Some results about operator perturbation of fusion frames in Hilbert spaces

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A R T I C L E  I N F O

Article history:
Received 22 January 2014
Available online xxx
Submitted by R. Curto

Keywords:
Fusion frames
Fusion frame sequences
Operator perturbation
Gap
Hilbert spaces

A B S T R A C T

The stabilities of fusion frames under operator perturbation (simple named operator perturbation of the fusion frames), is the study object of this paper. We provide a way as follows for studying the operator perturbation of fusion frames. Firstly, we consider the operator perturbation of fusion frame sequences by the gap between two closed subspaces and discuss the relationships between the operator perturbation of fusion frames and the operator perturbation of fusion frame sequences. Lastly, we transform the operator perturbation of fusion frames into the operator perturbation of fusion frame sequences based on these relationships. Our results obtained this way generalize the remarkable results which have been obtained by Casazza, Kutyniok, Asgari, Gavruta and Zhu.

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1. Introduction

In 1952, frames were first introduced by Duffin and Schaeffer [9]. In recent 30 years, it was widely studied and applied in sigma–delta quantization [3], filter bank theory [4], signal and image processing [5] and so on. Fusion frames which are the generalization of frames, were studied recently by Casazza, Fornasier and so on. Now many important results about fusion frames have been obtained by many authors (see [1,2,6,7,10,13–15]).

We note that $H$ and $K$ are two Hilbert spaces, $L(H,K)$ is the space of all bounded linear operators from $H$ to $K$, especially, $L(H)$ is the space of all bounded linear operators on $H$. $\{W_j\}_{j=1}^{\infty}$ is a family of closed subspaces of $H$ and $\{v_j\}_{j=1}^{\infty}$ is a family of positive weights, i.e. $v_j > 0$. $\pi_{W_j}$ is an orthogonal projection from $H$ onto $W_j$ for every $j \in Z^+$, where $Z^+$ is a set of positive integers. Let $I \subset Z^+$. The space $l^2(\{W_j\}_{j \in I})$ is defined by

\textsuperscript{∗} This work was supported by the National Natural Science Foundation of China (Grant No. 11071152), the Natural Science Foundation of Guangdong Province (Grant No. S2011010004511), and the Natural Science Foundation of Fujian Province, China (Grant No. 2012J01005).

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http://dx.doi.org/10.1016/j.jmaa.2014.07.068
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Please cite this article in press as: X.-B. Li et al., Some results about operator perturbation of fusion frames in Hilbert spaces, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.07.068
\[ l^2(\{W_j\}_{j \in I}) = \left\{ \{f_j\}_{j \in I} \big| f_j \in W_j \text{ and } \sum_{j \in I} \|f_j\|^2 < \infty \right\}, \]

with inner product given by
\[ \langle \{f_j\}_{j \in I}, \{g_j\}_{j \in I} \rangle = \sum_{j \in I} \langle f_j, g_j \rangle. \]

Let \( T \in L(H, K) \) be a closed range operator and \( n \) be a positive integer. Note that \( W_n = l^2(\{W_j\}_{j=1}^n) = \{\{f_j\}_{j=1}^n \big| f_j \in W_j \text{ and } 1 \leq j \leq n \} \) and \( l^2(\{H\}_{j=1}^n) = \{\{f_j\}_{j=1}^n \big| f_j \in H \text{ and } 1 \leq j \leq n \} \), we let \( \pi_{W_n} \) be an orthogonal projection from \( l^2(\{H\}_{j=1}^n) \) onto \( l^2(\{W_j\}_{j=1}^n) \), and let
\[ U_{n, T} = \left\{ \{v_j T^* \pi_{W_j} f\}_{j=1}^n \big| f \in K \right\}. \]

By some straightforward computations, we obtain
\[ \pi_{W_n}(\{f_j\}_{j=1}^n) = \{\pi_{W_j} f_j\}_{j=1}^n \quad (1.1) \]
for any \( \{f_j\}_{j=1}^n \in l^2(\{H\}_{j=1}^n) \).

**Definition 1.1.** Let \( \{W_j\}_{j=1}^\infty \) be a family of closed subspaces of \( H \), \( \{v_j\}_{j=1}^\infty \) be a family of positive weights, i.e. \( v_j > 0 \) for \( j = 1, 2, \cdots \). If there exist constants \( 0 < A \leq B < +\infty \) such that
\[ A\|f\|^2 \leq \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j}(f)\|^2 \leq B\|f\|^2 \]
for all \( f \in H \), then we call \( \{(W_j, v_j)\}_{j=1}^\infty \) a fusion frame for \( H \) with bounds \( A \) and \( B \).

**Definition 1.2.** A family \( \{(W_j, v_j)\}_{j=1}^\infty \) is a fusion frame sequence for \( H \), if \( \{(W_j, v_j)\}_{j=1}^\infty \) is a fusion frame for \( W \) with bounds \( A \) and \( B \), where
\[ W = \text{span}\{W_j\}_{j=1}^\infty = \left\{ \sum_{j \in I} f_j \big| f_j \in W_j, I \subseteq \mathbb{Z}^+ \text{ is finite set} \right\}. \]

The positive real constants \( A \) and \( B \) are called lower and upper bounds of the fusion frame sequence.

