THE TOPOLOGICAL STRUCTURE OF THE FUZZY SETS WITH THE TOPOLOGY INDUCED BY $\Gamma$-CONVERGENCE

LILI ZHANG$^1$, ZONGBEN XU$^2$ and ZHONGQIANG YANG$^3$

$^1$Department of Mathematics
Xi’an Jiaotong University
Xi’an, Shaanxi, 710049, P. R. China

$^1$Department of Mathematics and Physics
Xi’an Technological University
Xi’an, Shaanxi, 710032, P. R. China
e-mail: 313308zhll@163.com

$^2$Department of Mathematics
Xi’an Jiaotong University
Xi’an, Shaanxi, 710049, P. R. China

$^3$Department of Mathematics
Shantou University
Shantou, Guangzhou, 515063, P. R. China

Abstract

For a non-compact locally compact second countable Hausdorff space $X$, let $USCC(X)$ be the family of all the upper-semicontinuous fuzzy sets on $X$ with non-empty compact supports. We show that the space $USCC(X)$ with the topology induced by $\Gamma$-convergence is homeomorphic to the

2010 Mathematics Subject Classification: 54B20, 54F15, 57N20.

Keywords and phrases: fuzzy sets, fell topology, $\Gamma$-convergence, $\Sigma$, $Q_f$.

The first author is supported by the Special Fund of Shaanxi Provincial Education Department (Grant no. 09JK506) and Shaanxi Provincial Natural Science Fund (Grant no. 2009JM1017).

Received August 31, 2009
radial-interior $\sum = \{(x_n) \in Q : \sup |x_n| < 1\}$ of the Hilbert cube $Q = [-1,1]^\omega$
(respectively $\sum_f = \{(x_n) \in Q : x_n = 0$ except for finitely many $n\}$) if $X$ is non-discrete (resp. discrete).

1. Introduction and Main Results

The set of non-empty closed sets in a Hausdorff space $X$ is denoted by $\text{Cld}(X)$. For each $U \subset X$, we denote

$$U^- = \{A \in \text{Cld}(X) : A \cap U \neq \emptyset\}$$

and

$$U^+ = \{A \in \text{Cld}(X) : A \subset U\}.$$

The Fell topology on $\text{Cld}(X)$ is generated by the sets of the form $U^-$ and $(X \setminus K)^+$, where $U$ is open and $K$ is compact in $X$. As usual, a fuzzy set on $X$ is simply a function from $X$ to the unit interval $I$. A fuzzy set $f$ on $X$ is called upper semicontinuous (USC) if $f_t = \{x \in X : f(x) \geq t\}$ is closed in $X$ for every $t \in I$. Let $f$ be a fuzzy set on $X$. The support of $f$, denoted by $\text{supp}(f)$, is the closure $\{x \in X : f(x) > 0\}$ of the set $\{x \in X : f(x) > 0\}$ in $X$. The hypograph of $f$ is defined by

$$\text{hypo}(f) = \{(x, t) \in X \times I : t \leq f(x) \text{ and } x \in X\},$$

as subspace of the product space $X \times I$. Then $\text{hypo}(f)$ is closed in $X \times I$ if and only if $f$ is USC. We use $\text{USC}(X)$ and $\text{USCC}(X)$ to denote the families of all the USC fuzzy sets on $X$ and all the USC fuzzy sets on $X$ with non-empty compact supports, respectively. If $X$ is a metric space, then we can topologize $\text{USCC}(X)$ by letting hypograph distance $D(f, g)$ for $f, g \in \text{USCC}(X)$ be equal to the Hausdorff distance between $\text{hypo}(f)$ and $\text{hypo}(g)$ as subsets of $X \times I$. We use $\text{USCC}_H(X)$ to denote the metric space. A net $\{f_\lambda\}_{\lambda \in \Lambda}$ of fuzzy sets on $X$ converges to a fuzzy set $f$ if

$$f(x) = \lim_{V \in N(x)} \liminf_{\lambda \in \Gamma} \sup_{g \in V} f_{\lambda}(y) = \inf_{V \in N(x)} \limsup_{\lambda \in \Gamma} \sup_{y \in V} f_{\lambda}(y),$$

for all $x \in X$, where $N(x)$ is the neighborhood system of $x$. 

