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Topological classification of function spaces with the Fell topology IV ☆

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ABSTRACT

For a Tychonoff space X, let $\downarrow\mathrm{C}_F(X)$ denote the collection of the hypographs of all continuous maps from X to $[0,1]$ with the Fell topology. We show that, for a Tychonoff non-discrete k-space X, the function space $\downarrow\mathrm{C}_F(X)$ is homeomorphic to $c_0 \cup (Q \setminus \Sigma)$ if $\downarrow\mathrm{C}_F(X)$ is metrizable and the set of isolated points of X is dense in X, where $Q = [-1,1]^{\mathbb{N}}$ is the Hilbert cube, $\Sigma = \{(x_n) \in Q : \sup|x_n| < 1\}$ and $c_0 = \{(x_n) \in \Sigma : \lim x_n = 0\}$ are its subspaces. Combining results in the previous papers of the series, we give the topological classification for all metrizable function spaces $\downarrow\mathrm{C}_F(X)$ of k-spaces X.

All spaces are assumed to be Tychonoff topological spaces.

1. Introduction and the main theorem

For a space X, let $\mathrm{USC}(X)$ and $\mathrm{C}(X)$ be the sets of all upper semi-continuous maps and of all continuous maps from X to $\mathbf{I} = [0,1]$, respectively. For every $f \in \mathrm{USC}(X)$, define

$$\downarrow f = \{(x,t) \in X \times \mathbf{I} : t \leq f(x)\}.$$

The closed set $\downarrow f$ in the product space $X \times \mathbf{I}$ is called the *hypograph* of f. Let

$$\downarrow\mathrm{USC}(X) = \{\downarrow f : f \in \mathrm{USC}(X)\}, \quad \downarrow\mathrm{C}(X) = \{\downarrow f : f \in \mathrm{C}(X)\}.$$

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Then $\downarrow\mathrm{C}(X) \subset \downarrow\mathrm{USC}(X) \subset \mathrm{Cld}(X \times \mathbf{I})$, where $\mathrm{Cld}(X \times \mathbf{I})$ consists of all non-empty closed sets in the product space $X \times \mathbf{I}$. For a space Y, we recall that the *Fell topology* on the family $\mathrm{Cld}(Y)$ of all non-empty closed sets in Y is generated by

$$\{K^*, U^- : K \text{ and } U \text{ are compact and open, respectively in } Y\}$$

as a subbase, where

$$K^* = \{F \in \mathrm{Cld}(Y) : F \cap K = \emptyset\}, \quad U^- = \{F \in \mathrm{Cld}(Y) : F \cap U \neq \emptyset\}.$$

We use $\mathrm{Cld}_F(Y)$ to denote $\mathrm{Cld}(Y)$ with the Fell topology. And we use $\downarrow\mathrm{USC}_F(X)$ and $\downarrow\mathrm{C}_F(X)$ to denote the sets $\downarrow\mathrm{USC}(X)$ and $\downarrow\mathrm{C}(X)$ with the subspace topology of the space $\mathrm{Cld}_F(X \times \mathbf{I})$. Similarly, we can define the space $\uparrow\mathrm{C}_F(X)$. The topologies on $\downarrow\mathrm{C}_F(X)$ and $\uparrow\mathrm{C}_F(X)$ are called the *Fell hypograph topology* and *Fell hypergraph topology* in the set $\mathrm{C}(X)$, respectively. Note that the compact-open topology on the function space $\mathrm{C}(X)$ is the least upper bound of those two topologies on $\mathrm{C}(X)$.

As usual, cl_Y and int_Y denote the closure operator and interior operator in a space Y. We omit the subscript if no confusion. Let $\mathbb{N}$ be the set of all natural numbers with the discrete topology and

$$Q = [-1, 1]^{\mathbb{N}}$$

the Hilbert cube. Define its two subspaces as follows

$$\Sigma = \{(x_n) \in Q : \sup |x_n| < 1\} \ \text{ and } \ c_0 = \{(x_n) \in \Sigma : \lim_{n \to \infty} x_n = 0\}.$$

For two spaces X, Y and their subspaces A, B, respectively, the symbol $(X, A) \approx (Y, B)$ means that there exists a homeomorphism $h : X \to Y$ such that $h(A) = B$. Similarly, the symbol $X \approx Y$ means that X and Y are homeomorphic. Moreover, we use X_0 to denote the set of all isolated points in X.