**Definition 1.3.** Let \( W \) and \( V \) be two closed subspaces of \( H \). We define the gap between \( W \) and \( V \) by
\[ \delta(W, V) = \sup_{x \in W, \|x\|=1} \text{dist}(x, V) = \sup_{x \in W, \|x\|=1} \inf_{y \in V} \|x - y\|. \]

The thing to note here is that \( 0 \leq \delta(W, V) \leq 1 \). Especially, if \( W \subset V \) then \( \delta(W, V) = 0 \); if \( W \) and \( V \) are two orthogonal closed subspaces of \( H \) then \( \delta(W, V) = 1 \). Some results about the gap between two subspaces see [11].

**Definition 1.4.** Suppose that \( H, K \) are two Hilbert spaces, and \( T : H \to K \) is a bounded linear operator. Let
\[ \gamma(T) = \inf \{ \|Tx\| : x \in N_T^\perp, \|x\| = 1 \}, \]
then we call \( \gamma(T) \) the minimum modulus of operator \( T \).
In this paper, we want to study the operator perturbation of fusion frames, i.e. answer the following question:

- Let $T \in L(H)$ be a closed range operator and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame for $H$. In order to make the family $\{(TW_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame for $H$, what conditions (about $T$) need to be attached?

This question was first considered by Casazza and Kutyniok (see Proposition 3.20 in [6]). Soon afterwards, it was reconsidered by several scholars (such as Asgari [1], Gavruta [10] and Zhu [14]). By now, there are some remarkable results as follows.

Suppose that $\{(W_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame for $H$. P.G. Casazza, G. Kutyniok, S. Li and P. Gavruta obtained that if $T$ is an invertible operator on $H$, then $\{(TW_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame for $H$ (see Proposition 3.20 in [6], Theorem 2.11 in [7], and Theorem 2.4 in [10]). Then such a result was respectively generalized by M.S. Asgari, and J. Wang, Y. Zhu. Asgari obtained that if $T \in L(H)$ is surjective and $T^*T(W_j) \subset W_j$ for all $j \in \mathbb{Z}^+$, then $\{(TW_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame for $H$ (see Theorem 3.1 in [1]); Wang and Zhu obtained that if $T \in L(H)$ is surjective and $T^*(TW_j) \subset W_j$ for all $j \in \mathbb{Z}^+$, then $\{(TW_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame for $H$ (see Theorem 2 in [14]).

These scholars respectively found some conditions about $T$ to make the family $\{(TW_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame for $H$ when $\{(W_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame for $H$. Then, what is the connection among these conditions, and how to obtain such conditions (or more general conditions)? In this paper, we preliminary discuss the two questions, and provide a way for studying the operator perturbation of fusion frames as follows:

The first step, we consider the operator perturbation of fusion frame sequences, i.e. answer the following question which is very similar to the first question:

- Let $T \in L(H, K)$ be a closed range operator and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame for $H$. In order to make the family $\{(TW_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame sequence for $K$, what conditions (about $T$) need to be attached?

This question has been considered by M.S. Asgari (see Theorem 3.1 in [1]). Asgari obtained that if $T^*T(W_j) \subset W_j$ for all $j \in \mathbb{Z}^+$, then $\{(TW_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame sequence for $K$. In order to get more general results, we should further consider this question in Section 2 by the gap between two closed subspaces.

The second step, we discuss the relationships between the operator perturbation of fusion frames and the operator perturbation of fusion frame sequences. We find that the operator perturbation of fusion frames is equivalent to the operator perturbation of fusion frame sequences under some conditions (see Theorem 3.1).

The third step, we transform the operator perturbation of fusion frames into the operator perturbation of fusion frame sequences.

2. Operator perturbation of fusion frame sequences

**Lemma 2.1.** (See [8].) Let $H, K$ be two Hilbert spaces, and $U \in L(H, K)$ be a closed range operator. Then there exists a unique operator $U^+ \in L(K, H)$ such that

\[
N_{U^+} = U(H)^{\perp}, \quad N^+_U = U^+(K), \quad UU^+(f) = f, \quad f \in U(H).
\]

In this paper, we call $U^+$ the pseudo-inverse of $U$.

**Lemma 2.2.** (See [1,8,11,12].) Let $H$ be two Hilbert spaces, $T \in L(H, K)$ be a closed range operator. Then
Noticing that Lemma 2.5(1)

**Proof.** From Lemma 2.3 in [10], the part (1) of Lemma 2.3 holds. It is easy to know that TW \( \subset V \) if and only if \( \pi_V T \pi_W = T \pi_W \), \( \pi_V T \pi_W = T \pi_W \) if and only if \( \pi_W T^* \pi_V = (\pi_V T \pi_W)^* = (T \pi_W)^* = \pi_W T^* \). Hence the part (2) of Lemma 2.3 holds. \( \Box \)

**Remark 2.4.** The part (1) of Lemma 2.3 is a simple but beautiful equation which shows the relation between \( \pi_W \) and \( \pi_{TW} \). This equation is very useful for studying the operator perturbation of fusion frames. So, it would be helpful for the further study of the stabilities of fusion frames to consider a more general equation (see the part (2) of Lemma 2.3). In Theorem 3.6, we should consider the stabilities of fusion frames by this general equation.

