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IMPACT OF DEPENDENCE AMONG VALUATIONS ON EXPECTED REVENUE IN AUCTIONS

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SYNOPTIC ABSTRACT

This work studies the dependence and asymmetry among valuations in sealed-bid auctions with private valuations. In the framework of valuations with Archimedean copulas, it is found that a stronger positive dependence stochastically enhances the final price and lowers the winner’s expected rent in the two-bidder, second-price auction, and the degree of positive dependence has an opposite impact on the final price in the first-price auction. In the context of multiple bidders with valuations having proportional reversed hazards, reducing the asymmetry among negatively dependent valuations is proved to enhance the final price in the second-price auction, and a stronger (weaker) positive dependence along with less (more) asymmetry on valuations is showed to stochastically lower (increase) the final price in the first-price auction.

Key Words and Phrases: Archimedean copula, Kendall’s tau, majorization, superadditive.

1. Introduction

In the literature of private value auction, it is usually assumed that bidders’ valuations are statistically independent. However, because all involved bidders are interested in winning the object for bid and generally base their valuations on information revealed by the auctioneer and other individual-oriented factors, such as economic background, personal interests in the object, etc., the independence assumption is rather restricted and might

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not even be applicable in the real world. So, it is of both theoretical and practical interest to take the statistical dependence among valuations into consideration in this regard. In the context of dependence, several typical questions arise naturally: Will the expected payment of the winner be overestimated or underestimated if the dependence structure among valuations is ignored and the independence is assumed instead? What happens to the winner’s expected rent in the second-price auction (SPA) if the dependence among valuations is introduced? How does the dependence impact the final price in the first-price auction (FPA) and the SPA?

Ex ante asymmetry among bidders is widely known in auction markets. Take the art auction for example; bidders usually have quite different tastes. As documented in procurement auctions, such as the survey of Porter and Zona (1999) on the procurement of school milk, it was found that one or several firms were more competitive than the others, or in the procurement of oil leases, technology levels of petroleum extraction owned by different companies will lead to asymmetric valuations. The impact of asymmetry on the auctioneer’s expected revenue and that on bidders’ bidding strategies have been studied by several authors; see, for example, Lebrun (1996), Maskin and Riley (2003), and Cantillon (2008) for details. Note that in their frameworks, valuations are always assumed to be independent, then one may wonder what will happen to the winner’s payment in auctions when asymmetric valuations are considered to be interdependent. To blend dependence and asymmetry into bidders’ valuations, we employ the well-known Archimedean copulas to characterize the dependence structure among bidders’ valuations and conduct stochastic comparison on the winner’s payments and expected winner’s rent.

To the best of our knowledge, our model is the first in the literature to investigate the impact of interdependence among valuations on the revenue of private value sealed-bid auctions by using copulas. Compared to the symmetric affiliated private value proposed by Li et al. (2002), with the help of the copula, our model is able to characterize more dependence structures. It has some advantages for the quantitative analysis of interdependence. First, we prove that log convexity (log concavity) of the generator of an Archimedean copula corresponds to the positive
(negative) dependence among concerned marginal valuations, and by using Kendall’s $\tau$, one can conveniently measure the degree of dependence between two valuations. Second, we find that for two-bidder SPAs with common valuations, the negative (positive) dependence stochastically lowers (enlarges) the winner’s payment and affects the winner’s expected rent in quite an opposite way. Thus, independence assumption will incur a wrong evaluation of the auctioneer’s revenue. Third, for two-bidder SPAs with positively (negatively) dependent valuations, stronger dependence is found to play a positive (negative) role in enhancing the auctioneer’s revenue. Fourth, when it comes to multiple-bidder SPAs, although the role of dependence is not clearly understood yet, we reach the conclusion that under some common negative dependence structure, less asymmetric valuations will lead to a stochastically higher winner’s payment in SPAs with two or more bidders. Fifth, we also find that the stronger positive dependence between valuations does harm to the final price in two-bidder FPAs. Finally, for multiple-bidder FPAs with positively (negatively) dependent valuations, a higher degree of dependence along with less asymmetry implies a smaller (larger) winner’s payment. That is, the extent of asymmetry affects FPAs in quite the opposite way under different dependence structures. With these discoveries, we can handle the interdependence among valuations that might arise in different situations and arrive at a more accurate evaluation of the winner’s payment and expected revenue of the private value auctions.

The rest of this article is organized as follows: Section 2 introduces several concepts that will be used in the remaining sections. Section 3 has a discussion on auctions with two bidders. we first develop results regarding effects of dependent and asymmetric valuations on the final price and winner’s expected rent of SPAs, and then study FPAs with dependent and asymmetric bids. The multiple-bidder auctions are considered in Section 4. Concluding remarks are addressed in Section 5. For ease of reference, all proofs of the theoretical results are included in Appendix A.

For convenience, let $D = \{1, \cdots, d\}$ and denote $\phi^{(k)}$ the $k$th derivative of $\phi$. Throughout this manuscript, all random variables are assumed to be nonnegative and of finite expectations; also, by increasing and decreasing we mean nondecreasing and nonincreasing, respectively.
2. Preliminaries

This section first recalls some important concepts to be used in the subsequent sections for ease of reference and then introduces the model along with the benchmark auction environment.

2.1. Some Orders of Real Vectors and Random Variables

Let $a_1:d \leq \cdots \leq a_{d:d}$ be the ordered version of $a_1, \ldots, a_d$. Then, $a = (a_1, \ldots, a_d)$ is said to

(i) majorize $b \in \mathbb{R}^d$ (denoted as $b \preceq^m a$) if $\sum_{i=1}^d b_i = \sum_{i=1}^d a_i$ and $\sum_{i=1}^k b_{i:d} \geq \sum_{i=1}^k a_{i:d}$ for $k = 1, 2, \cdots, d-1$;

(ii) weakly supermajorize $b \in \mathbb{R}^d$ (denoted as $b \preceq^w a$) if $\sum_{i=1}^k b_{i:d} \geq \sum_{i=1}^k a_{i:d}$ for $k \in \mathcal{D}$;

(iii) weakly submajorize $b \in \mathbb{R}^d$ (denoted as $b \preceq_w a$) if $\sum_{i=k}^d b_{i:d} \leq \sum_{i=1}^k a_{i:d}$ for $k \in \mathcal{D}$.

The majorization orders play an important role in econometrics, operations research, and reliability. Succinctly speaking, $b \prec^w (\prec_w$ and $\prec_m) a$ asserts that the components of $b$ are more symmetric than those of $a$. For more details we refer readers to Marshall et al. (2009).

A random variable $X$ is said to be smaller than the random variable $Y$ in the

(i) stochastic order (denoted as $X \leq_{st} Y$) if $P(X > x) \leq P(Y > x)$ for all real $x$;

(ii) increasing convex (concave) order (denoted as $X \leq_{icx}$ ($\leq_{icv}$) $Y$) if $E[h(X)] \leq E[h(Y)]$ for all increasing convex (concave) functions $h$, provided the expectations exist.

Stochastic orders are quite useful in finance, insurance, and economics. For more on these orders, one may refer to Shaked and Shanthikumar (2007) and Li and Li (2013).

2.2. Archimedean Copula

Formally, for a random vector $X = (X_1, \ldots, X_d)$ with joint distribution function $F$ and univariate marginal distribution functions
$F_1, \ldots, F_d$, its copula is defined as

$$C_X(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \quad 0 < u_1, \ldots, u_d < 1,$$

where $F_i^{-1}$ is the left continuous inverse of $F_i$ for $i \in D$.

Because the copula does not contain any information of marginal distributions, it provides a particularly convenient way to impose a dependence structure onto predetermined marginal distributions in practice. Actually, a large number of excellent applications of copulas can be found in various areas, and so far, the copula has become more or less a standard tool in risk management, finance, econometrics, and actuarial science, etc. For more on the copula and its applications, we refer readers to monographs and surveys by Joe (1997) and Nelsen (2006). In the literature, there are a large number of copulas depicting various dependence structures; Hutchinson and Lai (1990) and Nelsen (2006) provide a wide range of families of bivariate copulas along with their properties. Archimedean copulas are rather popular because of the mathematical tractability and the capability of capturing wide ranges of dependence. Notably, since the statistical inference on Archimedean copulas has become well developed in this decade, excellent applications are coming to the fore in various areas. In this study, we will specifically employ Archimedean copulas to model the dependence structure among bidders’ valuations.

**Definition 2.1.** Suppose that $\phi : [0, +\infty) \mapsto [0, 1]$ is a non-increasing and continuous function such that $\phi(0) = 1$ and $\phi(+\infty) = 0$. Let $\phi^{-1}$ be its inverse, then the $d$-dimensional function

$$C_\phi(u_1, \ldots, u_d) = \phi(\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)),$$

for $(u_1, \ldots, u_d) \in [0, 1]^d$, (1)

is called an Archimedean copula with the generator $\phi$.