In [8,10,11,13,15], the authors gave the topological classification for all metrizable function spaces $\downarrow\mathrm{C}_F(X)$ under the condition that X is metrizable. That is,

Theorem 1. *For a metrizable space X, the function space $\downarrow\mathrm{C}_F(X)$ is metrizable if and only if X is a locally compact and separable metrizable space. Moreover, then*

$$(\downarrow\mathrm{USC}_F(X), \downarrow\mathrm{C}_F(X)) \approx \begin{cases} (\mathbf{I}^{|X|}, \mathbf{I}^{|X|}) & \textit{if } X \textit{ is discrete;} \\ (Q, c_0) & \textit{if } \mathrm{cl}\, X_0 \neq X; \\ (Q, c_0 \cup (Q \setminus \Sigma)) & \textit{otherwise.} \end{cases}$$

But there are also many non-metrizable spaces X such that $\downarrow\mathrm{C}_F(X)$ are metrizable, see the next section and the last section of this paper, [3,7,12].

In [14] and [9], the authors gave the topological structure of the metrizable function space $\downarrow\mathrm{C}_F(X)$ when X is a k-space and satisfies some conditions. That is:

Theorem 2. *For a k-space X, the following statements are equivalent:*

(a) $\downarrow\mathrm{C}_F(X) \approx c_0$*;*
(b) $\downarrow\mathrm{C}_F(X)$ *is metrizable but not Baire;*
(c) $\downarrow\mathrm{C}_F(X)$ *is metrizable and* $\mathrm{cl}\, X_0 \neq X$.

Theorem 3. *Let X be a space satisfying the following conditions:*

(i) $\downarrow\mathrm{C}_F(X)$ *is metrizable;*
(ii) X *is a non-discrete k-space;*
(iii) $\mathrm{cl}\, X_0 = X$*;*
(iv) $X \setminus X_0$ *is not compact.*

Then

$$\downarrow\mathrm{C}_F(X) \approx c_0 \cup (Q \setminus \Sigma).$$

In the present paper, we show that the above theorem also holds while the condition (iv) is removed. That is,

Main Theorem. *Let X be a non-discrete k-space. If the set of all isolated points in X is dense and $\downarrow\mathrm{C}_F(X)$ is metrizable, then we have*

$$\downarrow\mathrm{C}_F(X) \approx c_0 \cup (Q \setminus \Sigma).$$

Summarizing the above theorems, we have

Theorem 4. *For a k-space X, if the function space $\downarrow\mathrm{C}_F(X)$ is metrizable, then*

$$\downarrow\mathrm{C}_F(X) \approx \begin{cases} \mathbf{I}^{|X|} & \text{if } X \text{ is discrete;} \\ c_0 & \text{if } \mathrm{cl}\, X_0 \neq X; \\ c_0 \cup (Q \setminus \Sigma) & \text{otherwise.} \end{cases}$$

2. Preliminaries

In this section, we at first give some concepts and facts on general topology and infinite-dimensional topology. For more information, we refer to [1,2,5,6]. Secondly, for the sake of completeness, we recall some results and symbols in [9,12].

For a metric space (M, d), a closed subset A is called a *Z-set* if for open cover $\mathcal{U}$ of M there is a continuous map $f : M \to M \setminus A$ such that f and id_M are $\mathcal{U}$-close, that is, for every $x \in M$, there exists $U \in \mathcal{U}$ satisfying $U \ni f(x), x$. If A is compact in a metric space (M, d), A is a Z-set if and only if for every $\varepsilon > 0$ there is a continuous map $f : M \to M \setminus A$ with $d(f(x), x) < \varepsilon$ for every $x \in M$. A *Z_σ-set* in a space is a subset of the space which is a countable union of Z-sets in the space. A *Z-embedding* is an embedding with a Z-set image.

A subset A of a space Y is called *homotopy dense* if there exists a homotopy $H : Y \times \mathbf{I} \to Y$ such that $H(y, 0) = y$ and $H(y, t) \in A$ for every $y \in Y$ and $t \in (0, 1]$.

A space Y is defined to satisfy the *strong discrete approximation property* (briefly *SDAP*) if for every continuous map $f : Q \times \mathbb{N} \to Y$ and open cover $\mathcal{U}$ of Y, there exists a continuous map $g : Q \times \mathbb{N} \to Y$ such that g, f are $\mathcal{U}$-close and $\{g(Q \times \{n\}) : n \in \mathbb{N}\}$ is a discrete collection in Y.

