the primal subdivision scheme, the parametric shift satisfies $\tau = 0$. Then let $b(z) = \hat{a}_{m,n}(z^d) - 1$. Noting $\sigma_d(z) = (\frac{1+z+\cdots+z^{d-1}}{d})$, we have

$$\begin{aligned} b(z) &= h(\delta(z^d))^m \sum_{j=n+1}^{\infty} c_{m,n}\delta^j(z^d) \\ &= |\sigma_d(z^d)|^{2m} \sum_{j=n+1}^{\infty} c_{m,j}\left(\frac{(1-z)^2(1+z+\cdots+z^{d-1})^2}{-4z^d}\right)^j \\ &= (1-z)^{2n+2}|\sigma_d(z^d)|^{2m}\left(\frac{1+\cdots+z^{d-1}}{2iz^{d/2}}\right)^{2n+2}\sum_{j=0}^{\infty} c_{m,n+1+j}\delta^j(z^d). \end{aligned} \quad (3.4)$$

From Lemma 3.1 and $\frac{b(z)}{(1-z)^{2n+2}}|_{z=1} = c_{m,n+1}(-d^2/4)^{n+1} \neq 0$, we claim that the primal subdivision schemes with symbol $\hat{a}_{m,n}(z)$ reproduce polynomial up to degree $\min(2m, 2n+2)$.

Similarly, it is clear that the dual subdivision scheme with symbol $\widehat{\tilde{a}_{m,n}}(z)$ generates polynomial of degree at most $2m$. Note that the parametric shift satisfies $\tau = -1/2$ and

$$\begin{aligned} \widehat{\tilde{a}_{m,n}}(z) &= \frac{\sigma_d(z)}{z^{(d-1)/2}} h(\delta(z))^m \tilde{P}_n(\delta(z)) \\ &= h(\delta(z))^{m+1/2}\tilde{P}_n(\delta(z)) = 1 + h(\delta(z))^{m+1/2}\sum_{j=n+1}^{\infty}\tilde{c}_{m,n}\delta^j(z), \end{aligned}$$

we have

$$\begin{aligned} \tilde{b}(z) &= z^{d/2}\hat{a}_{m,n}(z^d) - 1 = h(\delta(z^d))^{m+1/2}\sum_{j=n+1}^{\infty}\tilde{c}_{m,n}\delta^j(z^d) \\ &= |\sigma_d(z)|^{2m+1}\sum_{j=n+1}^{\infty}\tilde{c}_{m,j}\left(\frac{(1-z)^2(1+z+\cdots+z^{d-1})^2}{-4z^d}\right)^j \\ &= (1-z)^{2n+2}|\sigma_d(z)|^{2m+1}\left(\frac{1+z+\cdots+z^{d-1}}{2iz^{d/2}}\right)^{2n+2}\sum_{j=0}^{\infty}\tilde{c}_{m,n+1+j}\delta^j(z^d). \end{aligned} \quad (3.5)$$

Now, we get the desired result from Lemma 3.1 and the fact

$$\left.\frac{\tilde{b}(z)}{(1-z)^{2n+2}}\right|_{z=1} = \tilde{c}_{m,n+1}\left(\frac{-d}{4}\right)^{n+1} \neq 0. \qquad \square$$

4. Exact Regularity of Symmetric Univariate Subdivision Schemes and the Application

In this section, we are concerned with the exact regularity of symmetric univariate subdivision schemes and applied the result to compare the regularities between different (dual) pseudo-splines. First, let us review some notations and results related to the characterization of regularity of refinable functions.

The Sobolev exponent and the Hölder exponent are two important regularity exponents. The critical *Sobolev exponent* $s_2(f)$ of the scaling function $f \in L^2(\mathbb{R})$ is defined by

$$s_2(f) = \sup\left\{\gamma : \int_{\mathbb{R}} |\hat{f}(\xi)|^2(1+|\xi|^2)^\gamma d\xi < \infty\right\}, \tag{4.1}$$