**Lemma 2.5.** Let \( W \) and \( V \) be two closed subspaces of \( H \), \( W_0 \) be a subspace of \( W \). Then the following two propositions hold for every real \( 0 \leq c \leq 1 \):

1. \( \delta(W, V) \leq c \iff \|\pi_V x\|^2 \geq (1 - c^2)\|x\|^2, \quad x \in W. \)
2. If \( \delta(W, V) \leq c \), then \( \delta(W_0, V) \leq c \).

**Proof.** (1) By Definition 1.3, we have

\[
\delta^2(W, V) = \left( \sup_{x \in W} \inf_{y \in V} \frac{\|x - y\|^2}{\|x\|_1} \right)^2 = \left( \sup_{x \in W} \inf_{y \in V} \frac{\|x - \pi_V x\|^2}{\|x\|_1} \right)^2 = \sup_{x \in W} \inf_{y \in V} \frac{\|x - \pi_V x\|^2}{\|x\|_1} = \sup_{x \in W} (\|x\|^2 - \|\pi_V x\|^2) = 1 - \inf_{x \in W} \|\pi_V x\|^2.
\]

It follows that \( \inf_{x \in W} \|\pi_V x\|^2 \geq 1 - c^2 \iff \delta(W, V) \leq c \). Combining with

\[
\inf_{x \in W} \|\pi_V x\|^2 \geq 1 - c^2 \iff \|\pi_V x\|^2 \geq (1 - c^2)\|x\|^2, \quad x \in W,
\]

we see that \( \delta(W, V) \leq c \iff \|\pi_V x\|^2 \geq (1 - c^2)\|x\|^2, \quad x \in W. \)

(2) If \( \delta(W, V) \leq c \), by Lemma 2.5(1), we see that \( \|\pi_V x\|^2 \geq (1 - c^2)\|x\|^2 \) for all \( x \in W_0 \subset W \). By using Lemma 2.5(1) again, it can be obtained that \( \delta(W_0, V) \leq c \). \( \Box \)

**Lemma 2.6.** Let \( T \in L(H, K) \) be a closed range operator, \( \{(W_j, v_j)\}_{j=1}^{\infty} \) be a fusion frame for \( H \) with lower bound \( A \). Then \( \sum_{j=1}^{\infty} v_j^* \|\pi_{TW_j} f\|^2 \geq A\|T\|^2\|T^+\|^2\|f\|^2 \) for all \( f \in T(H) = \overline{\text{span}\{TW_j\}_{j=1}^{\infty}} \).

**Proof.** Noticing that \( T(H) = \overline{\text{span}\{TW_j\}_{j=1}^{\infty}} \) if \( \{(W_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame for \( H \) and \( T \) is a bounded linear operator with closed range, we complete the proof of Lemma 2.6 from the proof of Theorem 3.1 in [1]. \( \Box \)

Please cite this article in press as: X.-B. Li et al., Some results about operator perturbation of fusion frames in Hilbert spaces, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.07.068
Theorem 2.7. Let $T \in L(H, K)$ be a closed range operator, and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame for $H$ with bounds $A$ and $B$. Then the following two conditions are equivalent:

1) The family $\{(\overline{T}W_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame sequence for $K$.
2) There exists a real $C > 0$ such that $\sum_{j=1}^{\infty} v_j^2 \|\pi_{\overline{T}W_j}f\|^2 \leq C \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j} T^* f\|^2$ for any $f \in T(H)$.

Proof. If the condition (1) holds, i.e. $\{(\overline{T}W_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame sequence for $K$ with upper bound $D$, then by Lemma 2.2, we obtain

$$\sum_{j=1}^{\infty} v_j^2 \|\pi_{\overline{T}W_j} f\|^2 \leq D\|f\|^2 \leq D\|T^+\|^2 \|T^* f\|^2 \leq DA^{-1}\|T^+\|^2 \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j} T^* f\|^2, \quad f \in T(H).$$

This shows that the condition (2) holds.

Conversely, if the condition (2) holds, then we have

$$\sum_{j=1}^{\infty} v_j^2 \|\pi_{\overline{T}W_j} f\|^2 \leq C \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j} T^* f\|^2 \leq CB\|T^*\|^2 \|f\|^2 = CB\|T\|^2 \|f\|^2, \quad f \in T(H).$$

Combining with Lemma 2.6, we obtain that $\{(\overline{T}W_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame for $T(H)$ with bounds $A\|T\|^{-2}\|T^+\|^{-2}$ and $CB\|T\|^2$. This completes the proof. □

Remark 2.8. Theorem 2.7 still holds if condition (2) is replaced by the following: There exist an operator $U \in L(H, K)$ and a real $C > 0$ such that $\sum_{j=1}^{\infty} v_j^2 \|\pi_{U\overline{T}W_j} f\|^2 \leq C \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j} U^* f\|^2$ for any $f \in T(H)$.