The $\Gamma$-convergence of nets of upper semicontinuous fuzzy sets on $X$ with non-empty compact supports plays an important role in applications of fuzzy set theory. This convergence is based on the Kuratowski limits and has been extensively applied by many authors (see, for example, De Giorgi [2, 3] and Attouch [1] and the references therein) to calculus of variations, differential equations, control theory, convex analysis, concave-convex functions and minimax theory, etc. When $X$ is a locally compact Hausdorff space, the convergence is topological [4]. In this case, the compatible topology, called hypo-topology, on $USCC(X)$ can be obtained by identifying each fuzzy set with its hypograph. The space $USCC(X)$ with the hypo-topology is then a subspace of $Cld(X)$ with the Fell topology. By identifying each fuzzy set with its hypograph, we use $USC_F(X)$ and $USC_F(X)$ to denote the subspaces of $Cld(X)$ with the Fell topology, respectively. In [10], it was shown that $USCC_H(X)$ is homeomorphic to ($\approx$) the Hilbert cube $Q = [-1, 1]^\omega$ if $X$ is an infinite compact metric space. In [7], it was shown that for a Hausdorff space $X$ the space $USC_F(X) \approx Q$ if and only if $X$ is locally compact, second countable and infinite. In [11], it was shown that when $X$ is a non-compact locally compact separable metric space, $USCC_H(X)$ is homeomorphic to $\sum$ (resp. $Q_f$) if $X$ non-discrete (resp. discrete), where $\sum = \{(x_n) \in Q : \sup x_n < 1\}$ is the radial-interior of the Hilbert cube $Q = [-1, 1]^\omega$ and $Q_f = \{(x_n) \in Q : x_n = 0$ except for finitely many $n\}$. It should be noted that the Fell topology and the topology induced by hypograph metric on $USCC(X)$ are identical if $X$ is an infinite compact metric space. In the present paper, we will show that when $X$ is non-compact locally compact and second countable, the natural bijection from $USCC_H(X)$ to $USC_F(X)$ defined by $f' \rightarrow f$ is continuous but not necessarily a homeomorphism, however, the two spaces are homeomorphic. The following is the main result.

**Theorem 1.** Let $X$ be a Hausdorff space. Then

(1) $USC_F(X) \approx Q_f$ if $X$ is non-compact, locally compact, second countable and discrete;

(2) $USCC(X) \approx \sum$ if $X$ is non-compact, locally compact, second countable and non-discrete.
2. Preliminaries

In this section, we at first recall some necessary fundamental concepts and facts. For more information on them, refer to [5] and [6]. At last, some symbols are given.

Let $X$ be a metric space with the metric $d$. Then a subset $Y$ of $X$ is said to be homotopy dense in $X$ if there exists a homotopy $h : X \times I \to X$ such that $h_0 = id_X$ and $h_t(X') \subset Y$ for every $t > 0$, where $h_t : X \to X$ is defined by $h_t(x) = h(x, t)$.

A closed subset $A$ of the space $X$ is said to be a Z-set if for every continuous map $h : X \to (0, \infty)$, there is a continuous map $f : X \to X \setminus A$ with $d(f(x), x) < \varepsilon(x)$ for every $x \in X$. It is trivial that every Z-set is closed nowhere dense but the converse is not necessarily true. If $X$ is compact, then obviously a map $\varepsilon$ can be replaced by an arbitrary positive real number $\varepsilon$. We call a subset $M \subset X$ a cap set (resp. an fd-cap set) for $X$ if

1. $M = \bigcup_{i=1}^{\infty} M_i$, where each $M_i$ is a (resp. finite-dimensional) compact Z-set in $X$ such that $M_i \subset M_{i+1}$, and

2. for each $\varepsilon > 0$, each integer $i > 0$ and each (resp. finite-dimensional) compact subset $K$ of $X$, there is an integer $j > i$ and an imbedding $h : K \to M_j$ such that $h|_{K \setminus M_i} = id$ and $d(h, id) < \varepsilon$.