According to Theorem 2.2 in McNeil and Nesálehořová (2009), (1) is an Archimedean copula generated by $\phi$ if and only if $(-1)^{k-2}\phi^{(k-2)}(x) \geq 0$ for $k = 2, \ldots, d$ and $(-1)^{d-2}\phi^{(d-2)}(x)$ is
TABLE 1 Some common Archimedean copulas.

<table>
<thead>
<tr>
<th>Generator $\phi$</th>
<th>Copula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independence $e^{-t}$</td>
<td>$C_1(u) = \prod_{i=1}^{d} u_i^{-1/\theta}$</td>
</tr>
<tr>
<td>Clayton $(\theta t + 1)^{-1/\theta}, \theta \geq 0$</td>
<td>$C_2(u) = \left( \prod_{i=1}^{d} u_i^{-\theta} - d + 1 \right)^{-1/\theta}$</td>
</tr>
<tr>
<td>Frank $-\frac{1}{\theta} \ln \left( 1 + e^{-t} (e^{-\theta} - 1) \right), \theta \geq 0$</td>
<td>$C_3(u) = -\frac{1}{\theta} \ln \left( 1 + \prod_{i=1}^{d} \frac{(e^{-\theta} u_i - 1)}{(e^{-\theta} - 1)^{d-1}} \right)$</td>
</tr>
<tr>
<td>AMH $\frac{1-\theta}{e^{1-\theta}}, \theta \in [0, 1)$</td>
<td>$C_4(u) = \frac{(1-\theta) \prod_{i=1}^{d} u_i}{\prod_{i=1}^{d} (1-\theta + \theta u_i)^{\theta} \prod_{i=1}^{d} u_i}$</td>
</tr>
</tbody>
</table>

nonincreasing and convex. The Archimedean family contains many useful copulas, including some that are well known. Table 1 lists several common Archimedean copulas.

Kendall’s $\tau$ is widely applied to quantify the strength of dependence between two random variables. It ranges on $[-1, 1]$, and the larger absolute value implies stronger dependence. A bivariate Archimedean copula $C_\phi$ with generator $\phi$ has its Kendall’s $\tau$ evaluated as $\tau_\phi = 1 + 4 \int_0^1 \frac{\phi^{-1}(t)}{|\phi^{-1}(t)|} \, dt$. And Kendall’s $\tau$ of the Clayton copula, Frank copula, AMH copula, and many other one-parameter Archimedean copulas (given in Table 4.1 of Nelsen, 2006) are monotonic in the dependence parameter.

Recall that a random vector $X$ is said to be positively lower orthant dependent (PLOD) if $P\{X_1 \leq x_1, \ldots, X_d \leq x_d\} \geq \prod_{i=1}^{d} P\{X_i \leq x_i\}$ for any $x \in \mathbb{R}^d$; and it is said to be negatively lower orthant dependent (NLOD) if the inequality is reversed. For more on PLOD and NLOD please refer to Denuit et al. (2005). The next lemma tells us that, among others, those Archimedean copulas with a log-convex (log-concave) generator entail the specific nonparametric positive (negative) dependence in terms of PLOD (NLOD).

**Lemma 2.2.** For a random vector $X$ with an Archimedean copula, if the generator is log-convex (log-concave), then it is PLOD (NLOD).

In fact, the log-convexity (log-concavity) of the generator $\phi$ means that $\phi_0^{-1} \circ \phi$ is subadditive (superadditive), where $\phi_0(t) = e^{-t}$ is the generator corresponding to the independence cop-
ula. Namely, $\phi^{-1} \circ \phi$ is in correspondence to the strength of dependence. In this regard, we are also interested in any two Archimedean copulas with generators $\phi_1$ and $\phi_2$ such that $\phi^{-1}_1 \circ \phi_2$ is superadditive (subadditive). In view of $\phi^{-1}_1 \circ \phi_2(0) = 0$, it follows from Proposition A.11 of Marshall and Olkin (2007) that the convexity (concavity) of $\phi^{-1}_1 \circ \phi_2$ implies the superadditivity (subadditivity). Take Clayton copulas, for example:

(i) for two generators $\phi_1(t) = (at + 1)^{-1/a}$ and $\phi_2(t) = (bt + 1)^{-1/b}$, we have

$$\phi^{-1}_1 \circ \phi_2 = \frac{-1 + (1 + bt)^{a/b}}{a},$$

which is convex (concave) whenever $a \geq (\leq) b$, and

(ii) $C_\phi$ with parameter $a$ has the Kendall’s $\tau = a/(a + 2)$, increasing in $a$.

That is, in the Clayton family, weakening (strengthening) the positive aspect of dependence (smaller (larger) Kendall’s $\tau$) actually leads to the convexity and hence superadditivity (subadditivity) of the transformation (2). In fact, a handful of one-parameter Archimedean copulas listed in Table 4.1 of Nelsen (2006) have this property, such as the Frank copula, the AMH copula and so on.

2.3. The Model Formulation

In this study, we equip valuations with Archimedean copulas and then study how the interdependence has an impact on SPA and FPA. In the following sections, we denote $V_{k:d}(\phi)$ as the $k$th smallest order statistics of $(V_1, \ldots, V_d)$ with the Archimedean copula generated by $\phi$ and $B_{d:d}(\phi)$ as the largest order statistics of the corresponding bids $(B_1(V_1), \ldots, B_d(V_d))$. Also, we utilize Kendall’s $\tau$ to measure the strength of dependence.

For a private value auction, assume $d$ risk-neutral bidders competing to obtain an object for sale through a sealed-bid FPA or SPA. Bidders’ valuations have their distribution functions $F_i$’s with $i \in D$, which are interdependent through some Archimedean
copula. The realization of valuations is private information but the dependence structure and distributions are common knowledge. Note that valuations’ marginal distributions, along with their Archimedean copula’s generator \((F_1, \ldots, F_d; \phi)\), completely characterize the auction environment. In our framework, we refer to \((F_1, \ldots, F_d; \phi)\) as a \(\phi\)-configuration. Particularly, \(\phi_0(t) = e^{-t}\), the \(\phi_0\)-configuration characterizes the mutually independent private valuations.

To understand how dependence among valuations affects the outcome in the FPAs and SPAs, we will compare the outcome in a dependent-value auction with that in the benchmark, which has the same marginal valuations as those of the auction with interdependent valuations, so that the difference between the final results only originates from the interdependence. That is, we will choose the \(\phi_0\)-configuration as a benchmark.

3. Auctions with two bidders

To begin with, we focus on auctions involving only two bidders and study the role that the interdependence between them plays in determining the outcome of auctions.

3.1. Dependence’s Impact on the SPA

Here, we investigate the role played by the dependence among valuations in the sealed-bid private value SPA. In this context, the bidder attaining the highest bid wins the object and pays the second-highest bid, and it is weakly dominant for a bidder to bid his/her true valuation, \(B_i(V_i) \equiv V_i\). Then, the winner’s payment has the probability distribution

\[
P\{V_{1:2} (\phi) \leq x\} = P\{V_1 \leq x\} + P\{V_2 \leq x\} - P\{V_1 \leq x, V_2 \leq x\}
= F_1 (x) + F_2 (x) - \phi (\phi^{-1} (F_1 (x)) + \phi^{-1} (F_2 (x)))) ,
\]

\[x \geq 0\]

(3)
and the winner’s expected rent
\[
\mathbb{E}[V_{2:2}(\phi) - V_{1:2}(\phi)] = \int_0^{+\infty} (1 - P(V_{2:2}(\phi) \leq x)) \, dx \\
- \int_0^{+\infty} (1 - P(V_{1:2}(\phi) \leq x)) \, dx \\
= \int_0^{+\infty} (P(V_{1:2}(\phi) \leq x) - P(V_{2:2}(\phi) \leq x)) \, dx \\
= \int_0^{+\infty} [F_1(x) + F_2(x) - 2\phi^{-1}(F_1(x)) + \phi^{-1}(F_2(x))] \, dx.
\]
(4)

The winner’s expected rent, just the expected surplus of the winning bidder, can be employed by auctioneers to increase their auction revenue. A substantial amount of literature on auctions is devoted to increasing auctioneers’ profit by using the winner’s expected rent; see Paul and Genaro (2004) and Li (2005) for references. Our first theorem conducts a comparison on the winner’s payments and expected rents in two SPAs with \(\phi\)-configuration and \(\phi_0\)-configuration in terms of the stochastic order.

**Theorem 3.1.** For sealed-bid SPAs with \(\phi\)-configuration \((F_1, F_2; \phi)\) and benchmark \((F_1, F_2; \phi_0)\), if \(\ln \phi\) is subadditive (superadditive), then

\[
V_{1:2}(\phi) \leq_{st} (\geq_{st}) V_{1:2}(\phi_0), \quad \text{and} \quad \mathbb{E}[V_{2:2}(\phi) - V_{1:2}(\phi)] \geq (\leq) \mathbb{E}[V_{2:2}(\phi_0) - V_{1:2}(\phi_0)].
\]

For many of Archimedean copula \(C_\phi\), the subadditivity (superadditivity) of \(\ln \phi\) implies the negative (positive) dependence of the corresponding random vector. Theorem 3.1 points out that for two SPAs with common valuations, the negative dependence will stochastically lower the winner’s payment while the positive dependence impacts the outcome in an opposite direction. And wrongly assuming the independence between bidders’ valuations could underestimate or overestimate the winner’s payment and, hence, the auctioneer’s revenue. As for the winner’s expected
rent, exactly the opposite: the negative dependence will enhance the winner’s expected rent, and positive dependence will decrease the winner’s expected rent.