Corollary 2.9. Let $T \in L(H, K)$ be a closed range operator, and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame for $H$ with bounds $A$ and $B$. Then $\{(\overline{T}W_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame sequence for $K$, if there exist a division $G$ of $Z^+$ and a real $C > 0$ such that $\sum_{j \in G} v_j^2 \|\pi_{\overline{T}W_j} f\|^2 \leq C \sum_{j \in G} v_j^2 \|\pi_{W_j} T^* f\|^2$ for any $f \in T(H)$ and $I \in G$.

Proof. Whether $\sum_{I \in G} \sum_{j \in I} v_j^2 \|\pi_{W_j} T^* f\|^2$ is convergence or not, we have

$$\sum_{j=1}^{\infty} v_j^2 \|\pi_{\overline{T}W_j} f\|^2 = \sum_{I \in G} \sum_{j \in I} v_j^2 \|\pi_{\overline{T}W_j} f\|^2 \leq C \sum_{I \in G} \sum_{j \in I} v_j^2 \|\pi_{W_j} T^* f\|^2 = C \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j} T^* f\|^2, \quad f \in T(H).$$

From Theorem 2.7, we obtain that $\{(\overline{T}W_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame sequence for $K$ with bounds $A\|T\|^{-2}\|T^+\|^{-2}$ and $CB\|T\|^2$. □

Corollary 2.10. Let $T \in L(H, K)$ be a closed range operator, and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame for $H$ with bounds $A$ and $B$. Then $\{(\overline{T}W_j, v_j)\}_{j=1}^{\infty}$ is a fusion frame sequence for $K$, if one of the following conditions holds:

1) There exists a strictly increasing positive integer sequence $\{n_k\}_{k=1}^{\infty}$ and a real $C > 0$ such that

$$\sum_{j=1}^{n_k} v_j^2 \|\pi_{\overline{T}W_j} f\|^2 \leq C \sum_{j=1}^{n_k} v_j^2 \|\pi_{W_j} T^* f\|^2, \quad f \in T(H).$$

(2) $\inf_{j \in Z^+} \gamma(\pi_{W_j} T^*) > 0$. 

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Proof. If the condition (1) holds, then we have
\[ \sum_{j=1}^{n_k} v_j^2 \| \pi_{TW_j} f \|^2 \leq C \sum_{j=1}^{n_k} v_j^2 \| \pi_{W_j} T^* f \|^2 \leq C \sum_{j=1}^{n_k} v_j^2 \| \pi_{W_j} T^* f \|^2, \quad f \in T(H) \text{ and } k \in \mathbb{Z}^+. \]

Letting \( k \to \infty \), we obtain that \( \sum_{j=1}^{\infty} v_j^2 \| \pi_{TW_j} f \|^2 \leq C \sum_{j=1}^{\infty} v_j^2 \| \pi_{W_j} T^* f \|^2 \) for any \( f \in T(H) \). By Lemma 2.6, \( \{ (TW_j, v_j) \}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \) with bounds \( A \| T \|^{-2} \| T^+ \|^{-2} \) and \( CB \| T \|^2 \).

If \( M = \inf_{j \in \mathbb{Z}^+} \gamma(\pi_{W_j} T^*) > 0 \), then for any \( j \in \mathbb{Z}^+ \),
\[ \| \pi_{W_j} T^* x \| \geq \gamma(\pi_{W_j} T^*) \geq M, \quad x \in N_{\pi_{W_j} T^*} = TW_j \text{ and } \| x \| = 1. \]
This implies that \( \| \pi_{W_j} T^* x \| \geq M \| x \| \) for all \( x \in TW_j \) and \( j \in \mathbb{Z}^+ \). By Lemma 2.3, we have
\[ \| \pi_{W_j} T^* x \| = \| \pi_{W_j} T^* \pi_{TW_j} x \| \geq M \| \pi_{TW_j} x \|, \quad x \in T(H) \text{ and } j \in \mathbb{Z}^+. \]

It follows that
\[ \sum_{j=1}^{\infty} v_j^2 \| \pi_{TW_j} f \|^2 \leq \frac{1}{M^2} \sum_{j=1}^{\infty} v_j^2 \| \pi_{W_j} T^* f \|^2 \quad \text{for any } f \in T(H). \]

By Lemma 2.6, \( \{ (TW_j, v_j) \}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \) with bounds \( A \| T \|^{-2} \| T^+ \|^{-2} \) and \( BM^{-2} \| T \|^2 \). This completes the proof. \( \square \)

Remark 2.11. In Corollary 2.10, we consider the operator perturbation of fusion frames from the norm and minimum modulus of operator. Next, we will consider the operator perturbation of fusion frames from gap between two closed subspaces (see Corollary 2.12).

Corollary 2.12. Let \( T \in L(H, K) \) be a closed range operator, and \( \{ (W_j, v_j) \}_{j=1}^{\infty} \) be a fusion frame for \( H \) with bounds \( A \) and \( B \). Then \( \{ (TW_j, v_j) \}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \), if one of the following conditions holds:

1. There exists a strictly increasing sequence \( \{ n_k \}_{k=1}^{\infty} \subset \mathbb{Z}^+ \) such that \( \sup_{k \in \mathbb{Z}^+} \delta(\mathcal{U}_{n_k, T}, W_{n_k}) < 1 \).
2. \( \sup_{j \in \mathbb{Z}^+} \delta(T^*TW_j, W_j) < 1 \).
3. \( T^*TW_j \subset W_j \) for any \( j \in \mathbb{Z}^+ \) (see Theorem 3.1 in [1]).