It is well known that for $M \subset Z$, $(Z, M) \approx (Q, \Sigma)$ (resp. $(Z, M) \approx (Q, Q_f)$) if and only if $Z \approx Q$ and $M$ is a cap set (resp. an fd-cap set) for $Z$ (cf. [6]). It is said that a tower $(M_i)_{i \in \mathbb{N}}$ has the deformation property in $X$ if there exists a homotopy $h : X \times I \to X$ such that $h_0 = id$ and, for each $t > 0$, $h(X \times [t, 1])$ is contained in some $M_i$. A tower $(M_i)_{i \in \mathbb{N}}$ is said to be finitely expensive if for each $i \in \mathbb{N}$, there exists some $j > i$ and an embedding $h : M_i \times I \to M_j$ such that $h(x, 0) = x$ for every $x \in M_i$.

Lemma 2 [8]. Let $M$ be a countable union of finite-dimensional Z-sets in $Q$. If $M$ contains a finitely expensive tower with the deformation property in $Q$, then $M$ is an fd-cap set for $Q$.

Lemma 3 [10]. The space $USCC_H(X)$ is homeomorphic to $Q$ if and only if $X$ is an infinite compact metrizable space.
Lemma 4 [7]. For a Hausdorff space $X$, $USC_F(X) \approx Q$ if and only if $X$ is locally compact, second countable and infinite.

Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be separable metric spaces. In case there are no confusions, $d$ stands for both metrics $d_X$ and $d_Y$. Put $B_d(a, \varepsilon) = \{x \in X : d(x, a) < \varepsilon\}$. We use $\mathbb{N}$ and $\mathbb{R}$ to denote the sets of all natural numbers and of all real numbers and, as stated before, $\mathbb{I} = [0, 1]$.

3. The Relationship between $USCC_H(X)$ and $USCC_F(X)$

Proposition 5. Let $X$ be a non-compact locally compact second countable Hausdorff space and $d$ be an admissible metric on $X$. The natural bijection from $USCC_H(X)$ to $USCC_F(X)$ defined by $f \rightarrow f'$ is continuous but not necessarily a homeomorphism.

Proof. First of all, we prove that the natural bijection is continuous. Let $\{f_n\}$ be a sequence converging to $f$ in $USCC_H(X)$. Let $U \subset X \times \mathbb{I}$ be open such that $\text{hypo}(f) \cap U \neq \emptyset$. Choose $(x_0, \lambda) \in \text{hypo}(f) \cap U$. Since $f_n \rightarrow f$ in $USCC_H(X)$, there exists a sequence $(x_n, \lambda_n)$ in $X \times \mathbb{I}$ such that $(x_n, \lambda_n) \in \text{hypo}(f_n)$ and $\lim_{n \rightarrow \infty} (x_n, \lambda_n) = (x_0, \lambda)$. Thus there exists $N \in \mathbb{N}$ such that $(x_n, \lambda_n) \in U$ for each $n > N$. Therefore, $\text{hypo}(f_n) \cap U \neq \emptyset$ for each $n > N$. On the other hand, let $K$ be a compact subset in $X \times \mathbb{I}$ such that $K \cap \text{hypo}(f_n) = \emptyset$. Then there exists $N \in \mathbb{N}$ such that $\text{hypo}(f_n) \cap K = \emptyset$ for each $n > N$. Otherwise, there exists a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that $\text{hypo}(f_{n_j}) \cap K \neq \emptyset$. Let $(x_{n_j}, \gamma_{n_j}) \in \text{hypo}(f_{n_j}) \cap K$. By the compactness of $K$, without loss of generality, we can assume that $\lim_{j \rightarrow \infty} (x_{n_j}, \gamma_{n_j}) = (x, \gamma) \in K$. By $f_n \rightarrow f$ in $USCC_H(X)$, we have $(x, \gamma) \in \text{hypo}(f)$. This is a contradiction with $K \cap \text{hypo}(f) = \emptyset$.