Next, let us compare the winner’s payments and expected rents in two SPAs, \((F_1, F_2; \phi_1)\) and \((F_1, F_2; \phi_2)\), with different dependence structures.

**Theorem 3.2.** For sealed-bid SPAs \((F_1, F_2; \phi_i)\) for \(i = 1, 2\), if \(\phi_1^{-1} \circ \phi_2\) is superadditive (subadditive), then

\[
V_{1:2}(\phi_1) \gtrless \left( \preceq \right) V_{1:2}(\phi_2), \quad \text{and} \\
\mathbb{E}[V_{2:2}(\phi_1) - V_{1:2}(\phi_1)] \leq \left( \geq \right) \mathbb{E}[V_{2:2}(\phi_2) - V_{1:2}(\phi_2)].
\]

Theorem 3.2 confirms that, for SPAs with positively interdependent valuations, stronger dependence plays a positive role in enhancing the auctioneer’s revenue; for SPAs with negatively interdependent valuations, auctioneer’s revenue behaves in the opposite direction as the valuations’ interdependence is strengthened. Therefore, the interdependence could either benefit or harm the winner’s payment and the winner’s expected rent in the SPAs.

Valuations are usually asymmetric due to bidders’ different economic backgrounds, knowledge about the object for sale, and other self-oriented factors. Recall that random variables \(X_1, \ldots, X_n\) are of proportional reversed hazards PRH\((H, \alpha)\) if \(X_i \sim H^{\alpha_i}(x)\) with the baseline distribution \(H\) and parameters \(\alpha_i > 0\), \(i = 1, \ldots, n\). In the rest of this subsection, we study SPAs of asymmetric bidders with a common dependence structure. Specifically, we assume valuations are of the proportional reverse hazard (PRH) model. In the presence of dependence, the next theorem investigates the impact of the degree of symmetry in terms of the weak supermajorization on the winner’s payment in SPAs.

**Theorem 3.3.** For sealed-bid SPAs \((\text{PRH}(H, \alpha); \phi)\) and \((\text{PRH}(H, \beta); \phi)\), if \(\phi\) is log-concave and \(\beta \prec_w \alpha\), then

\[
V_{1:2}(\phi, \alpha) \preceq \left( \succeq \right) V_{1:2}(\phi, \beta), \quad \text{and} \\
\mathbb{E}[V_{2:2}(\phi, \alpha) - V_{1:2}(\phi, \alpha)] \geq \left( \leq \right) \mathbb{E}[V_{2:2}(\phi, \beta) - V_{1:2}(\phi, \beta)].
\]
According to Lemma 2.2, the log-concavity of the generator corresponds to negative dependence. Theorem 3.3 reveals that, for two SPAs with common negatively dependent valuations, the one with less asymmetric valuations will have a stochastically larger final price and a smaller winner’s expected rent. Intuitively, when there are only two bidders, the winner’s payment is equal to the smallest bids, and the winner’s expected rent is the average difference between the maximum and the final price. Because asymmetry between valuations will make the chance of a small smallest bid become larger, it is harmful to the final price but is beneficial to the winner’s expected rent. Moreover, the negative dependence makes bidders’ bids behave in a less concordant way. Therefore, less asymmetry with common negative dependence will enhance the winner’s payment but reduce the winner’s expected rent in SPAs.

3.2. Dependence’s Impact on the FPA

Let us consider in the following the sealed-bid first-price auction, in which the bidder with the highest bid wins the auction and pays his own bid. In the literature, it is found that a monotone pure strategy equilibrium exists in the context of mutually independent private values with continuously differentiable distribution function on an interval; see Lebrun (1996) and Reny and Zamir (2004). For valuations with interdependence, we have no idea on the existence of the monotone equilibrium bidding strategies. Nevertheless, monotonicity of the expected payments in FPAs may still be addressed when bidders adopt strategies non-decreasing in their private valuations, and thus some light can be cast onto the impact on FPAs because of the dependence among valuations. Here and in Section 4.2, we always assume that bidders’ bidding strategies are nondecreasing in valuations. According to Proposition 4.4.4 in Denuit et al. (2005), the bids have the same copula as that of the valuations. Instead of comparing two FPAs based on distributions of valuations, we focus on the distributions of bids. For ease of reference, we will represent FPAs with \(\phi\)-configuration \((G_1, \ldots, G_d; \phi)\), of which \(G_i\) is the distribution of bids instead of valuations and \(\phi\) is the generator of valuations’ copulas.
In accordance with the first result, mistakenly assigning independence to valuations will either overestimate or underestimate the winner’s payment in an FPA.

**Theorem 3.4.** For a sealed-bid FPA with the configuration \((G_1, G_2; \phi)\), if \(\ln \phi\) is subadditive (superadditive), then the real winner’s payment is underestimated (overestimated) when valuations are mistakenly deemed to be independent.

Succinctly speaking, the subadditivity (superadditivity) of \(\ln \phi\) implies negatively (positively) dependent valuations. Theorem 3.4 tells us that, opposed to the independent valuations, the negatively (positively) dependent valuations result in a stochastically larger (smaller) winner’s payment in FPAs. So, it is of great importance to identify the dependence structure among valuations in practice. Our next result takes the strength of interdependence into a further consideration.

**Theorem 3.5.** For sealed-bid FPAs with their configurations \((G_1, G_2; \phi_1)\) and \((G_1, G_2; \phi_2)\). If \(\phi_1^{-1} \circ \phi_2\) is superadditive (subadditive), then \(B_{2:2}(\phi_1) \leq_{st} \geq_{st} B_{2:2}(\phi_2)\).

This theorem points out one interesting fact: in contrast to the impact of dependence in SPAs, the stronger positive dependence between valuations will do harm to the final price in FPAs. This is not counterintuitive because the positive dependence forces the bids to be more concerted, and hence, the largest bid may be driven down.

Intuitively higher bids lead to higher winner’s payment in FPAs, however, the next example shows that this is not always the case. When the dependence between higher bids gets stronger, the winner’s payment might be lowered instead of being raised.

**Example 3.6.** Let \(B_1 = (B_{11}, B_{12})\) and \(B_2 = (B_{21}, B_{22})\) be bids of bidders in two FPAs, \(B_{1i} \sim B(3, 2.1)\) and \(B_{2i} \sim B(3, 2)\), \(i = 1, 2\), where \(B(c_1, c_2)\) denotes the beta distribution function with parameters \(c_1\) and \(c_2\).

It is seen in Figure 1(a) that \(B_{1i} \leq_{st} B_{2i}\) for \(i = 1, 2\). Assume \((B_{11}, B_{12})\) and \((B_{21}, B_{22})\) have Clayton copulas generated by \(\phi_1(t) = (2t + 1)^{-1/2}\) and \(\phi_2(t) = (8t + 1)^{-1/8}\), respectively. It is
easy to verify that \( \phi_1^{-1} \circ \phi_2(t) = \left[(8t + 1)^{1/4} - 1\right]/2 \) is subadditive. Also, as is seen in Figure 1(b), distribution curves of \( B_{2:2}(\phi_1) \) and \( B_{2:2}(\phi_2) \) cross each other; this invalidates \( B_{2:2}(\phi_1) \leq_{st} B_{2:2}(\phi) \) directly.

Example 3.6 tells us that the maximum of bids with stronger dependence is not necessarily larger than that of the smaller bids with weaker dependence structure. Intuitively, with stronger positive dependence, bids behave more concordant, and hence, the chance for all of them to be small will be enlarged, which leads to a stochastically smaller final price.

Now, by assuming the PRH rates for bids, we consider the joint influence of dependence and asymmetry on FPAs.

**Theorem 3.7.** For sealed-bid FPAs \( (\text{PRH}(H, \alpha); \phi_1) \) and \( (\text{PRH}(H, \beta); \phi_2) \) with \( \beta \prec_w (\succ_w) \alpha \), if \( \phi_1 \)(\( \phi_2 \)) is log-convex (log-concave), and \( \phi_1^{-1} \circ \phi_2 \) is subadditive, then \( B_{2:2}(\phi_1, \alpha) \geq_{st} B_{2:2}(\phi_2, \beta) \).

Theorem 3.7 finds that in some FPAs with positively (negatively) dependent valuations, the higher the degree of dependence and the less asymmetry there is, the smaller (larger) the winner’s payment will be. That is to say, the extent of asymmetry affects FPAs in quite the opposite way under different dependence structures.