Proof. Suppose that the condition (1) holds. Let \( c = \sup_{k \in \mathbb{Z}^+} \delta(\mathcal{U}_{n_k, T}, W_{n_k}) \). By Lemma 2.2, Lemma 2.3, Lemma 2.5 and (1.1), we have
\[ \sum_{j=1}^{n_k} v_j^2 \| \pi_{W_j} T^* (f) \|^2 = \sum_{j=1}^{n_k} v_j^2 \| \pi_{W_j} T^* \pi_{TW_j} (f) \|^2 = \| \pi_{W_{n_k}} \{ v_j T^* \pi_{TW_j} f \}_{j=1}^{n_k} \|^2 \]
\[ \geq (1 - c^2) \| \{ v_j T^* \pi_{TW_j} f \}_{j=1}^{n_k} \|^2 = (1 - c^2) \sum_{j=1}^{n_k} v_j^2 \| T^* \pi_{TW_j} f \|^2 \geq (1 - c^2) \| T^+ \|^{-2} \sum_{j=1}^{n_k} v_j^2 \| T \|^2, \quad f \in T(H). \]

This shows that the condition (1) in Corollary 2.10 holds. Thus, \( \{ (TW_j, v_j) \}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \) with bounds \( A \| T \|^{-2} \| T^+ \|^{-2} \) and \( B(1 - c^2)^{-1} \| T^+ \|^{-2} \| T \|^2 \).
Suppose that the condition (2) holds. Let \( c = \sup_{j \in \mathbb{Z}_+} \delta(T^* TW_j, W_j) = \sup_{j \in \mathbb{Z}_+} \delta(T^* TW_j, W_j) \), by Lemma 2.2, Lemma 2.3 and Lemma 2.5, we have

\[
\sum_{j=1}^{\infty} v_j^2 \| \pi_{W_j} T^*(f) \|^2 = \sum_{j=1}^{\infty} v_j^2 \| \pi_{W_j} T^* \pi_{TW_j}(f) \|^2 \geq (1 - c^2) \sum_{j=1}^{\infty} v_j^2 \| T^* \pi_{TW_j} f \|^2 \\
\geq (1 - c^2) \| T^+ \|^{-2} \sum_{j=1}^{\infty} v_j^2 \| \pi_{TW_j} f \|^2, \quad f \in T(H).
\]

This shows that the condition (2) in Theorem 2.7 holds. Thus, \( \{(TW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \) with bounds \( A \| T \|^{-2} \| T^+ \|^{-2} \) and \( B(1 - c^2)^{-1} \| T^+ \|^2 \| T \|^2 \).

Suppose that the condition (3) holds, then \( T^* TW_j \subset W_j \) for any \( j \in \mathbb{Z}^+ \). It implies that

\[
\sup_{j \in \mathbb{Z}^+} \delta(T^* TW_j, W_j) = 0 < 1,
\]

i.e. the condition (2) holds. Thus, \( \{(TW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \) with bounds \( A \| T \|^{-2} \| T^+ \|^{-2} \) and \( B \| T^+ \|^2 \| T \|^2 \). This completes the proof. \( \square \)

**Theorem 2.13.** Let \( T, U \in L(H, K) \) be closed range operators and \( \{(W_j, v_j)\}_{j=1}^{\infty} \) be a fusion frame for \( H \). If \( \ker T = \ker U \), then \( \{(TW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \) if and only if \( \{(UW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \).

**Proof.** Since \( T, U \in L(H, K) \) are two closed range operators and \( \{(W_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame for \( H \), we obtain \( T(H) = \overline{\text{span}} \{(TW_j)_{j=1}^{\infty}\} \) and \( U(H) = \overline{\text{span}} \{(UW_j)_{j=1}^{\infty}\} \). Hence \( \{(TW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \) if and only if \( \{(UW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \) if and only if \( \{(TW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame sequence for \( U(H) \).

Similarly to the proof of Proposition 3.20 in [6], Theorem 2.11 in [7] or Theorem 2.4 in [10], we obtain that if there exists an invertible operator \( L : T(H) \to U(H) \) such that \( LT W_j = UW_j \) (i.e. \( LT W_j = UW_j \)) for any \( j \in \mathbb{Z}^+ \), then \( \{(TW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame for \( T(H) \) if and only if \( \{(UW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame for \( U(H) \).

Since \( T, U \in L(H, K) \) are closed range operators, we know that \( UT^+ \) is a bounded linear operator. Define the operator \( L : T(H) \to U(H) \) by \( Lx = UT^+ x, x \in T(H) \). Then \( L \) is a bounded linear operator. Actually, \( L \) is also an invertible operator and \( LT W_j = UW_j \) for any \( j \in \mathbb{Z}^+ \).