Now we prove whether $X$ is discrete or non-discrete, the natural bijection is not necessarily a homeomorphism by constructing an example. Take $X = \mathbb{R}$ or $\mathbb{N}$ with ordinary metric, let

\[
    f(x) = \begin{cases} 
        1, & \text{if } x = 0, \\
        0, & \text{otherwise} 
    \end{cases}
\]
and
\[ f_n(x) = \begin{cases} 1, & \text{if } x = 0, n, \\ 0, & \text{otherwise.} \end{cases} \]

Then \( D(f, f_n) = n \) for every \( n \). Hence, \( \lim_{n \to \infty} f_n \neq f \) in \( USCC_H(X) \). But, \( f_n \to f \) in \( USCC_F(X) \). In fact, for each open subset \( U \subset \mathbb{R} \times \mathbb{I} \) (or \( U \subset \mathbb{N} \times \mathbb{I} \)), it is clear that \( \text{hypo}(f_n) \cap U \neq \emptyset \) for each \( n \) if \( \text{hypo}(f) \cap U \neq \emptyset \). On the other hand, let \( K \) be a compact subset in \( \mathbb{R} \times \mathbb{I} \) (or \( \mathbb{N} \times \mathbb{I} \)) such that \( K \cap \text{hypo}(f) = \emptyset \). Then there exists \( N \in \mathbb{N} \) such that \( K \subset [-N, N] \times \mathbb{I} \). Hence, \( \text{hypo}(f_n) \cap K = \emptyset \) for large enough \( n \).

**Remark.** However, Theorem 1 shows that \( USCC_H(X) \) and \( USCC_F(X) \) are homeomorphic if \( X \) is a non-compact locally compact second countable Hausdorff space.

4. **Proof of Theorem 1(1)**

In this section, we use \( USC(X) \) to denote the space \( USCC_F(X) \). To show Theorem 1(1), we may, without loss of generality, assume that \( X = \mathbb{N} \) with the ordinary metric.

Let
\[ F_n = \{ f \in USC(X) : f(x) = 0 \text{ for every } x > n \}. \]

Then \( F_1 \subset F_2 \subset \cdots \) and \( USCC_F(X) = \bigcup_{n=1}^{\infty} F_n \).

**Lemma 6.** \( F_n \approx \mathbb{I}^n \) for every \( n \in \mathbb{N} \).

**Proof.** It is trivial.

**Lemma 7.** \( F_n \) is a Z-set of \( USC(X) \) for each \( n \in \mathbb{N} \).

**Proof.** We can define a map \( h : USC(X) \times \mathbb{I} \to USC(X) \) by
\[
    h(f, t)(x) = \begin{cases} 
        f(x), & \text{if } x \neq n + 1, \\
        \max\{f(x), t\}, & \text{if } x = n + 1. 
    \end{cases}
\]

Then \( h \) is well defined, \( h(f, 0) = f \) and \( h(f : t) \in USC(X) : F_n \) for \( t > 0 \). It is not hard to verify that \( h \) is continuous.
Lemma 8. For each \( n \in \mathbb{N} \), there exists an embedding \( h : F_n \times I \to F_{n+1} \) such that \( h(f, 0) = f \) for every \( f \in F_n \).

**Proof.** We define a map \( h : F_n \times I \to F_{n+1} \) as follows:

\[
h(f, t)(x) = \begin{cases} 
    f(x), & \text{if } x \neq n + 1, \\
    \max\{f(x), t\}, & \text{if } x = n + 1.
\end{cases}
\]

Obviously, the map \( h \) is our desired embedding.

Lemma 9. The tower \( (F_n)_{n \in \mathbb{N}} \) has the deformation property in \( USC(X) \).