and the *Hölder exponent* $\alpha(f)$ of a continuous function f is defined by

$$\alpha(f) = \sup\{\gamma : f \in C^\gamma\}, \tag{4.2}$$

where C^γ denotes the usual Hölder class. It is well known that $\alpha(f) < s_2(f) - 1/2$. Many researchers were interesting in estimating the Sobolev exponent and the Hölder exponent of the refinable distribution ϕ. Actually, $s_2(\phi)$ is commonly expressed in terms of the spectral radius of a transfer operator and measured by ℓ^2-norm joint spectral radius; for more details, see Refs. 1, 12, 13, 21. As already shown in Remark 1.2, we can derive that $\phi \in L^2(\mathbb{R})$ by checking $s_2(\phi) > 0$ with (dual) pseudo-splines ϕ. However, the convergence of the subdivision scheme will be studied in $(L^\infty, \|\cdot\|_\infty)$, which leads to the characterization of Hölder regularity of the subdivision scheme (3.1) and corresponding refinable functions in the sequel.

We say that the subdivision scheme (3.1) has Hölder regularity γ for some real $\gamma \geq 0$, if $f \in C^\beta$ for all initial data and for any $\beta < \gamma$, while $f \notin C^\beta$ for some initial data and for any $\beta > \gamma$. We suppose, after shifting if necessary, that symmetric finitely supported mask symbol $a(z)$ can be factorized as

$$a(z) = \sigma_d(z)^{r+1} b(z) \tag{4.3}$$

for some $r \geq 0$, where $b(z)$ is a symmetric Laurent polynomial, that is $b(z) := \sum_{k=-p}^{p} b_k z^k$, $p \in \mathbb{N}$ with coefficients $b_k \in \mathbb{R}$ and $b_{-k} = b_k$. For integer $s \geq 0$, denote $f_{\ell,k}^{[s]}$ the divided difference of the values $f_{\ell,k-s}, \ldots, f_{\ell,k}$ at the corresponding parameter values $t_{k-s}^\ell, \ldots, t_k^\ell$. Then

$$f_{\ell,k}^{[0]} = f_{\ell,k}$$

and

$$f_{\ell,k}^{[s]} = \frac{d^\ell}{s}\left(f_{\ell,k}^{[s-1]} - f_{\ell,k-1}^{[s-1]}\right), \quad s = 1, \ldots, r+1.$$

Define the associated Laurent polynomial $f_\ell^{[s]}(z)$ as

$$f_\ell^{[s]}(z) = \sum_k f_{\ell,k}^{[s]} z^k.$$

Then we have $f_\ell^{[s]}(z) = a^{[s]}(z)f_\ell^{[s]}(z^d)$ and $f_{\ell+1,k}^{[s]} = \sum_j a_{k-dj}^{[s]} f_{\ell,j}^{[s]}$ with $a^{[s]}(z) = \sigma_d(z)^{-s}a(z)$.

Further, define

$$g_{\ell,k} := f_{\ell,k}^{[r]} - f_{\ell,k-1}^{[r]} \quad \text{and} \quad g_\ell(z) := \sum_k g_{\ell,k} z^k.$$

Then

$$g_\ell(z) = b(z)g_{\ell-1}(z^d) = b_\ell(z)g_0(z^{d^\ell})$$

with $b_\ell(z) = b(z)b(z^d)\cdots b(z^{d^{\ell-1}})$. It is easy to see that $b_\ell(z)$ is the symbol of $b_{\ell,k}$ with $b_{0,k} = \delta_k$ and $b_{\ell+1,k} = \sum_j b_{k-dj}b_{\ell,j}$. Therefore,

$$g_{\ell,k} = \sum_j b_{\ell,k-d^\ell j}g_{0,j}. \tag{4.4}$$

One can show that if

$$\sup_{k\in\mathbb{Z}} |g_{\ell,k}^r| \le C\lambda^\ell \tag{4.5}$$

for some constant C, $\lambda < 1$ and sufficiently large ℓ, then the subdivision scheme converges and $f \in C^r$. Moreover, if $1/d < \lambda < 1$, then $f \in C^{r-\log_d(\lambda)}$. In fact, if (4.5) holds for any λ with $1/d < \lambda < d^r$, then $f \in C^{r-\log_d \lambda}$. For the proof of $d = 2$, one can refer Lemma 1 of Ref. 11.

Noting the relationship between the finite symmetric sequence and its Fourier series, we have the following Lemma 4.1.