Let Z be a separable metrizable space and $\mathcal{C}$ a class of separable metrizable spaces. We say that Z is *strongly $\mathcal{C}$-universal* provided that for every $C \in \mathcal{C}$, every continuous map $f : C \to Z$, every closed subset K of Z such that $f|K : K \to C$ is a Z-embedding and for every open cover $\mathcal{U}$ of Z, there is a Z-embedding $g : C \to Z$ such that $g|K = f|K$ and g, f are $\mathcal{U}$-close.

Let $\mathcal{M}_0$ denote the class of metrizable compacta, and for a topological class $\mathcal{C}$, let $(\mathcal{M}_0, \mathcal{C})$ denote the class of pairs (Z, C) such that $Z \in \mathcal{M}_0$ and $C \in \mathcal{C}$ with $Z \supset C$. Let (M, d) be a copy of the Hilbert cube Q. We say that a subspace Z of M is *strongly $\mathcal{C}$-universal in M* provided that for every $(Y, C) \in (\mathcal{M}_0, \mathcal{C})$, every continuous map $f : Y \to M$, every closed subset K of Y such that $f|K : K \to M$ is a Z-embedding and for every open cover $\mathcal{U}$ of M, there is a Z-embedding $g : Y \to M$ such that $g|K = f|K$, $g^{-1}(Z) \setminus K = C \setminus K$ and g, f are $\mathcal{U}$-close.

By [1, 1.7.9 Theorem], we have

Lemma 1. *If Y is both homotopy dense and strongly $\mathcal{C}$-universal in a copy M of the Hilbert cube and Y has SDAP, then Y is strongly $\mathcal{C}$-universal.*

A space Y is called a *$\mathcal{C}$-coabsorber* if

(i) Y is a strongly $\mathcal{C}$-universal ANR (absolutely neighborhood retract) satisfying SDAP;
(ii) every Z-set in the space Y is in $\mathcal{C}$;
(iii) Y is a co-Z_σ-space, that is, Y contains a homotopy dense absolute G_δ-subset.

The fundamental property of coabsorber is as follows, see [1, 1.6.4 Theorem].

Lemma 2. *Let $\mathcal{C}$ be a class of spaces. Two $\mathcal{C}$-coabsorbers are homeomorphic if and only if they are homotopy equivalent. In particular, two $\mathcal{C}$-coabsorbing AR's (absolutely retracts) are homeomorphic.*

Next, we introduce some concepts needed in this paper.

A separable metrizable space Y is called an *absolute $F_{\sigma\delta}$-space* if it is an $F_{\sigma\delta}$-set in any metrizable space which contains Y as a subspace. Let $\mathcal{F}_{\sigma\delta}$ be the class of all absolute $F_{\sigma\delta}$-spaces. We will prove that $(Q \setminus \Sigma) \cup c_0$ is an $\mathcal{F}_{\sigma\delta}$-coabsorber.

A space X is called a *k-space* provided for a subset A of X, A is closed in X if and only if $A \cap C$ is closed in C for every compact subspace C of X. It is well-known that a space is a k-space if and only if it is a quotient space of a locally compact space, see [2, Theorem 3.3.18]. A space X is called *weakly locally compact* if for every compact set K in X there exists an open set U in X such that $K \subset \mathrm{cl}\, U$ and $\mathrm{cl}\, U$ is compact. A space X is said to be *hemicompact* if there exists a countable family $\mathcal{K}$ of compact sets in X such that every compact set in X is included in some element of $\mathcal{K}$. A space is called an *$\aleph_0$-space* if there exists a countable k-network, where a family $\mathcal{K}$ of sets in a space X is called a *k-network*[1] if, for every compact set C and every open set U in X with $U \supset C$, there exists $K \in \mathcal{K}$ such that $C \subset K \subset U$. A continuous map $r : Y \to X$ is called *compact-covering* if for every compact set C in X, there exists a compact set D in Y such that $r(D) = C$, or, equivalently, $r(D) \supset C$. Note that a space is an $\aleph_0$-space if and only if it is an image of a separable metrizable space under compact-covering map, see [4]. A continuous map $r : Y \to X$ is called *weakly open* if for every non-empty open set U in Y, $\mathrm{int}\,\mathrm{cl}\, r(U)$ is non-empty in X.