**Proof.** To complete the proof of the lemma, we need to verify that there exists a continuous map \( h : USC(X) \times I \to USC(X) \) such that \( h_0 = id_{USC(X)} \) and for every \( t > 0 \), there exists an integer \( n > 0 \) such that \( h(USC(X) \times [t, 1]) \subseteq F_n \). For any \( 0 < t \leq 1 \), there exists unique \( n_t \in \mathbb{N} \) such that \( n_t \leq \frac{1}{t} < n_t + 1 \). Now we define a map \( h : USC(X) \times I \to USC(X) \) as follows:

\[
G(f, t)(x) = \begin{cases} 
    f(x), & \text{if } t = 0 \text{ and } x \in X, \\
    f(x), & \text{if } t > 0 \text{ and } x < n_t + 1, \\
    (-n_t(n_t + 1) + n_t + 1)f(x), & \text{if } t > 0 \text{ and } x = n_t + 1, \\
    0, & \text{if } t > 0 \text{ and } x > n_t + 1.
\end{cases}
\]

By the definition of \( h \), \( h_0 = id_{USC(X)} \) and for each \( t > 0 \), there exists an integer \( n \in \mathbb{N} \) such that \( h(USC(X) \times [t, 1]) \subseteq F_n \). To show the continuity of \( h \), let \( \{(f_n, t_n)\} \) be a sequence in \( USC(X) \times I \) with the limit \( (f, t) \). We shall show that \( \lim_{n \to \infty} h(f_n, t_n) = h(f, t) \). We consider the following cases:

**Case A.** \( t = 0 \). Then \( h(f, t)(x) = f(x) \) for every \( x \in X \). Without loss of generality, we can assume that \( t_n > 0 \) for each \( n \). Suppose that \( V \subseteq X \times I \) is open and \( hypo(f) \cap V \neq \emptyset \). Let \( (m, \lambda) \in hypo(f) \cap V \). Since \( \mathbb{N} \) is locally compact and discrete, there exists \( \delta > 0 \) such that \( \{m\} \times [\lambda - \delta, \lambda + \delta] \subseteq V \). Thus \( hypo(f) \cap (\{m\} \times (\lambda - \delta, \lambda + \delta)) \neq \emptyset \). By \( f_n \to f \), we have that there exists \( N_1 \in \mathbb{N} \) such
that \( \text{hypo}(f_n) \cap \{(m) \times (\lambda - \delta, \lambda + \delta)\} \neq \emptyset \) for each \( n > N_1 \). By \( t_n \to 0 \), there exists \( N_2 \in \mathbb{N} \) such that \( n_{n_2} > m \) for each \( n > N_2 \). Let \( N = \max\{N_1, N_2\} \). Then
\[
h(f_n, t_{n_2})(m) = f_n(m) \quad \text{for each} \quad n > N, \quad \text{that is,} \quad \text{hypo}(f_n, t_{n_2}) \cap V \neq \emptyset \quad \text{for each} \quad n > N.
\]
On the other hand, let \( K \subset X \times \mathbb{I} \) be compact such that \( \text{hypo}(f) \cap K = \emptyset \). It follows that there exists an integer \( n \in \mathbb{N} \) such that \( \text{hypo}(f_n) \cap K = \emptyset \) for each \( n > N \) from \( f_n \to f \). By \( h(f_n, t_{n}) \leq f_n(x) \) for each \( x \in X \), we have
\[
\text{hypo}(h(f_n, t_{n})) \cap K = \emptyset \quad \text{for each} \quad n > N.
\]