### 4. Auctions with Multiple Bidders

So far we have discussed the impact of interdependent valuations in two-bidder auctions. As a continuation, this section studies the auctions with multiple bidders.
Unlike the two-bidder SPA, the strength of dependence plays a different role in impacting the winner’s payment in multiple-bidder SPAs. Example 4.1 shows that even for symmetric valuations with more than 2 bidders, the winner’s payment in two SPAs with different dependence structures may not be comparable in the sense of stochastic order.

Example 4.1. Consider (i) two three-bidder SPAs \((F, F, F; \phi_1)\) and \((F, F, F; \phi_2)\); (ii) two four-bidder SPAs \((F, F, F; \phi_1)\) and \((F, F, F; \phi_2)\), where \(F\) is the uniform distribution on \([0, 100]\), \(\phi_1\) and \(\phi_2\) are generators of Clayton copula with parameters 10 and 1, respectively. Figure 2 plots the differences between distributions of winner’s payments of SPAs in (i) and (ii) separately. As can be seen, both differences range from negative to positive, indicating that the SPA with stronger positive dependence doesn’t have a stochastically larger winner’s payment. Moreover, we can calculate the difference between expected winner’s payments as

\[
\mathbb{E}[V_{2:3}(\phi_1) - V_{2:3}(\phi_2)] = -0.6480,
\]

\[
\mathbb{E}[V_{3:4}(\phi_1) - V_{3:4}(\phi_2)] = 4.2182.
\]

That is, the impact of the interdependence strength on the expected final price of SPA is subject to the number of bidders, even under identical valuations.

Although the impact of the strength of interdependence on SPAs is unclear, when valuations are of PRH rates, the following theorem articulates the impact of asymmetry on the final price in an SPA.
Theorem 4.2. For sealed-bid SPAs \((\text{PRH}(H, \alpha); \phi)\) and \((\text{PRH}(H, \beta); \phi)\) with \(d \geq 2\) bidders, if \(\phi\) is log-concave and \(\beta \prec^w \alpha\), then

\[
V_{d-1:d}(\phi, \alpha) \leq\text{st } V_{d-1:d}(\phi, \beta) \quad \text{and} \quad \mathbb{E}[V_{d:d}(\phi, \alpha) - V_{d-1:d}(\phi, \alpha)] \geq \mathbb{E}[V_{d:d}(\phi, \beta) - V_{d-1:d}(\phi, \beta)].
\]

Per Lemma 2.2, the log-concave generator implies the negative dependence NLOD. Theorem 4.2 further concludes that with the presence of common negative dependence structure, the more asymmetric the valuations are, the lower the winner’s payment will be. That is to say, the asymmetry is harmful to the auctioneer’s revenue.

Example 4.3. (Independent valuations with proportional reversed hazard) For two private value SPAs with mutually independent valuations having \(\text{PRH}(H, \alpha)\) and \(\text{PRH}(H, \beta)\), respectively, Cantillon (2008, Theorem 2) and Waehrer and Perry (2003) showed that the majorization \(\beta \prec_m \alpha\) implies the stochastic order \(V_{d-1:d}(\alpha) \leq\text{st } V_{d-1:d}(\beta)\). For two SPAs \((\text{PRH}(H, \alpha); \phi_0)\) and \((\text{PRH}(H, \beta); \phi_0)\), since \(\phi_0(t) = e^{-t}\) is log-concave, and \(\beta \prec_m \alpha\) implies \(\beta \prec^w \alpha\), by Theorem 4.2, it can be concluded that \(V_{d-1:d}(\phi_0, \alpha) \leq\text{st } V_{d-1:d}(\phi_0, \beta)\). That is, the result due to Cantillon (2008, Theorem 2) and Waehrer and Perry (2003) is just a special case of Theorem 4.2.

Theorem 4.2 extends Theorem 3.2 to auctions with more than 2 bidders. One naturally wonders what will happen if the log-concavity of the generator (corresponding to negative dependence) is replaced by log-convexity. The following example tells us that with the log-convex generator and, hence, positively dependent valuations, the effect of asymmetry may be undetermined.

Example 4.4. For three SPAs \((\text{PRH}(H, \alpha); \phi)\), \((\text{PRH}(H, \beta); \phi)\), and \((\text{PRH}(H, \zeta); \phi)\) with

\[
H \sim B(3, 2), \quad \alpha = (1, 3, 8),
\]
FIGURE 3 Difference between distribution functions of winner’s payment.

\[ \beta = (2, 2, 8), \quad \zeta = (2, 4, 6), \]

and \( \phi \) is the generator of the Clayton copula with parameter 2, which can be easily verified to be log-convex. Clearly, it holds that \( \alpha \succ_m \beta \succ_m \zeta \) and \( V_{2,3}(\phi, \alpha), V_{2,3}(\phi, \beta), \) and \( V_{2,3}(\phi, \zeta) \) denote the winner’s payments of the three SPAs, respectively.

As is seen in Figure 3 (a) and (b), the winner’s payments might not be stochastically dominated by each other. In fact, the winner’s expected payments of the three SPAs can be evaluated as 0.7694, 0.7499, and 0.7958. Hence, in this context, the winner’s payments of the three SPAs with asymmetric valuations might not be comparable even by using the increasing convex order or increasing concave order, let alone the stochastic order.

4.2. FPA with Dependent Valuations

Now we move our focus to FPAs with multiple bidders. Again, we assume the bids follow the PRH rate model. Unlike results in the previous subsection about SPAs, we can generalize Theorem 3.7 from the two-bidder case to the multiple-bidder case directly.

**Theorem 4.5.** For sealed-bid FPAs \((\text{PRH}(H, \alpha), \phi_1)\) and \((\text{PRH}(H, \beta), \phi_2)\) with \( \beta \prec_w (\succ_w) \alpha \), if \( \phi_1 \) (\( \phi_2 \)) is log-convex (log-concave) and \( \phi_1^{-1} \circ \phi_2 \) is subadditive, then \( B_{d:d}(\phi_1, \alpha) \preceq_s B_{d:d}(\phi_2, \beta) \).

In brief, Theorem 4.5 states that (i) for positively interdependent valuations, bids that are more symmetric along with stronger positive dependence will be harmful to the final price, and (ii)
the FPA with bids that are less symmetric and weaker negatively dependent valuations will have a smaller final price. This finding is not counterintuitive. In fact, with positive dependence, more asymmetry among bids guarantees one stochastically larger component, and the weaker positive dependence enhances the chance of a larger maximum. And with negative dependence, the symmetry among bids leads to averagely larger realization of maximum, and the stronger negative dependence enhances the chance of a larger maximum.

5. Concluding Remarks

This work further studies the impact of the interdependence within bidders’ valuations on winner’s payment in FPAs and SPAs from the viewpoint of the auctioneer. A benchmark with independent valuations is introduced to display the role the interdependence plays in this context, and it is found that mistakenly ignoring the interdependence structure of valuations incurs either conservative or panglossian evaluation of the final price. Briefly speaking, in FPAs, positive (negative) dependence among valuations may undermine (reinforce) the auctioneer’s profit. Also, we show that the auctioneer is better off with bids in FPAs that are less dependent and more symmetric. However, for SPAs, an almost opposite result is reached: stronger dependence could result in a positive effect on the winner’s payment. Moreover, stronger dependence may decrease the winner’s expected rent.

It is of interest to further pursue the following several aspects in future work:

- To the best of our knowledge, the impact of the strength of dependence on the final prices of two SPAs is not clear yet. Although Example 4.1 tells us that, for uniformly distributed valuations, the SPA with stronger positive dependence may result in a smaller expected payment when the number of bidders is large, it is still an open question in general.
- It is also of interest to investigate the role of interdependence in other auction formats, such as the open ascending price (English) auction and the open descending price (Dutch) auction.
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References


Appendix A

A.1. Lemma 2.2 Let $X$ have the Archimedean copula $C_\phi$ and the univariate marginal distributions $F_i$, $i = 1, \cdots, d$. Since $\phi$ is log-convex, $\phi'/\phi = (\ln \phi)'$ is increasing, and hence,

$$
\left[ \phi^{-1}(e^{-t}) \right]' = -\frac{e^{-t}}{\phi'(\phi^{-1}(e^{-t}))} = -\frac{\phi(\phi^{-1}(e^{-t}))}{\phi'(\phi^{-1}(e^{-t}))}
$$

is increasing in $t \geq 0$. Namely $\phi^{-1}(e^{-t})$ is convex with respect to $t \in [0, \infty)$. In view of $\phi^{-1}(e^0) = 0$, it follows from Proposition A.11 of Marshall and Olkin (2007) that $\phi^{-1}(e^{-t})$ is superadditive. Now note that $\phi$ is decreasing, we have, for any real $x_i$, $i = 1, \cdots, d$,

$$
P\{X_1 \leq x_1, \cdots, X_d \leq x_d\} = \prod_{i=1}^d F_i(x_i)
$$

$$
= \phi\left(\sum_{i=1}^d \phi^{-1}(F_i(x_i))\right) - \phi \circ \phi^{-1}\left(\exp\left\{\sum_{i=1}^d \ln(F_i(x_i))\right\}\right)
$$

$$
\geq \phi\left(\sum_{i=1}^d \phi^{-1}(F_i(x_i))\right) - \phi\left(\sum_{i=1}^d \phi^{-1} \circ \exp\{\ln(F_i(x_i))\}\right) = 0.
$$

Therefore, $X$ is PLOD.