Using the above concepts, in [3], cf. [12], McCoy and Ntantu presented a sufficient and necessary condition for $\downarrow\mathrm{C}_F(X)$ being metrizable. That is,

Lemma 3. *The space $\downarrow\mathrm{C}_F(X)$ is metrizable if and only if X is a weakly locally compact and hemicompact $\aleph_0$-space.*

In [7], it was proved that X is metrizable if $\downarrow\mathrm{C}_F(X)$ is metrizable and X is either first-countable or locally compact. But the metrization of $\downarrow\mathrm{C}_F(X)$ does not imply that the metrization of X since there exists a weakly locally compact and hemicompact $\aleph_0$-space, such as the sequential fan space $S_{\mathbb{N}}$ in the next section, which is neither first-countable nor locally compact.

[1] In some literatures, a family $\mathcal{K}$ of sets in a space X is called a k-network if, for every compact set C and every open set U in X with $U \supset C$, there exists a finite subfamily $\mathcal{K}_0$ of $\mathcal{K}$ such that $C \subset \bigcup \mathcal{K}_0 \subset U$. Such difference does not make any trouble.

From now to the end of this paper, we fix a space X and assume that X is at least

a weakly locally compact and hemicompact $\aleph_0$-space.

For such a space X, in [12], the authors constructed a locally compact separable metrizable space M and a compact-covering and weakly open map $r : M \to X$. The map induces a map $r^\# : \mathrm{USC}_F(X) \to \mathrm{USC}_F(M)$ as follows:

$$r^\#(f) = f \circ r, \text{ for every } f \in \mathrm{USC}(X).$$

Therefore, we can obtain a map $r^\# : {\downarrow}\mathrm{USC}_F(X) \to {\downarrow}\mathrm{USC}_F(M)$ defined by

$$r^\#({\downarrow}f) = {\downarrow}(f \circ r), \text{ for } f \in \mathrm{USC}(X).$$

Let

$$A(X) = \mathrm{cl}_{{\downarrow}\mathrm{USC}_F(M)}\, r^\#({\downarrow}\mathrm{C}_F(X)).$$

In [12,14], the following results were proved:

Lemma 4 *(The Embedding Theorem). For every weakly locally compact and hemicompact infinite $\aleph_0$-space X, we have that $r^\# : {\downarrow}\mathrm{C}_F(X) \to r^\#({\downarrow}\mathrm{C}_F(X))$ is a homeomorphism, $A(X) \approx Q$ and $r^\#({\downarrow}\mathrm{C}_F(X))$ is homotopy dense in $A(X)$.*

Lemma 5. *For every weakly locally compact and hemicompact infinite $\aleph_0$-space X, the following statements are equivalent:*

(a) *$r^\#({\downarrow}\mathrm{C}_F(X))$ is closed in ${\downarrow}\mathrm{C}_F(M)$;*
(b) *$r : M \to X$ is a quotient map;*
(c) *X is a k-space.*

Hence $r^\#({\downarrow}\mathrm{C}_F(X))$ is an absolute $F_{\sigma\delta}$-space if X is a k-space since, by Theorem 1, ${\downarrow}\mathrm{C}_F(M)$ is an absolute $F_{\sigma\delta}$-space.

Let

$${\downarrow}\mathrm{C}_0(X) = \{{\downarrow}f \in A(X) : f(z) = 0 \text{ for every non-isolated point } z \in M\}.$$

The following results were essentially proved in [9, Lemmas 8, 10, 11],

Lemma 6. *Let X be a weakly locally compact and hemicompact infinite non-discrete $\aleph_0$-space. If X is a k-space and $\mathrm{cl}\, X_0 = X$, then*

(1) *$(A(X), {\downarrow}\mathrm{C}_0(X)) \approx (Q, Q \setminus \Sigma)$;*
(2) *${\downarrow}\mathrm{C}_0(X) \subset r^\#({\downarrow}\mathrm{C}(X))$;*
(3) *${\downarrow}\mathrm{C}_0(X)$ satisfies SDAP and is homotopy dense in $r^\#({\downarrow}\mathrm{C}(X))$;*
(4) *If $x_\infty \in X \setminus X_0$, then there exists a sequence $\{x_k\}$ of points in X_0 such that $\lim_{k\to\infty} x_k = x_\infty$.*

Proof. The statements (1) and (2) follow from [9, Lemmas 8, 10] and [5, Section 6.5]. Since $Q \setminus \Sigma \approx \mathbb{R}^\mathbb{N}$ and $\mathbb{R}^\mathbb{N}$ satisfies SDAP (see [1, 1.3.2 Theorem]), ${\downarrow}\mathrm{C}_0(X)$ satisfies SDAP. It is well-known that $(Q, Q \setminus \Sigma) \approx$

$(Q, (-1,1)^{\mathbb{N}})$ (see [5, Section 6.5]) and $(-1,1)^{\mathbb{N}}$ is homotopy dense in Q. Thus, (3) holds. The statement (4) follows from [9, Lemma 11]. □

3. A characterization of the sequential fan space

To prove The Main Theorem, we give a characterization of the sequential fan space $S_{\mathbb{N}}$ in this section.