**Case B.** \( t > 0 \). Without loss of generality, we can assume that \( |n_t - n_{t_n}| \leq 1 \) for each \( n \in \mathbb{N} \). Then there exists \( N \in \mathbb{N} \) such that \( h(f, t)(x) = 0 \) and \( h(f_n, t_{n})(x) = 0 \) for each \( x > N \). By \( f_n \to f \), \( \{h(f_n, t_{n})\} \) uniformly converges to \( h(f, t) \).
Thus, \( h(f_n, t_{n}) \to h(f, t) \). In fact, suppose that \( V \subset X \times \mathbb{I} \) is open and \( \text{hypo}(h(f, t)) \cap V \neq \emptyset \). Let \( (l, \gamma) \in \text{hypo}(h(f, t)) \cap V \). Without loss of generality, we can assume that \( \gamma > 0 \). Since \( V \) is open, there exists \( \delta > 0 \) such that \( (l, \gamma - \delta) \in V \). Moreover, by \( h(f, t)(l) \to h(f_n, t_{n})(l) \), there exists \( N_1 \in \mathbb{N} \) such that \( h(f_n, t_{n})(l) > (\gamma - \delta) \), that is, \( \downarrow h(f_n, t_{n}) \cap V \neq \emptyset \) for each \( n > N_1 \). On the other hand, let \( K \subset X \times \mathbb{I} \) be compact such that \( \text{hypo}(h(f, t)) \cap K = \emptyset \). Then there exists \( N_2 \in \mathbb{N} \) such that \( K \subset \{1, 2, ..., N_2\} \times \mathbb{I} \). Thus, there exists \( \delta_1 > 0 \) such that \( B_d(K, \delta_1) \cap \text{hypo}(h(f, t)) = \emptyset \). Since \( \{h(f_n, t_{n})\} \) uniformly converges to \( h(f, t) \), we have that \( \text{hypo}(h(f_n, t_{n})) \cap K = \emptyset \) for large enough \( n \).

**Proof of Theorem 1(1).** By Lemma 4, \( \text{USC}(X) \cong \mathcal{Q} \). It follows directly from Lemmas 2, 6, 7, 8 and 9 that \( \text{USCC}(X) \) is an fd-cap set for \( \text{USC}(X) \). Hence, \( (\text{USC}(X), \text{USCC}_F(X)) \cong (\mathcal{Q}, Q_f) \).

5. **Proof of Theorem 1(2)**

Let \( X \) be a non-compact locally compact second countable non-discrete space. Then \( X \) is a separable metrizable space. We use \( \alpha X = X \cup \{x\} \) to denote the one-point compactification of \( X \). In this section, we assume that \( d \) is an admissible metric on \( \alpha X \). For simplicity’s sake, we use \( \text{USCC}(\alpha X) \) to denote the space \( \text{USCC}_F(\alpha X) \).

We define a map \( \alpha : \text{USC}(X) \to \text{USCC}(\alpha X) \) as follows:
\[ \alpha(f)(x) = \begin{cases} f(x), & \text{if } x \in X, \\ 1, & \text{if } x = \infty, \end{cases} \]

for every \( f \in USC(X) \). It is not difficult to verify that \( \alpha : USC_F(X) \to USCC(\alpha X) \) is an embedding. Moreover, it is easily seen that

\[
\alpha(USC(X)) = \{ f \in USCC(\alpha X) : f(\infty) = 1 \},
\]

\[
\alpha(USCC(X)) = \{ f \in USCC(\alpha X) : \exists \text{ open set } U \ni \infty \text{ such that } f(x) = 0 \quad \forall x \in U \setminus \{x\} \text{ and } f(\infty) = 1 \}.
\]

Thus, we shall identify the spaces \( USC_F(X) \) and \( USCC_F(X) \) with the subspaces \( \alpha(USC_F(X)) \) and \( \alpha(USCC_F(X)) \) of \( USCC(\alpha X) \), respectively, that is, we shall assume that \( USC_F(X) \) and \( USCC_F(X) \) are subspaces of \( USCC(\alpha X) \). Hence, we may study the pair-space \( (USC_F(X), USCC_F(X)) \) in \( USCC(\alpha X) \).

Since \( X \) is non-discrete, we can choose \( \varepsilon_1 \in (0, 1) \) satisfying \( \alpha X \setminus B_d(\varepsilon_1, \varepsilon_1) \) is infinite. Define a decreasing sequence \( 1 > \varepsilon_1 > \varepsilon_2 > \cdots > 0 \) such that \( \lim_{n \to \infty} \varepsilon_n = 0 \) and \( B_d(\varepsilon_n, \varepsilon_{n+1}) \neq \emptyset \) for every \( n \). Let \( F_n = \{ f \in USCC(\alpha X) : f(x) = 0 \} \) for every \( x \in B_d(\varepsilon_n, \varepsilon_n) \setminus \{\infty\} \) and \( f(\infty) = 1 \). Then \( F_1 \subseteq F_2 \subseteq \cdots \) and \( USC_F(X) = \bigcup_{n=1}^{\infty} F_n \).