The case with the log-concave part can be proved in a similar manner.

A.2. Theorem 3.1 We prove only the subadditive case, the superadditive case can be proved in a similar manner. Since $\phi_0$, $\phi$ and $F_i(x)$ are all continuous, there exists some $\epsilon_1(x)$ such that

$$
\phi_0(\phi^{-1}_0(F_1(x)) + \phi^{-1}_0(F_2(x)) + \epsilon_1(x)) = \phi(\phi^{-1}(F_1(x)) + \phi^{-1}(F_2(x))).
$$

Moreover, the subadditivity of $\ln \phi$ is equivalent to the superadditivity of $-\ln \phi = \phi^{-1}_0 \circ \phi$, this implies, for any $x$,

$$
\epsilon_1(x) = \phi^{-1}_0 \circ \phi(\phi^{-1}(F_1(x)) + \phi^{-1}(F_2(x)))
$$

$$
- \phi_0^{-1}(F_1(x)) - \phi_0^{-1}(F_2(x))
$$
\[= \phi_0^{-1} \circ \phi(\phi^{-1}(F_1(x))\phi^{-1}(F_2(x))) - \phi_0^{-1} \circ \phi(\phi^{-1}(F_1(x)))
- \phi_0^{-1} \circ \phi(\phi^{-1}(F_2(x))) \geq 0,\]

and hence \(\phi_0(\phi_0^{-1}(F_1(x)) + \phi_0^{-1}(F_2(x))) \geq \phi(\phi^{-1}(F_1(x)) + \phi^{-1}(F_2(x))).\) As a result, from (3) and (4) it follows that, for any \(x \geq 0,\)

\[P\{V_{1:2}(\phi) \leq x\} - P\{V_{1:2}(\phi_0) \leq x\}
= \phi_0(\phi_0^{-1}(F_1(x)) + \phi_0^{-1}(F_2(x))) - \phi(\phi^{-1}(F_1(x))
+ \phi^{-1}(F_2(x))) \geq 0,\]

and

\[E[V_{2:2}(\phi) - V_{1:2}(\phi)] - E[V_{2:2}(\phi_0) - V_{1:2}(\phi_0)]
= 2 \int_0^{+\infty} \left[ \phi_0(\phi_0^{-1}(F_1(x)) + \phi_0^{-1}(F_2(x))) - \phi(\phi^{-1}(F_1(x))
+ \phi^{-1}(F_2(x))) \right] dx \geq 0.\]

That is, \(V_{1:2}(\phi) \leq_s V_{1:2}(\phi_0)\) and \(E[V_{2:2}(\phi) - V_{1:2}(\phi)] \geq E[V_{2:2}(\phi_0) - V_{1:2}(\phi_0)].\) \(\square\)

**A.3. Theorem 3.2** Since \(F_1, F_2, \phi_1\) and \(\phi_2\) are all continuous, for any \(x \geq 0,\) there exists some \(\varepsilon_2(x),\) such that

\[\phi_1(\phi_1^{-1}(F_1(x)) + \phi_1^{-1}(F_2(x)) + \varepsilon_2(x)) = \phi_2(\phi_2^{-1}(F_1(x)) + \phi_2^{-1}(F_2(x))).\]

Then, the superadditivity of \(\phi_1^{-1} \circ \phi_2\) implies that, for any real \(x,\)

\[\varepsilon_2(x) = \phi_1^{-1} \circ \phi_2(\phi_2^{-1}(F_1(x)) + \phi_2^{-1}(F_2(x)))
- \phi_1^{-1}(F_1(x)) - \phi_1^{-1}(F_2(x))
= \phi_1^{-1} \circ \phi_2(\phi_2^{-1}(F_1(x)) + \phi_2^{-1}(F_2(x)))
- \phi_1^{-1} \circ \phi_2(\phi_2^{-1}(F_1(x))) - \phi_1^{-1} \circ \phi_2(\phi_2^{-1}(F_2(x))) \geq 0.\]

Note that \(\phi_1\) is decreasing, by (3), it holds that, for all \(x,\)

\[P\{V_{1:2}(\phi_1) \leq x\} - P\{V_{1:2}(\phi_2) \leq x\} \]
\[= \phi_2(\phi_1^{-1}(F_1(x)) + \phi_2^{-1}(F_2(x))) - \phi_1(\phi_1^{-1}(F_1(x))
+ \phi_1^{-1}(F_2(x)))
\]
\[= \phi_1(\phi_1^{-1}(F_1(x)) + \phi_1^{-1}(F_2(x)) + \varepsilon_2(x)) - \phi_1(\phi_1^{-1}(F_1(x))
+ \phi_1^{-1}(F_2(x))) \leq 0.\]

So, we have \(P\{V_{1:2}(\phi_1) \leq x\} \leq P\{V_{1:2}(\phi_2) \leq x\}\) for all \(x\) and hence \(V_{1:2}(\phi_1) \geq_{sl} (\leq_{sl}) V_{1:2}(\phi_2)\). Also by (4), we have
\[E[V_{2:2}(\phi_1) - V_{1:2}(\phi_1)] - E[V_{2:2}(\phi_2) - V_{1:2}(\phi_2)]
= 2 \int_0^\infty \left[ \phi_2(\phi_2^{-1}(F_1(x)) + \phi_2^{-1}(F_2(x))) - \phi_1(\phi_1^{-1}(F_1(x))
+ \phi_1^{-1}(F_2(x))) \right] dx
\]
\[= 2 \int_0^\infty \left[ \phi_1(\phi_1^{-1}(F_1(x)) + \phi_1^{-1}(F_2(x)) + \varepsilon_2(x))
- \phi_1(\phi_1^{-1}(F_1(x)) + \phi_1^{-1}(F_2(x))) \right] dx \leq 0.\]

That is, \(E[V_{2:2}(\phi_1) - V_{1:2}(\phi_1)] \leq E[V_{2:2}(\phi_2) - V_{1:2}(\phi_2)]\).

The case of the subadditivity can be proved in a similar manner.

\[\hfill\Box\]

**A.4. Theorem 3.3.** We first recall an important definition. For a real vector \(a = (a_1, \ldots, a_d)\) with components arrayed in the descending order and some constant \(\gamma \in [1/2, 1]\), the mapping
\[T_\gamma(a) = (a_1, \ldots, a_{j-1}, \gamma a_j + (1 - \gamma) a_k, a_{j+1}, \ldots, a_{k-1}, \gamma a_k
+ (1 - \gamma) a_j, a_{k+1}, \ldots, a_d)\]
is called a \(T\)-transform of \(a\). For more on \(T\)-transforms, please refer to Marshall et al. (2011).

Let us use the notations in the proof of Theorem 3.2. Without loss of generality, we assume \(\alpha_1 \geq \alpha_2\) and \(\beta_1 \geq \beta_2\). Denote \(T_\gamma(\alpha) = (\alpha_1', \ldots, \alpha_d')\) the \(T\)-transform of \(\alpha\). In view of
\[P\{V_{1:2}(\phi, \alpha) \leq x\} = H^{\alpha_1}(x) + H^{\alpha_2}(x) + \phi(\phi^{-1}(H^{\alpha_1}(x))
- \phi^{-1}(H^{\alpha_2}(x))),\]
we have
\[ \Delta(x, \gamma) = P\{V_{1:2}(\phi, T_{\gamma}(\alpha)) \leq x\} - P\{V_{1:2}(\phi, \alpha) \leq x\} \]
\[ = H^{\alpha_1}(x) + H^{\alpha_2}(x) - \phi^{-1}(H^{\alpha_1}(x)) + \phi^{-1}(H^{\alpha_2}(x)) \]
\[ - H^{\alpha_1}(x) - H^{\alpha_2}(x) + \phi^{-1}(H^{\alpha_1}(x)) + \phi^{-1}(H^{\alpha_2}(x)). \]

Taking derivative with respect to \( \gamma \), we have \( \frac{\partial}{\partial \gamma} \Delta(x, \gamma) = D(x, \gamma) \) with
\[
\frac{D(x, \gamma)}{(\alpha_1 - \alpha_2) \ln H(x)} = [\phi^{-1}]'(H^{\alpha_2}(x)) H^{\alpha_2}(x) \phi^{-1}(H^{\alpha_1}(x)) \\
+ \phi^{-1}(H^{\alpha_2}(x)) \phi^{-1}(H^{\alpha_1}(x)) \\
\times \phi^{-1}(H^{\alpha_1}(x)) + \phi^{-1}(H^{\alpha_2}(x)) \\
+ H^{\alpha_1}(x) - H^{\alpha_2}(x). \]