Let $\{S_i : i \in \mathbb{N}\}$ be countable pair-disjoint convergent sequences. For every $i \in \mathbb{N}$, assume that S_i converges to a_i. Then quotient space of $\bigoplus_{i\in\mathbb{N}} S_i$ collapsing the set $\{a_i : i \in \mathbb{N}\}$ to a point is called the *sequential fan space*, denoted by $S_{\mathbb{N}}$.

We have the following proposition.

Proposition 1. *For a space Y, $Y \approx S_{\mathbb{N}}$ if and only if Y is a non-metrizable weakly locally compact and hemicompact $\aleph_0$-space and a k-space with unique non-isolated point.*

Proof. The necessity is obvious. We will prove the sufficiency. Let x_∞ be the unique non-isolated point in Y. By [12, Lemma 3], there exists a k-network $\{C_n : n \in \mathbb{N}\}$ in Y such that every C_n is compact Y. Define $\mathcal{S}$ to be the family of images sets of sequences converging to x_∞ in Y. Then we have:

Fact 1: Every compact subset of Y is either a finite set or in $\mathcal{S}$.

Fact 2: The set $A = \{n : C_n \in \mathcal{S}\}$ is infinite.

Since Y is an $\aleph_0$-space, every compact set in Y is metrizable. Fact 1 follows.

Next we prove Fact 2. Assume that $A = \{n : C_n \in \mathcal{S}\}$ is finite. Let $C = \{x_\infty\} \cup \bigcup_{n\in A} C_n$. Then C is a metrizable compact subspace of Y. For every $n \in \mathbb{N}$, $(Y \setminus C) \cap C_n$ is finite and hence it is closed in C_n. Since $\{C_n\}_{n=1}^{\infty}$ is a k-network and Y is a k-space, $Y \setminus C$ is an open and closed discrete subspace of Y. Hence $Y = C \oplus (Y \setminus C)$ is metrizable. It is a contradiction.

We define $\{Q_n : n = 1, 2, \cdots\}$ by induction as follows. Let

$$Q_1 = C_1, \text{ and for } \ n > 1$$

$$Q_n = \begin{cases} (C_n \setminus \bigcup_{j=1}^{n-1} Q_j) \cup \{x_\infty\} & \text{if } C_n \setminus \bigcup_{j=1}^{n-1} Q_j \text{ is infinite,} \\ C_n \setminus \bigcup_{j=1}^{n-1} Q_j & \text{otherwise.} \end{cases}$$

Then for every $n \in \mathbb{N}$, Q_n is either finite or $Q_n \in \mathcal{S}$. Similarly to the proof of Fact 2, we have

$$\{n : Q_n \in \mathcal{S}\} = \{n_1 < n_2 < \cdots < n_i < \cdots\}$$

is infinite. Let

$$S_i = \bigcup_{n_{i-1} < j \le n_i} Q_j,$$

where $n_0 = 0$. Then for every $i \ne j$,

$$S_i \in \mathcal{S}, \quad S_i \cap S_j = \{x_\infty\}.$$

The natural map $r : \bigoplus_{i=1}^{\infty} S_i \to Y$ is continuous since $r|S_i$ is imbedding for every i.

At last, we show that r is a quotient map, which complete our proof. Assume that $A \subset Y$ and $r^{-1}(A)$ is closed in $\bigoplus_{i=1}^{\infty} S_i$. Then $A \cap S_i$ is closed in hence compact for every i. Thus, $r(A \cap S_i) = A \cap S_i$ is compact. Note that every C_n is included in a finite union of S_i's. It follows that $A \cap C_n$ is compact for every n. Since Y is a k-space and $\{C_n\}$ is a k-network for Y, A is closed in Y. We conclude that $r : \bigoplus_{i=1}^{\infty} S_i \to Y$ is a quotient map. □

4. $\downarrow \mathrm{C}_F(X)$ is strongly-$\mathcal{F}_{\sigma\delta}$-universal in $A(X)$

In this section, we verify that $\downarrow \mathrm{C}_F(X)$ is strongly-$\mathcal{F}_{\sigma\delta}$-universal in $A(X)$ if X is also a non-discrete k-space and $\operatorname{cl} X_0 = X$.