In fact, since \( \ker T = \ker U \), we have \( T^*(K) = (\ker T)^\perp = (\ker U)^\perp = U^*(K) \). By Lemma 2.1 and Lemma 2.2, we obtain

\[
\ker UT^+ = \ker TT^+ = T(H)^\perp, \\
T^+ T = \pi_{T^*(K)} = \pi_{U^*(K)} = U^+ U, \\
LT = UT^+ T = UU^+ U = U.
\]

It implies that \( LT W_j = UW_j \) for any \( j \in \mathbb{Z}^+ \) and

\[
\ker L = \ker UT^+ \cap T(H) = T(H)^\perp \cap T(H) = \{0\}, \quad LT(H) = U(H).
\]

This completes the proof. \( \square \)

**Corollary 2.14.** Let \( T \in L(H, K) \) be a closed range operator, and \( \{(W_j, v_j)\}_{j=1}^{\infty} \) be a fusion frame for \( H \) with bounds \( A \) and \( B \). Then \( \{(TW_j, v_j)\}_{j=1}^{\infty} \) is a fusion frame sequence for \( K \), if one of the following conditions holds:

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(1) $\sup_{j \in \mathbb{Z}^+} \delta((T+TW_j)W_j, W_j) < 1$.
(2) $\sup_{j \in \mathbb{Z}^+} \delta((T^*T)^NW_j, W_j) < 1$ for some integer $N \geq 2$.
(3) There exists a positive integer $N \geq 2$ such that $(T^*T)^NW_j \subset W_j$ for any $j \in \mathbb{Z}^+$.

**Proof.** Suppose the condition (1) holds, i.e.

$$\sup_{j \in \mathbb{Z}^+} \delta((T+TW_j)W_j, W_j) = \sup_{j \in \mathbb{Z}^+} \delta((T+TW_j)W_j, W_j) < 1.$$ 

By Corollary 2.12, we have that $\{(T+TW_j,v_j)\}_{j=1}^\infty$ is a fusion frame sequence for $H$. Since $\ker T+T = \ker T$, by Theorem 2.13, we have $\{(TW_j,v_j)\}_{j=1}^\infty$ is a fusion frame sequence for $K$.

Suppose the condition (2) holds. If $N$ is an odd number, then $\ker(T(T^*T)^{(N-1)/2}) = \ker T$ and

$$\sup_{j \in \mathbb{Z}^+} \delta((T(T^*T)^{(N-1)/2})^*T(T^*T)^{(N-1)/2}W_j, W_j) = \sup_{j \in \mathbb{Z}^+} \delta((T^*T)^NW_j, W_j) < 1.$$ 

If $N$ is an even number, then $\ker(T^*T)^{N/2} = \ker T$ and

$$\sup_{j \in \mathbb{Z}^+} \delta((T^*T)^{N/2})^*(T^*T)^{N/2}W_j, W_j) = \sup_{j \in \mathbb{Z}^+} \delta((T^*T)^NW_j, W_j) < 1.$$ 

By Corollary 2.12 and Theorem 2.13, we have $\{(TW_j,v_j)\}_{j=1}^\infty$ is a fusion frame sequence for $K$.

Obviously, if the condition (3) holds, then $\sup_{j \in \mathbb{Z}^+} \delta((T^*T)^NW_j, W_j) = 0 < 1$, i.e. the condition (2) holds. This completes the proof. \square

3. Operator perturbation of fusion frames

In this section, some new results about the operator perturbation of fusion frames are given.

**Theorem 3.1.** Let $T \in L(H)$ be a surjection, $U \in L(H,K)$ be a closed range operator, $\{(W_j,v_j)\}_{j=1}^\infty$ be a fusion frame for $H$. If $\ker T = \ker U$, then $\{(TW_j,v_j)\}_{j=1}^\infty$ is a fusion frame for $H$ if and only if $\{(UW_j,v_j)\}_{j=1}^\infty$ is a fusion frame sequence for $K$.

**Proof.** It follows immediately from Theorem 2.13. \square

**Remark 3.2.** Theorem 3.1 shows that the operator perturbation of fusion frames can be transformed into the operator perturbation of fusion frame sequences. In fact, there exist many closed range operators like $U$ with $\ker U = \ker T$. For example, $T+T$, $T^*T$, $(T^*T)^2$, $(T^*T)^3$ and so on. By Theorem 3.1, if we can find some conditions about $T$ such that one of $\{(T+TW_j,v_j)\}_{j=1}^\infty$, $\{(T^*TW_j,v_j)\}_{j=1}^\infty$, $\{(T^*T)^2W_j,v_j)\}_{j=1}^\infty$, \ldots is a fusion frame sequence for $K$, then $\{(TW_j,v_j)\}_{j=1}^\infty$ is a fusion frame for $H$. The following two theorems show part of results (about operator perturbation of fusion frames) obtained by this way.

**Theorem 3.3.** Let $T \in L(H)$ be a surjection, and $\{(W_j,v_j)\}_{j=1}^\infty$ be a fusion frame for $H$. Then the following conditions are equivalent:

(1) The family $\{(TW_j,v_j)\}_{j=1}^\infty$ is a fusion frame for $H$.
(2) There exists a real $C > 0$ such that

$$\sum_{j=1}^{\infty} v_j^2 \|\pi_{TW_j}f\|^2 \leq C \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j}T^*f\|^2,$$

for any $f \in T(H)$. 