Lemma 4.1. *Let $b(z)$ be defined as in* (4.3) *and $B(\xi) := b(e^{-i\xi}) = b_0 + 2\sum_{j=1}^p b_j \cos(j\xi) \ge 0$ for all $\xi \in [-\pi, \pi]$. Then*

$$\max_k |b_{\ell,k}| = b_{\ell,0}, \quad \forall j \ge 0.$$

By (4.4) and Lemma 4.1, it follows that (4.3) holds if $b_{\ell,0} \le C\lambda^\ell$. Define

$$M = (b_{k-dj})_{k,j=-q,\ldots,q} \tag{4.6}$$

with $q = \lfloor \frac{p-1}{d-1} \rfloor$($\lfloor x \rfloor$ denotes the biggest integer not greater than $x \in \mathbb{R}$) and the vector of coefficients

$$\mathbf{b}_\ell = (b_{\ell,-q}, \ldots, b_{\ell,q})^T$$

generated by $\mathbf{b}_{\ell+1} = M\mathbf{b}_\ell$. Moreover, we have the following proposition, which generalizes the main results for the general (dual) pseudo-splines with dilation $d \ge 2$. The proof is similar to [Theorems 1 and 2 of Ref. 11] and we delete it here.

Proposition 4.2. *Let $b(z)$ and $B(\xi)$ be given as in Lemma* 4.1*. Then*

$$\lim_{j\to\infty} b_{\ell,0}^{1/\ell} = \rho, \tag{4.7}$$

where ρ is the spectral radius of M. Moreover, if $\rho > 1/d$, then a lower bound for the regularity of the corresponding subdivision scheme is $r - \log_d \rho$. Further, the lower bound is optimal if $B(\xi) > 0$ for all $\xi \in [-\pi, \pi]$.

Table 1. Regularities for $\widehat{a}_{m,n}$ with dilation $d = 3, 4$.

		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
	$m=2$	1.8173						
	$m=3$	3.5350	2.3199					
	$m=4$	5.3199	3.8940	2.5900				
$d=3$	$m=5$	7.1460	5.5486	4.0825	2.7330			
	$m=6$	9.0000	7.2581	5.6547	4.1782	2.8267		
	$m=7$	10.8743	9.0075	7.2850	5.6972	4.2417	2.8999	
	$m=8$	12.7638	10.7872	8.9594	7.2727	5.7239	4.2936	2.9618
	$m=2$	1.7075						
	$m=3$	3.4563	2.0996					
	$m=4$	5.2703	3.7324	2.3515				
$d=4$	$m=5$	7.1226	5.4399	3.9241	2.5497			
	$m=6$	9.0000	7.1968	5.5676	4.0862	2.7228		
	$m=7$	10.8953	8.9888	7.2616	5.6872	4.2351	2.8809	
	$m=8$	12.8038	10.8071	8.9937	7.3370	5.8059	4.3757	3.0281

Table 2. Regularities for $\widehat{\tilde{a}_{m,n}}$ with dilation $d = 3, 4$.

		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
	$m=2$	2.6653						
	$m=3$	4.4211	3.0947					
	$m=4$	6.2288	4.7132	3.3245				
$d=3$	$m=5$	8.0701	6.3976	4.8603	3.4449			
	$m=6$	9.9350	8.1285	6.4636	4.9297	3.5247		
	$m=7$	11.8173	9.8940	8.1174	6.4787	4.9754	3.5883	
	$m=8$	13.7132	11.6863	9.8101	8.0778	6.4858	5.0143	3.6431
	$m=2$	2.5710						
	$m=3$	4.3573	2.9043					
	$m=4$	6.1926	4.5787	3.1273				
$d=4$	$m=5$	8.0587	6.3132	4.7385	3.3086			
	$m=6$	9.9457	8.0891	6.4092	4.8798	3.4707		
	$m=7$	11.8481	9.8951	8.1235	6.5068	5.0141	3.6209	
	$m=8$	13.7621	11.7241	9.8712	8.1766	6.6091	5.1437	3.7623

By Proposition 4.2, we can calculate the regularities of schemes defined by (dual) pseudo-splines, see Tables 1 and 2. Moreover, for (dual) pseudo-splines ϕ, we can obtain $\phi \in L^2(\mathbb{R})$ by calculating $r - \log_d \rho > 0$.