Due to \( \gamma \in [1/2, 1] \), it holds that \( \alpha_1' \geq \alpha_2' \) and hence \( H^{\alpha_1} \leq H^{\alpha_2} \) with the baseline distribution \( H \). Since \( \phi' \) is increasing, and the log-concavity of \( \phi \) implies that \( t[\phi^{-1}]'(t) \) is decreasing, \( \ln H(x) < 0 \) along with \([\phi^{-1}]'(H^{\alpha_2}(x)) H^{\alpha_2}(x) \leq [\phi^{-1}]'(H^{\alpha_1}(x)) H^{\alpha_1}(x) \leq 0 \) implies
\[ [\phi^{-1}]'(H^{\alpha_2}(x)) H^{\alpha_2}(x) \ln H(x) \geq [\phi^{-1}]'(H^{\alpha_1}(x)) H^{\alpha_1}(x) \]
\[ \ln H(x) \geq 0. \]

That is, \( D(x, \gamma) \) is also nonnegative. Consequently, \( \frac{\partial}{\partial \gamma} \Delta(x, \gamma) \geq 0 \) and thus \( \Delta(x, \gamma) \leq \Delta(x, 1) = 0 \). Equivalently, it holds that \( V_{1:2}(\phi, T_{\gamma}(\alpha)) \succeq_{t} V_{1:2}(\phi, \alpha) \). By D.2. Proposition in Chapter 2 of Marshall et al. (2011), the weakly supermajorization order \( \beta \prec^{w} \alpha \) guarantees that there exists a finite number of \( T \)-transforms \( T_{\gamma_1}, \ldots, T_{\gamma_l} \) such that \( \beta \succeq T_{\gamma_l} \circ \cdots \circ T_{\gamma_1}(\alpha) = \zeta \). Then, \( \phi^{-1}(H^{\beta_j}) \succeq \phi^{-1}(H^{\zeta_j}) \) for \( j = 1, 2 \), and hence,
\[
(\phi^{-1}(H^{\beta_1}), \phi^{-1}(H^{\beta_2})) \prec^{w} (\phi^{-1}(H^{\zeta_1}), \phi^{-1}(H^{\zeta_2})). \quad (5)
\]
For $x_i \geq 0$, $i = 1, 2$, denote $h(x_1, x_2) = \phi(x_1) + \phi(x_2) - \phi(x_1 + x_2)$. Obviously, $h(x_1, x_2)$ is symmetric, and for $x_1, x_2 \geq 0$,

$$(x_2 - x_1)\left(\frac{\partial h}{\partial x_2} - \frac{\partial h}{\partial x_1}\right) = (x_2 - x_1)\left[\phi'(x_2) - \phi'(x_1 + x_2) - \phi'(x_1)
+ \phi'(x_1 + x_2)\right] = (x_2 - x_1)\left[\phi'(x_2) - \phi'(x_1)\right] \geq 0,$$

where the last inequality is from the increasing property of $\phi'$. Moreover,

$$\frac{\partial h(x_1 + \varepsilon, x_2)}{\partial \varepsilon} = \phi'(x_1 + \varepsilon) - \phi'(x_1 + x_2 + \varepsilon) \leq 0,$$

for $\varepsilon \in [x_2 - x_1, 0]$ with $x_1 \geq x_2$. (6)


$$h(\phi^{-1}(H^{\beta_1}(x)), \phi^{-1}(H^{\beta_2}(x))) \leq h(\phi^{-1}(H^{\xi_1}(x)), \phi^{-1}(H^{\xi_2}(x))).$$

Hence, $V_{1:2}(\phi, \zeta) \preceq_{st} V_{1:2}(\phi, \beta)$. According to the previous proof, we may conclude that

$$V_{1:2}(\phi, \beta) \succeq_{st} V_{1:2}(\phi, T_1 \circ \cdots \circ T_1(\alpha)) \succeq_{st} \cdots \succeq_{st} V_{1:2}(\phi, T_1(\alpha)) \succeq_{st} V_{1:2}(\phi, \alpha).$$

The difference between winner’s expected rents of SPAs (PRH($H, \sigma$); $\phi$) and (PRH($H, \eta$); $\phi$)

$$E[V_{2:2}(\phi, \sigma) - V_{1:2}(\phi, \sigma)] - E[V_{2:2}(\phi, \eta) - V_{1:2}(\phi, \eta)]$$

$$= \int_0^\infty \left[H^{\sigma_1}(x) + H^{\sigma_2}(x) - 2\phi(\phi^{-1}(H^{\sigma_1}(x)) + \phi^{-1}(H^{\sigma_2}(x))) - H^{\eta_1}(x) - H^{\eta_2}(x) + 2\phi(\phi^{-1}(H^{\eta_1}(x)) + \phi^{-1}(H^{\eta_2}(x)))\right] dx,$$

then, by applying a similar method in the previous part of the proof, we can verify the conclusion on the winner’s expected rent. This completes the proof.
A.5. **Theorem 3.4** Denote bids of the FPA \((G_1, G_2; \phi)\) as \(B_1, B_2\). Then the winner’s payment has the distribution

\[
P\{B_{2:2}(\phi) \leq x\} = P\{B_1 \leq x, B_2 \leq x\} = \phi(\phi^{-1}(G_1(x)) + \phi^{-1}(G_2(x))), \quad x \geq 0.
\]

Given \((B_1, B_2)\), if one mistakenly specifies the independence between valuations, then the corresponding final price is \(B_{2:2}(\phi_0)\).

Since \(\phi_0, \phi\) and \(G_i(x)\) are continuous, there exists some \(\varepsilon_3(x)\) such that, for \(x \geq 0\),

\[
\phi_0(\phi_0^{-1}(G_1(x)) + \phi_0^{-1}(G_2(x)) + \varepsilon_3(x)) = \phi(\phi^{-1}(G_1(x)) + \phi^{-1}(G_2(x))). \quad (8)
\]

Then, the subadditivity of \(\ln \phi\) and equivalently the superadditivity of \(\phi_0^{-1} \circ \phi = - \ln \phi\) implies

\[
\varepsilon_3(x) = \phi_0^{-1} \circ \phi(\phi^{-1}(G_1(x)) + \phi^{-1}(G_2(x))) - \phi_0^{-1}(G_1(x))
\]

\[
- \phi_0^{-1}(G_2(x))
\]

\[
= \phi_0^{-1} \circ \phi(\phi^{-1}(G_1(x)) + \phi^{-1}(G_2(x)))
\]

\[
- \phi_0^{-1} \circ \phi(\phi^{-1}(G_1(x))) - \phi_0^{-1} \circ \phi(\phi^{-1}(G_2(x)))
\]

\[
\geq 0, \text{ for all } x \geq 0,
\]

which invokes, for all \(x \geq 0\),

\[
P\{b_{2:2}(\phi) \leq x\} = \phi(\phi^{-1}(G_1(x)) + \phi^{-1}(G_2(x)))
\]

\[
\leq \phi_0(\phi_0^{-1}(G_1(x)) + \phi_0^{-1}(G_2(x))) = P\{b_{2:2}(\phi_0) \leq x\}.
\]

Hence, \(B_{2:2}(\phi) \geq_{st} B_{2:2}(\phi_0)\). Namely, the winner’s payment is stochastically underestimated.

The case of the superadditive \(\ln \phi\) can be similarly proved. ■

A.6. **Theorem 3.5** Denote the bids of two FPAs \((G_1, G_2; \phi_1)\) and \((G_1, G_2; \phi_2)\) as \((B_{11}, B_{12})\) and \((B_{21}, B_{22})\), respectively. Since the two bidding vectors have common marginal distributions, we have,

\[
P\{B_{2:2}(\phi_1) \leq x\} - P\{B_{2:2}(\phi_2) \leq x\} = P\{B_{1:2}(\phi_2) \leq x\}
\]
Then, in the manner similar to the proof of Theorem 3.2, we can reach the desired conclusion.

**A.7. Theorem 3.7.** Denote the bidding strategies as $B_1$ and $B_2$ in correspondence with PRH($H, \alpha$) and PRH($H, \beta$), respectively. Without loss of generality, we assume $\alpha_1 \geq \alpha_2$ and $\beta_1 \geq \beta_2$.