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(3) There exists a real $C > 0$ such that
\[ \sum_{j=1}^{\infty} v_j^2 \|\pi_{T+TW_j} f\|^2 \leq C \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j} (T^*T)^\ast f\|^2, \]
for any $f \in T^*T(H)$.

(4) There exists a real $C > 0$ and integer $N > 0$ such that
\[ \sum_{j=1}^{\infty} v_j^2 \|\pi_{(T^*T)^N W_j} f\|^2 \leq C \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j} (T(T^*)^N)^\ast f\|^2, \]
for any $f \in (T^*T)^N(H)$.

(5) There exists a real $C > 0$ and integer $N > 0$ such that
\[ \sum_{j=1}^{\infty} v_j^2 \|\pi_{(T^*T)^N W_j} f\|^2 \leq C \sum_{j=1}^{\infty} v_j^2 \|\pi_{W_j} ((T^*T)^N)^\ast f\|^2, \]
for any $f \in (T^*T)^N(H)$.

**Proof.** It follows immediately from Theorem 2.7 and Theorem 3.1. \qed

**Theorem 3.4.** Let $T \in L(H)$ be a surjection, and $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame for $H$ with bounds $A$ and $B$. Then $\{\langle TW_j, v_j \rangle\}_{j=1}^{\infty}$ is a fusion frame for $H$, if one of the following conditions holds:

(1) $\sup_{j \in \mathbb{Z}^+} \delta(T^*TW_j, W_j) < 1$.
(2) $\sup_{j \in \mathbb{Z}^+} \delta((T^*T)^N W_j, W_j) < 1$ for some positive integer $N$.
(3) There exists a positive integer $N$ such that $(T^*T)^N W_j \subset W_j$ for any $j \in \mathbb{Z}^+$.

**Proof.** It follows immediately from Corollary 2.12, Corollary 2.14 and Theorem 3.1. \qed

The following theorem shows Theorem 3.4 in this paper is more general than the results (see Proposition 3.20 in [6], Theorem 2.11 in [7], and Theorem 2.4 in [10], Theorem 3.1 in [1], Theorem 2 in [14]) obtained by Casazza, Kutyniok, Asgari, Gavruta and Zhu.

**Theorem 3.5.** Let $T \in L(H)$ be a surjection, $\{(W_j, v_j)\}_{j=1}^{\infty}$ be a fusion frame for $H$. If one of the following conditions holds, then $\sup_{j \in \mathbb{Z}^+} \delta(T^*TW_j, W_j) < 1$ and $\sup_{j \in \mathbb{Z}^+} \delta(T^*TW_j, W_j) < 1$:

(a) For every $j \in \mathbb{Z}^+$, $T^*TW_j \subset W_j$ or $T^*TW_j \subset W_j$.
(b) $T$ is an invertible operator.