Next, we require some pre-requisites. For integers $r, \tilde{r} \geq 0$, consider two subdivision schemes defined by the mask symbols

$$\hat{a}(z) := \sigma_d(z)^{r+1}\hat{b}(z) \quad \text{and} \quad \hat{\tilde{a}}(z) := \sigma_d(z)^{\tilde{r}+1}\hat{\tilde{b}}(z), \tag{4.8}$$

respectively, where the symmetric Laurent polynomials $\hat{b}(z)$ and $\hat{\tilde{b}}(z)$ are defined as in (4.3) with $B(\xi) := \hat{b}(e^{-i\xi}) > 0$ and $\tilde{B}(\xi) := \hat{\tilde{b}}(e^{-i\xi}) > 0$ for all $\xi \in [-\pi, \pi]$. Let γ and $\tilde{\gamma}$ be the corresponding regularities. To compare with the regularities between d-refinable primal and dual pseudo-splines, the following lemma is needed. The proof is similar to [Lemma 5 of Ref. 11].

Lemma 4.3. *Let $b(z)$ and $\widetilde{b}(z)$ be defined as in* (4.8). *If there exists a constant $C \geq 1$ such that*

$$\widetilde{B}(\xi) \leq CB(\xi), \quad \xi \in [-\pi, \pi],$$

then $\tilde{\gamma} \geq \gamma + \tilde{r} - r - \log_d C$.

For integer d, we denote $\mathrm{odd}(d) = 1$ if d is odd and $\mathrm{odd}(d) = 0$ if d is even. We now present some properties of regularity for (dual) pseudo-splines. Note that similar properties have been discussed in Refs. 11, 9 and 6 for $d = 2$.

Theorem 4.4. *Let $\gamma_{m,n}$ be the regularity of the d-refinable pseudo-splines scheme defined by $\widehat{a}_{m,n}(\xi)$ and denote $\gamma_{m,m-1}$ by symbol γ_m. Then*

(i) *$\gamma_{m,n}$ is decreasing with the variable n; furthermore,*

$$\gamma_{m,n-1} - \log_d(C) \leq \gamma_{m,n} \leq \gamma_{m,n-1}$$

with

$$C := \begin{cases} 1 + D, & d > 2, \\ \dfrac{m+n}{n}, & d = 2, \end{cases}$$

and

$$D := \left[\frac{n-1}{n} \sin^{-2}\left(\frac{\pi}{d}\right) + \frac{m}{n} \sum_{h=1}^{d-1} \sin^{-2}\left(\frac{h\pi}{d}\right)\right] \times \left(\sin^{-2(n-1)}\left(\frac{(d-1)\pi}{2d}\right)\right)^{\mathrm{odd(d)}}.$$

(ii) *$\gamma_{m,n}$ is increasing with the variable m and*

$$\gamma_{m,n} + 2 - (d-1)\log_d\left[1 + \frac{n}{m(d-1)}\right] \leq \gamma_{m+1,n} \leq \gamma_{m,n} + 2.$$

(iii) *γ_m is increasing with the variable m for $d = 2$ and*

$$\gamma_m + 2 - \log_d(C) \leq \gamma_{m+1} \leq \gamma_m + 2$$

with

$$C := \begin{cases} \left(1 + \dfrac{m-1}{m(d-1)}\right)^{d-1} [1 + D], & d > 2 \\ \dfrac{2(2m+1)}{m+1}, & d = 2 \end{cases}$$

and

$$D := \left[2\sin^{-2}\left(\frac{\pi}{d}\right) + \frac{m+1}{m}\sum_{h=2}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right)\right] \times \left(\sin^{-2(m-1)}\left(\frac{(d-1)\pi}{2d}\right)\right)^{\mathrm{odd}(d)}.$$