Suppose $\phi_1$ is log-convex. For two configurations (PRH($H, \alpha$), $\phi_1$) and (PRH($H, \alpha'$), $\phi_2$) with the T-transform $\alpha' = T_{\gamma}(\alpha)$, it is plain that $\alpha'_1 \geq \alpha'_2$, and hence, $H^{\alpha'_1} \leq H^{\alpha'_2}$. Note that $\phi_1$, $\phi_2$, and $H$ are all continuous; for any $x > 0$, there exists some $\epsilon_4(x)$ such that

$$
\phi_1(\phi_1^{-1}(H^{\alpha_1}(x)) + \phi_1^{-1}(H^{\alpha_2}(x)) + \epsilon_4(x)) = \phi_2(\phi_2^{-1}(H^{\alpha'_1}(x)) + \phi_2^{-1}(H^{\alpha'_2}(x))). (6)
$$

Since $\phi_1^{-1} \circ \phi_2$ is subadditive, it holds that

$$
\epsilon_4(x) = \phi_1^{-1} \circ \phi_2(\phi_2^{-1}(H^{\alpha'_1}(x)) + \phi_2^{-1}(H^{\alpha'_2}(x))) - \phi_1^{-1}(H^{\alpha_1}(x))$

$$
= \phi_1^{-1}(H^{\alpha_2}(x))$

$$
\leq \phi_1^{-1}(H^{\alpha_1}(x)) + \phi_1^{-1}(H^{\alpha_2}(x)) - \phi_1^{-1}(H^{\alpha_1}(x))$

$$
= \phi_1^{-1}(H^{\alpha_1}(x)) - \phi_1^{-1}(H^{\alpha_2}(x))$

$$
= \phi_1^{-1}(H^{\gamma \alpha_1 + (1-\gamma) \alpha_2}(x)) + \phi_1^{-1}(H^{\gamma \alpha_2 + (1-\gamma) \alpha_1}(x))$

$$
= \epsilon_5(x, \gamma).
$$

Due to the log-convexity of $\phi_1$ and the decreasing property of $\phi_1^{-1}$, it holds that

$$
t[\phi_1^{-1}](t) = \frac{t}{\phi_1^{-1}(t)} = \frac{\phi_1(\phi_1^{-1}(t))}{\phi_1(\phi_1^{-1}(t))}
$$
is increasing. So, by $\alpha'_1 \geq \alpha'_2$ we have

$$\frac{\partial \varepsilon_5(x, \gamma)}{\partial \gamma} = \left[ \phi_1^{-1}(H^{\gamma\alpha_1(x)}(x)) (\alpha_1 - \alpha_2) \right.$$ 

$$\times H^{\gamma\alpha_1(x)}(x) \ln H(x)$$ 

$$+ \left[ \phi_1^{-1}(H^{\gamma\alpha_2(x)}(x)) (\alpha_2 - \alpha_1) \right.$$ 

$$\times H^{\gamma\alpha_2(x)}(x) \ln H(x)$$ 

$$= (\alpha_1 - \alpha_2) \ln H(x) \left[ H^{\alpha'_1}(x) \left[ \phi_1^{-1}(H^{\alpha'_1}(x)) \right] \right.$$

$$- H^{\alpha'_2}(x) \left[ \phi_1^{-1}(H^{\alpha'_2}(x)) \right] \left. \right] \geq 0,$$

implying that $\varepsilon_5(x, \gamma)$ increases in $\gamma \in [1/2, 1]$. So, $\varepsilon_4(x) \leq \varepsilon_5(x, \gamma) \leq \varepsilon_5(x, 1) = 0$, and hence,

$$P\left\{ b_{2,2}(\phi_1, \alpha) \leq x \right\} - P\left\{ b_{2,2}(\phi_2, T_\gamma(\alpha)) \leq x \right\}$$

$$= \phi_1(\phi_1^{-1}(H^{\alpha_1}(x)) + \phi_1^{-1}(H^{\alpha_2}(x))) - \phi_2(\phi_2^{-1}(H^{\alpha'_1}(x))$$

$$+ \phi_2^{-1}(H^{\alpha'_2}(x))) \leq 0.$$

This actually invokes

$$B_{2,2}(\phi_1, \alpha) \geq_{st} B_{2,2}(\phi_2, T_\gamma(\alpha)). \quad (7)$$

Setting $\phi_1 = \phi_2$ in (7), we have

$$B_{2,2}(\phi_2, \sigma) \geq_{st} B_{2,2}(\phi_2, T_\gamma(\sigma)) \quad \text{for any } \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}_+^2. \quad (8)$$

By C.6. Proposition in Chapter 2 of Marshall et al. (2011), the weakly submajorization order $\beta \sim_w \alpha$ guarantees that there exist a finite number of $T$-transforms $T_{\gamma_1}, \ldots, T_{\gamma_r}$ such that $\beta \leq \zeta = T_{\gamma_r} \circ \cdots \circ T_{\gamma_1}(\alpha)$. Thus, for all $x$, $H^{\beta_j}(x) \geq H^{\zeta_j}(x)$ for $j = 1, 2$ and

$$\phi_2(\phi_2^{-1}(H^{\beta_1}(x)) + \phi_2^{-1}(H^{\beta_2}(x))) \geq \phi_2(\phi_2^{-1}(H^{\zeta_1}(x)) + \phi_2^{-1}(H^{\zeta_2}(x))) = \phi_2(\phi_2^{-1}(H^{\alpha_1}(x)) + \phi_2^{-1}(H^{\alpha_2}(x))).$$
That is,

\[ B_{2:2}(\phi_2, \zeta) \geq_{st} B_{2:2}(\phi_2, \beta). \] (9)

In combination with (7), (8), and (9) we have

\[ B_{2:2}(\phi_1, \alpha) \geq_{st} B_{2:2}(\phi_2, T_{\gamma_1}(\alpha)) \geq_{st} \cdots \geq_{st} B_{2:2}(\phi_2, T_{\gamma_r} \circ \cdots \circ T_{\gamma_1}(\alpha)) \geq_{st} B_{2:2}(\phi_2, \beta). \]

Now, let us consider the case that \( \phi_2 \) is log-concave. Then, \[
\frac{1}{(\ln \phi_2(s))'} = \frac{\phi_2(s)}{\phi_3(s)} \text{ is increasing and } \left( \phi_2^{-1}(e^t) \right)' = \frac{\phi_2(\phi_2^{-1}(e^t))}{\phi_2' \phi_2^{-1}(e^t)} \text{ is decreasing in } t \leq 0, \text{ and thus, } -\phi_2^{-1}(e^t) \text{ is increasing and convex. Moreover, it is plain that } \alpha \prec^w \beta \text{ implies } (\alpha_1 \ln H(x), \alpha_2 \ln H(x)) \prec_w (\beta_1 \ln H(x), \beta_2 \ln H(x)). \]

Then, according to A.4. of Marshall et al. (2011, Chapter 1), we have

\[
-\phi_2^{-1} \circ \exp \{ \ln H^{\beta_1}(x) \} - \phi_2^{-1} \circ \exp \{ \ln H^{\beta_2}(x) \} \geq -\phi_2^{-1} \circ \exp \{ \ln H^{\alpha_1}(x) \} - \phi_2^{-1} \circ \exp \{ \ln H^{\alpha_2}(x) \}. \] (10)

Again, by the continuity of \( \phi_1, \phi_2, H \), there exists \( \epsilon_6(x) \) such that

\[
\phi_1(\phi_1^{-1}(H^{\alpha_1}(x)) + \phi_1^{-1}(H^{\alpha_2}(x)) + \epsilon_6(x)) = \phi_2(\phi_2^{-1}(H^{\beta_1}(x)) + \phi_2^{-1}(H^{\beta_2}(x))). \] (11)

Since \( \phi_1^{-1} \circ \phi_2 \) is subadditive and increasing by (10), we have

\[
\epsilon_6(x) = \phi_1^{-1} \circ \phi_2(\phi_2^{-1}(H^{\beta_1}(x)) + \phi_2^{-1}(H^{\beta_2}(x))) - \phi_1^{-1} \circ H^{\alpha_1}(x)
\]

\[
= \phi_1^{-1} \circ \phi_2(\phi_2^{-1}(H^{\beta_1}(x)) + \phi_2^{-1}(H^{\beta_2}(x)))
\]

\[
- \phi_1^{-1} \circ \phi_2(\phi_2^{-1}(H^{\alpha_1}(x))) - \phi_1^{-1} \circ \phi_2(\phi_2^{-1}(H^{\alpha_2}(x)))
\]

\[
\leq 0.
\]
Then, for all $x$,

$$
P\{b_{2:2}(\phi_1, \alpha) \leq x\} - P\{b_{2:2}(\phi_2, \beta) \leq x\} = \phi_1\left(\phi_1^{-1}(H^{\alpha_1}(x)) + \phi_1^{-1}(H^{\alpha_2}(x))\right) - \phi_2\left(\phi_2^{-1}(H^{\beta_1}(x)) + \phi_2^{-1}(H^{\beta_2}(x))\right) \leq 0.
$$

Namely, $B_{2:2}(\phi_1, \alpha) \geq_{st} B_{2:2}(\phi_2, \beta)$.