**Proof.** It is easy to show that $T^*TW_j = T^*TW_j$ and $T^*TW_j = T^*TW_j$ for any $j \in \mathbb{Z}^+$. This implies that $\sup_{j \in \mathbb{Z}^+} \delta(T^*TW_j, W_j) = \sup_{j \in \mathbb{Z}^+} \delta(T^*TW_j, W_j) = \sup_{j \in \mathbb{Z}^+} \delta(T^*TW_j, W_j)$ and $\sup_{j \in \mathbb{Z}^+} \delta(T^*TW_j, W_j) = \sup_{j \in \mathbb{Z}^+} \delta(T^*TW_j, W_j)$. For any $j \in \mathbb{Z}^+$, from $T^*TW_j \subset W_j$, which implies $T^*TW_j \subset W_j$, we have $\delta(T^*TW_j, W_j) = 0$ and
\[ \pi_{W_j}(T^*TW_j f) = T^*TW_j f \quad \text{for every } f \in H, \]
i.e. $\pi_{W_j} T^*TW_j = T^*TW_j$. It follows that
\[ \pi_{TW_j} (T^*) = (\pi_{TW_j})^\ast = (\pi_{W_j} T^*TW_j)^\ast = \pi_{TW_j} (T^*) \pi_{W_j}, \]
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From Lemma 2.2, we obtain
\[ \pi_{TW_j} (T^+) \pi_{W_j} (T^+ \pi_{TW_j}) = \pi_{TW_j} (T^+) \pi_{TW_j} = \pi_{TW_j}. \]
Since
\[ \| \pi_{TW_j} (T^+) \pi_{W_j} (T^+ \pi_{TW_j}) f \| \leq \| \pi_{TW_j} (T^+) \| \cdot \| \pi_{W_j} (T^+ \pi_{TW_j}) f \| \leq \| T^+ \| \cdot \| \pi_{W_j} (T^+ \pi_{TW_j}) f \| \]
and
\[ \| \pi_{TW_j} (T^+) \pi_{W_j} (T^+ \pi_{TW_j}) f \| = \| \pi_{TW_j} f \| \leq \| T^+ \|^{-1} \| T^+ \pi_{TW_j} f \| = \| T \|^{-1} \cdot \| T^+ \pi_{TW_j} f \| \]
for every \( f \in H \), we obtain
\[ \| \pi_{W_j} (T^+ \pi_{TW_j}) f \| \geq \| T^+ \|^{-1} \| T \|^{-1} \cdot \| T^+ \pi_{TW_j} f \|, \quad f \in H. \]
By Lemma 2.5, we obtain that \( \delta(T^+ TW_j, W_j) \leq \sqrt{1 - \| T^+ \|^{-2} \| T \|^{-2}} < 1. \)
For any \( j \in Z^+ \), from \( T^+ TW_j \subset W_j \), which implies \( T^+ TW_j \subset W_j \), we have \( \delta(T^+ TW_j, W_j) = 0 \) and \( \pi_{W_j} (T^+ \pi_{TW_j}) = T^+ \pi_{TW_j} \). It follows that
\[ \pi_{TW_j} T = (T^+ \pi_{TW_j})^* = (\pi_{W_j} T^+ \pi_{TW_j})^* = \pi_{TW_j} T \pi_{W_j}. \]
From Lemma 2.2, we obtain
\[ \pi_{TW_j} T \pi_{W_j} (T^+ \pi_{TW_j}) = \pi_{TW_j} T T^+ \pi_{TW_j} = \pi_{TW_j}. \]
Since
\[ \| \pi_{TW_j} T \pi_{W_j} T^+ \pi_{TW_j} f \| \leq \| \pi_{TW_j} T \| \cdot \| \pi_{W_j} (T^+ \pi_{TW_j}) f \| \leq \| T \| \cdot \| \pi_{W_j} (T^+ \pi_{TW_j}) f \| \]
and
\[ \| \pi_{TW_j} T \pi_{W_j} T^+ \pi_{TW_j} f \| = \| \pi_{TW_j} f \| \leq \| T^+ \|^{-1} \cdot \| T^+ \pi_{TW_j} f \| \]
for every \( f \in H \), we obtain
\[ \| \pi_{W_j} (T^+ \pi_{TW_j}) f \| \geq \| T^+ \|^{-1} \| T \|^{-1} \cdot \| T^+ \pi_{TW_j} f \| \]
for every \( f \in H \). By Lemma 2.5, we obtain \( \delta(T^+ TW_j, W_j) \leq \sqrt{1 - \| T^+ \|^{-2} \| T \|^{-2}} < 1. \)
Since \( T \) is an invertible operator, we have \( T^+ T (W_j) = T^{-1} T (W_j) \subset W_j \) for all \( j \in Z^+ \), it follows that \( \delta(T^+ TW_j, W_j) = 0 \) and \( \delta(T^+ TW_j, W_j) \leq \sqrt{1 - \| T^+ \|^{-2} \| T \|^{-2}} \) for all \( j \in Z^+ \). This completes the proof. □

**Theorem 3.6.** Let \( \{ (W_j, v_j) \}_{j=1}^\infty \) be a fusion frame for \( H \) and \( \{ V_j \}_{j=1}^\infty \) be a family of closed subspaces for \( K \). If there exists a surjection \( T \in L(H, K) \) such that \( \sup_{j \in Z^+} \delta(T^* V_j, W_j) < 1 \) and \( TW_j \subset V_j \) for any \( j \in Z^+ \), then \( \{ (V_j, v_j) \}_{j=1}^\infty \) is a fusion frame for \( K \).

**Proof.** Suppose \( A \) and \( B \) are the bounds of \( \{ (W_j, v_j) \}_{j=1}^\infty \). Since \( TW_j \subset V_j \) for any \( j \in Z^+ \), we have
\[ \| \pi_{TW_j} f \| = \| \pi_{TW_j} \pi_{V_j} f \| \leq \| \pi_{V_j} f \| \]
for any \( j \in Z^+ \) and \( f \in K \). By Lemma 2.6, we obtain

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\[ \sum_{j=1}^{\infty} v_j^2 \| \pi V_j f \|^2 \geq \sum_{j=1}^{\infty} v_j^2 \| \pi W_j f \|^2 \geq A \| T^+ \|^{-2} \| T \|^{-2} \| f \|^2 \]

for any \( f \in T(H) = K \). Since \( \delta(T^* V_j, W_j) \leq c < 1 \) for any \( j \in Z^+ \), by Lemma 2.3(2),

\[ B \| T^* \| \| f \|^2 \geq B \| T^* f \|^2 \geq \sum_{j=1}^{\infty} v_j^2 \| \pi W_j T^* f \|^2 = \sum_{j=1}^{\infty} v_j^2 \| \pi W_j T^* \pi V_j f \|^2 \]

\[ \geq (1 - c^2) \sum_{j=1}^{\infty} v_j^2 \| T^* V_j f \|^2 \geq (1 - c^2) \sum_{j=1}^{\infty} v_j^2 \| \pi V_j f \|^2, \]

where \( f \in K \). Hence we conclude that \( \sum_{j=1}^{\infty} v_j^2 \| \pi V_j f \|^2 \leq B(1 - c^2)^{-1} \| T \| \| T^+ \| \| f \|^2 \) for any \( f \in K \). This completes the proof. \( \square \)

References