Proof. Part (i) follows from Lemma 4.3 and Proposition 2.4. Since $P_{m,n-1}(\xi) \leq P_{m,n}(\xi)$ for $\xi \in [-\pi, \pi]$, Lemma 4.3 implies $\gamma_{m,n}$ is decreasing with the variable n, then the second inequality can be easy to check. To prove the first inequality, we look for a constant $C \geq 1$ such that

$$P_{m,n-1}(\xi) \leq CP_{m,n}(\xi), \quad \xi \in [-\pi, \pi], \tag{4.9}$$

or equivalently, such that

$$p(y) := C\sum_{j=0}^{n-1} F_{m,j}y^j - \sum_{j=0}^{n} F_{m,j}y^j \geq 0, \quad 0 \leq y \leq 1.$$

Letting

$$c_j := (C-1)F_{m,j}, \quad 0 \leq j \leq n-1,$$

we have

$$p(y) = \sum_{j=0}^{n-1} c_j y^j - F_{m,n}y^n = \sum_{j=0}^{n-1} c_j(y^j - y^n) - c_n y^n$$

with $c_n = \sum_{j=0}^{n-1} c_j - F_{m,n}$. For $y \in [0,1]$, $p(y) \geq 0$ if $c_j \geq 0, 0 \leq j \leq n$. Since $c_j \geq 0$ if $C > 1$ for $0 \leq j \leq n-1$, we only need to prove that $c_n \geq 0$.

When d is even, by Proposition 2.3 and $h = d/2$, we have

$$\sum_{k=0}^{n-1} F_{m,k} = E_{m,d/2,n-1}.$$

This together with (2.12) follows:

$$\begin{aligned} c_n &= \sum_{j=0}^{n-1} c_j - F_{m,n} \\ &= \sum_{j_1+\cdots+j_{d-1}=n-1}\left[(C-1)\frac{m+j_{d/2}}{m} - \sum_{\ell=1}^{d-1}\frac{m+j_\ell}{n}\sin^{-2}\left(\frac{\ell\pi}{d}\right)\right] \\ &\quad \times \prod_{k=1}^{d-1}\binom{m-1+j_k}{j_k}\sin^{-2j_k}\left(\frac{k\pi}{d}\right). \end{aligned} \tag{4.10}$$

Similar to Proposition 2.4, we have

$$\frac{m}{n}\sum_{\ell=1}^{d-1}\frac{m+j_\ell}{m+j_{d/2}}\sin^{-2}\left(\frac{\ell\pi}{d}\right) \leq \frac{n-1}{n}\sin^{-2}\left(\frac{\pi}{d}\right) + \frac{m}{n}\sum_{h=1}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right)$$

for even integer $d \geq 4$. Thus, $c_n \geq 0$ holds for

$$C := \begin{cases} 1 + \left[\dfrac{n-1}{n}\sin^{-2}\left(\dfrac{\pi}{d}\right) + \dfrac{m}{n}\displaystyle\sum_{h=1}^{d-1}\sin^{-2}\left(\dfrac{h\pi}{d}\right)\right], & d > 2, \\ \dfrac{m+n}{n}, & d = 2. \end{cases}$$

When d is odd, by Proposition 2.3 and $h_0 = \frac{d-1}{2d}$, we have,

$$\sum_{k=0}^{n-1} F_{m,k} \geq \sum_{k=0}^{n-1} \sin^{2k}\left(\frac{h_0\pi}{d}\right) F_{m,k} = \sin^{2(n-1)}\left(\frac{h_0\pi}{d}\right) E_{m,h_0,n-1}.$$

This together with (2.12) deduces

$$\begin{aligned} c_n &= \sum_{j=0}^{n-1} c_j - F_{m,n} \\ &\geq \sum_{j_1+\cdots+j_{d-1}=n-1} \left[(C-1)\frac{m+j_{h_0}}{m}\sin^{2(n-1)}\left(\frac{h_0\pi}{d}\right) - \sum_{\ell=1}^{d-1}\frac{m+j_\ell}{n}\sin^{-2}\left(\frac{\ell\pi}{d}\right)\right] \\ &\quad \times \prod_{k=1}^{d-1}\binom{m-1+j_k}{j_k}\sin^{-2j_k}\left(\frac{k\pi}{d}\right). \end{aligned} \tag{4.11}$$

Similarly, we obtain $c_n \geq 0$ with

$$C = 1 + \left[\frac{n-1}{n}\sin^{-2}\left(\frac{\pi}{d}\right) + \frac{m}{n}\sum_{h=1}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right)\right]\sin^{-2(n-1)}\left(\frac{(d-1)\pi}{2d}\right)$$

for odd integer $d \geq 3$.