For $i \in \mathbb{N}, j \in \mathbb{N} \cup \{\infty\}$, let $a_{i,j} = (i, \frac{1}{2^j})$, where $\frac{1}{2^\infty} = 0$. And let

$$M = \{a_{i,j} \in \mathbb{R}^2 : j \in \mathbb{N} \cup \{\infty\}, i \in \mathbb{N}\}$$

be a metric subspace of Euclidean space $\mathbb{R}^2$ with the usual metric d. Then

$$d((a,t),(b,s)) = \max\{d(a,b), |t-s|\}$$

is a metric on the space $M \times \mathbf{I}$. We consider the sequential fan $S_\mathbb{N}$ as a quotient space of M collapsing the set $\{(i,0) : i \in \mathbb{N}\}$ to the point m_∞. And $r : M \to S_\mathbb{N}$ is the quotient map corresponding. We can define $A(S_\mathbb{N})$ as the same method in Section 2 by them. Moreover, we can define an admissible metric ρ on the space $A(S_\mathbb{N})$ such that for every pair $\downarrow f, \downarrow g \in A(S_\mathbb{N})$,

$$\rho(\downarrow f, \downarrow g) \le d_H(\downarrow f, \downarrow g),$$

where d_H is the Hausdorff metric on $\mathrm{Cld}(M \times \mathbf{I})$ induced by the metric d on $M \times \mathbf{I}$, see [9]. (Note that d_H is not an admissible metric on the space $A(S_\mathbb{N})$.) We might as well identify $r(a_{i,j})$, $r^\#(\mathrm{C}(S_\mathbb{N}))$ and $a_{i,j}, \mathrm{C}(S_\mathbb{N})$, respectively. Then we have:

Lemma 7. *The space $\downarrow \mathrm{C}_F(S_\mathbb{N})$ is strongly-$\mathcal{F}_{\sigma\delta}$-universal in $A(S_\mathbb{N})$.*

Proof. Let Y be a compact metrizable space and C, K an $F_{\sigma\delta}$-set and a closed set in Y, respectively. For an arbitrary $\varepsilon \in (0,1)$ and every continuous map $\downarrow \Phi : Y \to A(S_\mathbb{N})$, while its restriction $\downarrow \Phi|_K$ is a Z-embedding, we will define a continuous map $\downarrow \Psi : Y \to A(S_\mathbb{N})$ such that

(1) $\downarrow \Psi : Y \to A(S_\mathbb{N})$ is a Z-embedding;
(2) $\downarrow \Psi | K = \downarrow \Phi | K$;
(3) $\downarrow \Psi^{-1}(\downarrow \mathrm{C}(S_\mathbb{N})) \setminus K = C \setminus K$;
(4) $\rho(\downarrow \Psi(y), \downarrow \Phi(y)) < \varepsilon$ for every $y \in Y$.

By the Mapping Replacement Theorem [6, Theorem 5.3.11], we may assume that

$$\downarrow \Phi(K) \cap \downarrow \Phi(Y \setminus K) = \emptyset.$$

First, we define $\delta : Y \to [0,1]$ as

$$\delta(y) = \frac{1}{4} \min\{\varepsilon, \rho(\downarrow \Phi(y), \downarrow \Phi(K))\}.$$

Then, $y \in K$ if and only if $\delta(y) = 0$. By [6, Proposition 4.1.7] and Lemma 4, there exists a homotopy $\downarrow H : A(S_\mathbb{N}) \times \mathbf{I} \to A(S_\mathbb{N})$ such that for every $\downarrow f \in A(S_\mathbb{N})$ and $t \in (0,1]$,

$$\downarrow H_0 = id_{A(S_\mathbb{N})}, \downarrow H_t(A(S_\mathbb{N})) \subset \downarrow \mathrm{C}(S_\mathbb{N}), \text{ and } \rho(\downarrow H_t(\downarrow f), \downarrow f) \le t.$$

For every $y \in Y$, define $\downarrow h(y) \in A(S_\mathbb{N})$ as follows

$$h(y)(a_{i,j}) = \max\{\delta(y), H(\Phi(y), \delta(y))(a_{i,j})\}.$$

Then

(5) $\downarrow h|K = \downarrow \Phi|K$;
(6) $\downarrow h(Y \setminus K) \subset \downarrow \mathrm{C}(S_{\mathbb{N}})$;
(7) $\rho(\downarrow h(y), \downarrow \Phi(y)) \leq \delta(y)$;
(8) $h(y)(a_{i,j}) > 0$ for every $y \in Y \setminus K$ and $a_{i,j} \in M$.