**Lemma 10.** \( F_n \approx Q \) for every \( n \in \mathbb{N} \).

**Proof.** Note that for every \( n \), \( F_n \) is homeomorphic to \( USC_H(X \setminus B_d(\varepsilon_n, \varepsilon_n)) \) and \( X \setminus B_d(\varepsilon_n, \varepsilon_n) \) is an infinite compact metric space. By Lemma 3, we are done.

**Lemma 11.** \( F_n \) is a \( Z \)-set in \( USC_F(X) \) for every \( n \in \mathbb{N} \).

**Proof.** By the definition of the sequence \( \{\varepsilon_n\} \), we can choose a point \( x_n \) satisfying \( 0 < d(x_n, \infty) < \varepsilon_n \) for every \( n \). For every \( \varepsilon > 0 \), let \( \varphi : USC_F(X) \to USC_F(X) \setminus F_n \) be the map defined by

\[
\varphi(g)(x) = \begin{cases} g(x), & \text{if } x \neq x_n, \\ \max \left( g(x), \frac{\varepsilon}{2} \right), & \text{if } x = x_n. \end{cases}
\]

Then \( \varphi \) is continuous, \( \varphi(USC_F(X)) \cap F_n = \emptyset \) and \( d(\varphi, id_{USC_F(X)}) < \varepsilon \). \( \square \)
Lemma 12. For any $\varepsilon > 0$, there is an integer $n > 0$ and a continuous map $\varphi : \text{USC}(X) \to F_n$ which is $\varepsilon$-close to id.

Proof. Choose a point $x_0 \in X$ such that $0 < d(x_0, \infty) < \varepsilon$. Let $\delta = d(x_0, \infty)$.

Now we define the desired map $h : \text{USC}(X) \to \text{USC}(X)$ as follows:

$$h(f)(x) = \begin{cases} f(x), & \text{if } d(x, \infty) > \delta \text{ or } x = \infty, \\ 1, & \text{if } d(x, \infty) = \delta, \\ 0, & \text{if } 0 < d(x, \infty) < \delta. \end{cases}$$

First of all, note that for every $f \in \text{USC}(X)$, $h(f)(x) = 0$ for each $x$ with $0 < d(x, \infty) < \delta$, we have that there exists an integer $n > 0$ such that $h(\text{USC}(X)) \subset F_n$.

To verify that $h$ is $\varepsilon$-close to id, let $f \in \text{USC}(X)$ and $x \in \alpha X$. If $d(x, \infty) \leq \delta$, then $d((x, t), (\infty, t)) \leq \delta < \varepsilon$, which implies that $\{x\} \times [0, 1] \subset B_d(\text{hypo}(f), \varepsilon)$ and $\{x\} \times [0, 1] \subset B_d(\text{hypo}(h(f)), \varepsilon)$. If $d(x, \infty) > \delta$, then $f(x) = h(f)(x)$. Thus, $\text{hypo}(h(f)) \subset B_d(\text{hypo}(f), \varepsilon)$ and $\text{hypo}(f) \subset B_d(\text{hypo}(h(f)), \varepsilon)$, that is, $h$ is $\varepsilon$-close to id.

The remainder is to verify that $h : \text{USC}_F(X) \to \text{USC}_F(X)$ is continuous. Suppose $f_n \to f$ in $\text{USC}_F(X)$, we have the following facts:

**Fact 1.** If $(x_n, \lambda_n) \in \text{hypo}(h(f_n))$ and $\lim_{n \to \infty} (x_n, \lambda_n) = (x, \lambda)$, then $(x, \lambda) \in \text{hypo}(h(f))$, or equivalently, $\lambda \leq h(f)(x)$.