For part (ii), we apply Lemma 4.3 with $r = 2m$ and $\tilde{r} = 2m+2$. Since $P_{m,n}(\xi) \leq P_{m+1,n}(\xi)$ for $\xi \in [-\pi, \pi]$, the second inequality follows. To prove the first inequality, we look for a constant $C \geq 1$ such that

$$P_{m+1,n}(\xi) \leq C P_{m,n}(\xi), \quad \xi \in [-\pi, \pi], \tag{4.12}$$

or equivalently, such that

$$p(y) := C\sum_{j=0}^{n} F_{m,j} y^j - \sum_{j=0}^{n} F_{m+1,j} y^j \geq 0, \quad 0 \leq y \leq 1.$$

Since

$$\begin{aligned} p(y) &= \sum_{j=0}^{n}\sum_{j_1+\cdots+j_{d-1}=j}\left[\prod_{k=1}^{d-1} C\binom{m-1+j_k}{j_k} - \prod_{k=1}^{d-1}\binom{m+j_k}{j_k}\right]\sin^{-2j_k}\frac{k\pi}{d} \\ &= \sum_{j=0}^{n}\sum_{j_1+\cdots+j_{d-1}=j}\prod_{k=1}^{d-1}\binom{m-1+j_k}{j_k}\left(C - \prod_{k=1}^{d-1}\frac{m+j_k}{m}\right)\sin^{-2j_k}\frac{k\pi}{d} \end{aligned} \tag{4.13}$$

and

$$\prod_{k=1}^{d-1} \frac{m+j_k}{m} \le \left(\frac{m(d-1)+j}{m(d-1)}\right)^{d-1}$$

(this inequality can be obtained by arithmetic and geometric means inequality), we deduce from Lemma 4.3 that the first inequality in (ii) holds with $C = (1 + \frac{n}{m(d-1)})^{d-1}$. Moreover, $\gamma_{m,n}$ is increasing with the variable m and can be derived by

$$2 - (d-1)\log_d \left[1 + \frac{n}{m(d-1)}\right] \le 2 - \log_d e > 0.$$

For part (iii), we apply again the Lemma 4.3 with $r = 2m$ and $\tilde{r} = 2m+2$. Since $P_{m,m-1}(\xi) \le P_{m+1,m}(\xi)$ for $\xi \in [-\pi,\pi]$, the second inequality follows. To prove the first inequality, we look for a constant $C \ge 1$ such that

$$P_{m+1,m}(\xi) \le CP_{m,m-1}(\xi), \quad \xi \in [-\pi,\pi], \tag{4.14}$$

or equivalently, such that

$$p(y) := C\sum_{j=0}^{m-1} F_{m,j}y^j - \sum_{j=0}^{m} F_{m+1,j}y^j \ge 0, \quad 0 \le y \le 1.$$

Define

$$c_j := C(F_{m,j} - F_{m+1,j}), \quad 0 \le j \le m-1.$$

Then

$$p(y) = \sum_{j=0}^{m-1} c_j y^j - F_{m+1,m}y^m = \sum_{j=0}^{m-1} c_j(y^j - y^m) - c_m y^m$$

with $c_m = \sum_{j=0}^{m-1} c_j - F_{m+1,m}$. Since

$$c_j = \sum_{j_1+\cdots+j_{d-1}=j} \left[C - \prod_{\ell=1}^{d-1} \frac{m+j_\ell}{m}\right] \prod_{k=1}^{d-1} \binom{m-1+j_k}{j_k} \sin^{-2j_k} \frac{k\pi}{d},$$

then $c_j \ge 0, 0 \le j \le m-1$ for $C \ge (1 + \frac{m-1}{m(d-1)})^{d-1}$. Note that if $c_j \ge 0$ for $0 \le j \le m$, then $p(y) \ge 0$ for $y \in [0,1]$. Thus, we only need to prove that $c_m \ge 0$.