**A.8. Theorem 4.2.** Let us use the same notations as in the proof of Theorem 3.2. Note that

$$
P\{V_{d-1:d}(\phi, \alpha) \leq x\} = \sum_{i=1}^{d} P\{V_j \leq x, j \neq i\} - (d - 1)P\{V_{d:d}(\phi, \alpha) \leq x\} = \sum_{i=1}^{d} \phi\left(\sum_{j \neq i} \phi^{-1}(H^{\alpha_j}(x))\right) - (d - 1)\phi\left(\sum_{i=1}^{d} \phi^{-1}(H^{\alpha_i}(x))\right),
$$

then, we have

$$\Delta(x, \gamma) = P\{V_{d-1:d}(\phi, T_\gamma(\alpha)) \leq x\} - P\{V_{d-1:d}(\phi, \alpha) \leq x\}
= \sum_{i=1}^{d} \phi\left(\sum_{m \neq i} \phi^{-1}(H^{\alpha_m}(x))\right) - (d - 1)\phi\left(\sum_{i=1}^{d} \phi^{-1}(H^{\alpha_i}(x))\right)
- \sum_{i=1}^{d} \phi\left(\sum_{m \neq i} \phi^{-1}(H^{\alpha_m}(x))\right)
+ (d - 1)\phi\left(\sum_{i=1}^{d} \phi^{-1}(H^{\alpha_i}(x))\right),$$

here, $\alpha_j' = \gamma \alpha_j + (1 - \gamma)\alpha_k$, $\alpha_k' = \gamma \alpha_k + (1 - \gamma)\alpha_j$, $\alpha_j \geq \alpha_k$ and $\alpha_i' = \alpha_i$ for $i \neq j, k$. Taking derivative with respect to $\gamma$, we have
\[ \frac{\partial}{\partial \gamma} \Delta(x, \gamma) = A(x, \gamma) + D(x, \gamma) \] with

\[ A(x, \gamma) = \sum_{i \neq j, k} (\alpha_j - \alpha_k) \phi'(\sum_{m \neq i} \phi^{-1}(H_{\alpha_m}^i(x))) \]

\[ - \phi'\left(\sum_{m=1}^{d} \phi^{-1}(H_{\alpha_m}^i(x))\right) \]

\[ \cdot \left\{ [\phi^{-1}]'(H_{\alpha_j}^i(x)) H_{\alpha_j}^i(x) - [\phi^{-1}]'(H_{\alpha_k}^i(x)) H_{\alpha_k}^i(x) \right\} \ln H(x), \]

and

\[ D(x, \gamma) \]

\[ \frac{(\alpha_j - \alpha_k) \ln H(x)}{\ln H(x)} \]

\[ = [\phi^{-1}]'(H_{\alpha_j}^i(x)) H_{\alpha_j}^i(x) \phi'\left(\sum_{m=1}^{d} \phi^{-1}(H_{\alpha_m}^i(x))\right) \]

\[ - \phi'\left(\sum_{m \neq j} \phi^{-1}(H_{\alpha_m}^i(x))\right) \]

\[ - [\phi^{-1}]'(H_{\alpha_j}^i(x)) H_{\alpha_j}^i(x) \phi'\left(\sum_{m=1}^{d} \phi^{-1}(H_{\alpha_m}^i(x))\right) \]

\[ - \phi'\left(\sum_{m \neq k} \phi^{-1}(H_{\alpha_m}^i(x))\right). \]

Since \( \gamma \in [1/2, 1] \), we have \( \alpha_j' \geq \alpha_k' \), and hence, \( H_{\alpha_j}^i \leq H_{\alpha_k}^i \) with the baseline distribution \( H \). Note that \( \phi' \) is increasing, and the log-concavity of \( \phi \) implies that \( [\phi^{-1}]'(t) t \) is decreasing; it is easy to calculate that \( A(x, \gamma) \geq 0 \). Note that \( \phi^{-1} \) is decreasing; we have

\[ \phi'\left(\sum_{m=1}^{d} \phi^{-1}(H_{\alpha_m}^i(x))\right) - \phi'\left(\sum_{m \neq j} \phi^{-1}(H_{\alpha_m}^i(x))\right) \]
\[
\geq \phi'(\sum_{m=1}^{d} \phi^{-1}(H^{\alpha_{m}}(x))) - \phi'(\sum_{m \neq k} \phi^{-1}(H^{\alpha_{m}}(x))) \geq 0.
\]

Moreover, \(\ln H(x) < 0\) along with \([\phi^{-1}]'(H^{\alpha_{i}}(x))H^{\alpha_{i}}(x) \leq [\phi^{-1}]'(H^{\alpha_{j}}(x))H^{\alpha_{j}}(x) \leq 0\) implies

\[
[\phi^{-1}]'(H^{\alpha_{i}}(x))H^{\alpha_{i}}(x) \ln H(x) \geq [\phi^{-1}]'(H^{\alpha_{j}}(x))H^{\alpha_{j}}(x) \ln H(x) \geq 0.
\]

That is, \(D(x, \gamma)\) is also nonnegative. Consequently, \(\frac{\partial}{\partial \gamma} \Delta(x, \gamma) \geq 0\), and thus, \(\Delta(x, \gamma) \leq \Delta(x, 1) = 0\). Equivalently, \(V_{d-1:d}(\phi, T\gamma(\alpha)) \geq_{st} V_{d-1:d}(\phi, \alpha)\).

By D.2. Proposition in Chapter 2 of Marshall et al. (2011), the weakly super-majorization order \(\beta \prec_{w} \alpha\) guarantees that there exists a finite number of \(T\)-transforms \(T_{\gamma_{1}}, \ldots, T_{\gamma_{r}}\) such that \(\beta \geq \zeta = T_{\gamma_{r}} \circ \cdots \circ T_{\gamma_{1}}(\alpha)\). Then, \(\phi^{-1}(H^{\beta_{j}}(x)) \geq \phi^{-1}(H^{\zeta_{j}}(x))\) for \(j \in I_{d}\), and hence,

\[
(\phi^{-1}(H^{\beta_{1}}), \ldots, \phi^{-1}(H^{\beta_{d}})) \prec_{w} (\phi^{-1}(H^{\zeta_{1}}), \ldots, \phi^{-1}(H^{\zeta_{d}})).
\]

For \(x = (x_{1}, \ldots, x_{d}) \in \mathbb{R}^{d}_{+}\), denote \(h(x) = \sum_{i=1}^{d} \phi(\sum_{j \neq i} x_{j}) - (d-1)\phi(\sum_{i=1}^{d} x_{i})\). Obviously, \(h(x)\) is symmetric, and for \(x_{1}, x_{2} \geq 0\),

\[
(x_{2} - x_{1})\left(\frac{\partial h(x)}{\partial x_{2}} - \frac{\partial h(x)}{\partial x_{1}}\right)
\]

\[
= (x_{2} - x_{1}) \left[ \sum_{i \neq 2} \phi' \left( \sum_{j \neq i} x_{j} \right) - (d-1)\phi' \left( \sum_{i=1}^{d} x_{i} \right) 
- \sum_{i \neq 1} \phi' \left( \sum_{j \neq i} x_{j} \right) + (d-1)\phi' \left( \sum_{i=1}^{d} x_{i} \right) \right] 
= (x_{2} - x_{1}) \left[ \phi' \left( \sum_{j \neq 1} x_{j} \right) - \phi' \left( \sum_{j \neq 2} x_{j} \right) \right] \geq 0,
\]
where the last inequality is due to the increasing property of $\phi'$. Moreover, assume $x_1 \geq x_2 \geq \cdots \geq x_d$, then for $x_2 - x_1 \leq \varepsilon \leq 0$,

\[
\frac{\partial h(x_1 + \varepsilon, x_2, \cdots, x_d)}{\partial \varepsilon} = \sum_{i \neq 1} \phi' \left( \sum_{j \neq i} x_j + \varepsilon \right) - (d - 1) \phi' \left( \sum_{i = 1}^{d} x_i + \varepsilon \right) \leq 0.
\]


\[
h(\phi^{-1}(H^{\beta_1}(x)), \cdots, \phi^{-1}(H^{\beta_d}(x))) \leq h(\phi^{-1}(H^{\zeta_1}(x)), \cdots, \phi^{-1}(H^{\zeta_d}(x))).
\]

Hence, $V_{d-1:d}(\phi, \zeta) \leq_{st} V_{d-1:d}(\phi, \beta)$. According to the previous proof, we may conclude that

\[
V_{d-1:d}(\phi, \beta) \geq_{st} V_{d-1:d}(\phi, T_r \circ \cdots \circ T_1(\alpha)) \geq_{st} \cdots \geq_{st} V_{d-1:d}(\phi, \alpha).
\]

Note that the difference between the winner’s expected rents of any two SPAs $(\text{PRH}(H, \sigma); \phi)$ and $(\text{PRH}(H, \eta); \phi)$ can be easily calculated as

\[
E[V_{d:d}(\phi, \sigma) - V_{d-1:d}(\phi, \sigma)] - E[V_{d:d}(\phi, \eta) - V_{d-1:d}(\phi, \eta)]
\]

\[
= \int_0^{+\infty} \left[ \sum_{i=1}^{d} \phi \left( \sum_{m \neq i} \phi^{-1}(H^{\sigma_m}(x)) \right) - d\phi \left( \sum_{i=1}^{d} \phi^{-1}(H^{\sigma_i}(x)) \right) 
\]

\[
- \sum_{i=1}^{d} \phi \left( \sum_{m \neq i} \phi^{-1}(H^{\eta_m}(x)) \right) + d\phi \left( \sum_{i=1}^{d} \phi^{-1}(H^{\eta_i}(x)) \right) \right] dx.
\]

Then, by applying a similar method in the previous part of proof, the result regarding the winner’s expected rent is proved. This completes the proof.

**A.9. Theorem 4.5.** It can be proved by applying the method in Theorem 3.7 to $\alpha$ and $\beta$ with dimension $d > 2$. 