**Case A.** $d(x, \infty) = \delta$ or $x = \infty$. It is trivial.

**Case B.** $0 < d(x, \infty) < \delta$. Then $0 < d(x_n, \infty) < \delta$ for large enough $n$. Hence, we have $\lim_{n \to \infty} \lambda_n = \lambda = 0$, that is, $(x, \lambda) \in \text{hypo}(h(f))$.

**Case C.** $d(x, \infty) > \delta$. Then $d(x_n, \infty) > \delta$ for large enough $n$. Thus $h(f)(x_n) = f(x_n)$. By $f_n \to f$, we have $(x, \lambda) \in \text{hypo}(h(f))$.

**Fact 2.** For each $(x, \lambda) \in \text{hypo}(h(f))$, there exists a sequence $(x_n, \lambda_n)$ in $\alpha X \times I$ such that $(x_n, \lambda_n) \in \text{hypo}(h(f_n))$ and $\lim_{n \to \infty} (x_n, \lambda_n) = (x, \lambda)$.
Case A. \( d(x, \infty) \leq \delta \). It is trivial.

Case B. \( d(x, \infty) > \delta \). Then \( h(f)(x) = f(x) \). By \( f_n \to f \), there exists a sequence \( \{(x_n, \lambda_n)\} \) in \( \alpha X \times \Pi \) such that \((x_n, \lambda_n) \in \text{hypo}(f_n)\) and \( \lim_{n \to \infty} (x_n, \lambda_n) = (x, \lambda) \). Since \( d(x, \infty) > \delta \), \( d(x_n, \infty) > \delta \) for large enough \( n \). Thus, \( f(x_n) = h(f)(x_n) \) for large enough \( n \).

Lemma 13. For each integer \( n > 0 \), \( F_n \) is a Z-set in \( F_{n+1} \).

Proof. By the definition of the sequence \( \{F_n\} \), for each \( n \in \mathbb{N} \) there exists a point \( x_n \) in \( X \) such that \( x_n \in B_d(\infty, e_n) \cap B_d(\infty, e_{n+1}) \). For every \( \varepsilon > 0 \), let \( \varphi : F_{n+1} \to F_{n+1} \setminus F_n \) be the map defined by

\[
\varphi(g)(x) = \begin{cases} 
\max\{g(x), \frac{\varepsilon}{2}\}, & \text{if } x = 0, \\
\frac{\varepsilon}{2}, & \text{if } x = 0.
\end{cases}
\]

Then \( \varphi \) is continuous and \( \varphi(F_{n+1}) \cap F_n = \emptyset \) and \( d(\varphi, id_{F_{n+1}}) < \varepsilon \). □

Proof of Theorem 1(2). Let \( A \) be a compact set in \( USC_F(X) \), \( 0 < m \in \mathbb{N} \) and \( \varepsilon > 0 \). By Lemma 10, \( F_n \equiv Q \) for any \( n \in \mathbb{N} \). By using Lemma 12 and Mapping Replacement Theorem [5, 5.3.11], there is an integer \( k_0 > m \) and an embedding \( h : USC_F(X) \to F_{k_0} \) which is \( \frac{1}{2} \varepsilon \)-close to \( id \). Since \( h(F_m) \) and \( F_m \) are Z-sets in \( F_{k_0+1} \), there is a homeomorphism \( \phi : F_{k_0+1} \to F_{k_0+1} \) such that \( \phi h|_{F_m} = id \) and \( \phi \) is \( \frac{1}{2} \varepsilon \)-close to \( id \) by the Homeomorphism Extension Theorem [5, 5.3.7]. Then we have an embedding \( \phi h|_{A} : A \to F_{k_0+1} \) such that \( \phi h|_{F_m} = id \) and \( \phi h|_{A} \) is \( \varepsilon \)-close to \( id \). Moreover, \( USC_F(X) = \bigcup_{n=1}^{\infty} F_n \) and each \( F_n \) is a Z-set in \( USC_F(X) \) by Lemma 11. Hence, \( USC_F(X) \) is a cap set for \( USC_F(X) \). By Lemma 4, \( USC_F(X) \) is homeomorphic to \( Q \). Then we have \( (USC_F(X), USC_F(X)) \approx (Q, \Sigma) \).

References