Case (i) d is even. Denote $M_m := \prod_{\ell=1}^{d-1} \frac{m+j_\ell}{m}$. By Proposition 2.4 and the statement in part (i), we have

$$\begin{aligned} c_m = \sum_{j=0}^{m-1} c_j - F_{m+1,m} = \sum_{j_1+\cdots+j_{d-1}=m-1} \Bigg[& C\frac{m+j_h}{m} - \frac{m+1+j_h}{m+1} M_m \\ & - M_m \sum_{\ell=1}^{d-1} \frac{m+1+j_\ell}{m} \sin^{-2}\left(\frac{\ell\pi}{d}\right)\Bigg] \prod_{k=1}^{d-1} \binom{m-1+j_k}{j_k} \sin^{-2j_k}\left(\frac{k\pi}{d}\right). \end{aligned} \tag{4.15}$$

By calculation, we can obtain the desired result.

Case (ii), d is odd. Denote

$$C = \left(1 + \frac{m-1}{m(d-1)}\right)^{d-1} \times \left[1 + \left(2\sin^{-2}\left(\frac{\pi}{d}\right) + \frac{m+1}{m}\sum_{h=2}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right)\right)\sin^{-2(m-1)}\left(\frac{(d-1)\pi}{2d}\right)\right].$$

By similarly calculating Case (i), one obtains $c_m \geq 0$. Then the conclusion follows. □

Basing on Proposition 2.5, we can obtain the following result by applying the same method to prove Theorem 4.4.

Theorem 4.5. *Let $\widetilde{\gamma}_{m,n}$ be the regularity of d-refinable pseudo-splines scheme defined by $\widehat{\widetilde{a}}_{m,n}(\xi)$ and denote $\widetilde{\gamma}_{m,m-1}$ by $\widetilde{\gamma}_m$. Then*

(i) *$\widetilde{\gamma}_{m,n}$ is decreasing with variable n, furthermore,*

$$\widetilde{\gamma}_{m,n-1} - \log_d(C) \leq \widetilde{\gamma}_{m,n} \leq \widetilde{\gamma}_{m,n-1}$$

with

$$C := \begin{cases} 1 + D, & d > 2, \\ \dfrac{m+n+\frac{1}{2}}{n}, & d = 2 \end{cases}$$

and

$$D := \left[\frac{n-1}{n}\sin^{-2}\left(\frac{\pi}{d}\right) + \frac{m+\frac{1}{2}}{n}\sum_{h=1}^{d-1}\sin^{-2}\left(\frac{h\pi}{d}\right)\right] \times \left(\sin^{-2(n-1)}\left(\frac{(d-1)\pi}{2d}\right)\right)^{\mathrm{odd(d)}}.$$

(ii) *$\widetilde{\gamma}_{m,n}$ is increasing with the variable m and*

$$\widetilde{\gamma}_{m,n} + 2 - (d-1)\log_d\left[1 + \frac{n}{\left(m+\frac{1}{2}\right)(d-1)}\right] \leq \widetilde{\gamma}_{m+1,n} \leq \widetilde{\gamma}_{m,n} + 2.$$

(iii) *$\widetilde{\gamma}_m$ is increasing with the variable m for $d = 2$, and*

$$\widetilde{\gamma}_m + 2 - \log_d(C) \leq \widetilde{\gamma}_{m+1} \leq \widetilde{\gamma}_m + 2.$$

with

$$C := \begin{cases} \left(1 + \dfrac{m-1}{\left(m+\frac{1}{2}\right)(d-1)}\right)^{d-1} [1+D], & d > 2 \\ \dfrac{\left(2m+\frac{1}{2}\right)\left(2m+\frac{3}{2}\right)}{m\left(m+\frac{3}{2}\right)}, & d = 2 \end{cases} \quad \textit{and}$$

$$D := \left[\frac{2m-\frac{1}{2}}{m+\frac{1}{2}} \sin^{-2}\left(\frac{\pi}{d}\right) + \sum_{h=2}^{d-1} \sin^{-2}\left(\frac{h\pi}{d}\right)\right] \times \left(\sin^{-2(m-1)}\left(\frac{(d-1)\pi}{2d}\right)\right)^{\mathrm{odd(d)}}.$$

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