Define $L : (Y \setminus K) \times (S_{\mathbb{N}} \setminus \{m_\infty\}) \to \mathbf{I}$ by

$$L(y, a_{i,j}) = \max\{h(y)(a_{i,p}) : j \leq p \leq \infty\}.$$

Note that $h(y)$ is continuous for every $y \in Y \setminus K$ and $\{a_{i,p} : j \leq p \leq \infty\}$ is compact for every pair i, j. It follows that $L(y, a_{i,j})$ is well-defined. Obviously L is continuous. By [10, Lemma 2.9], there exists an embedding map $g : Y \to Q_u = [0,1]^{\mathbb{N}}$ such that $g^{-1}(c_1) = C$, where

$$c_1 = \{(x_n) \in [0,1]^{\mathbb{N}} : \lim x_n = 1\}.$$

Let

$$Y_k = \{y \in Y : \frac{1}{2^k} \leq \delta(y) \leq \frac{1}{2^{k-1}}\}.$$

For every $y \in Y_k$, we construct a function $\downarrow \Psi_k(y) \in A(S_{\mathbb{N}})$ as follows. For $a_{i,j} \in M$, let

$$\Psi_k(y)(a_{i,j}) = \begin{cases} h(y)(a_{i,j}) & j \leq 2k; \\ (1-t) \cdot \delta(y) + t \cdot h(y)(a_{i,j}) & j = 2k+1; \\ (1-t) \cdot \frac{\delta(y)}{2} + t \cdot h(y)(a_{i,j}) & j = 2k+2; \\ \delta(y) & j = 2k+3; \\ t \cdot \frac{\delta(y)}{2} & j = 2k+4; \\ (1-t) \cdot B(y)(a_{i,j}) + t \cdot \delta(y) & j = 2k+5; \\ 0 & j = 2k+6; \\ B(y)(a_{i,j}) & j = 2k+7; \\ (1-t) \cdot \delta(y) & j = 2k+8; \\ (1-t) \cdot \delta(y) + t \cdot B(y)(a_{i,j}) & j = 2k+9; \\ t \cdot \delta(y) & j = 2k+10; \\ \delta(y) & j = 2k+11; \\ (1-t) \cdot \delta(y) \cdot g(y)(1) & j = 2k+12; \\ \delta(y) \cdot [(1-t) \cdot g(y)(l+1) + t \cdot g(y)(l)] & j > 2k+12, \\ & j \text{ is even}; \\ \delta(y) & \text{otherwise}, \end{cases}$$

where $t = 2 - 2^k \delta(y)$, $l = 2^{-1}j - k - 6$ and $B(y)(a_{i,j}) = (1-t) \cdot L(y, a_{i,2k-1}) + t \cdot L(y, a_{i,2k+1})$.

For every $m \geq 2k + 10$, define

$$\Psi_k^m(y)(a_{i,j}) = \begin{cases} \Psi_k(y)(a_{i,j}) & j \leq m; \\ \delta(y) & j > m. \end{cases}$$

Then $\Psi_k^m(y) \in \mathrm{C}(S_{\mathbb{N}})$ and $\downarrow \Psi_k(y) = \lim_{m\to\infty} \downarrow \Psi_k^m(y)$. It follows that $\downarrow \Psi_k(y) \in A(S_{\mathbb{N}})$ for every $y \in Y \setminus K$.

Now, define a function $\downarrow\Psi : Y \to A(S_{\mathbb{N}})$ by

$$\Psi(y) = \begin{cases} \Phi(y) = h(y) & y \in K; \\ \Psi_k(y) & y \in Y_k. \end{cases}$$

The function Ψ is as required.

Firstly, $\downarrow\Psi : Y \to A(S_{\mathbb{N}})$ is well-defined and $\downarrow\Psi^{-1}(\downarrow\mathrm{C}(S_{\mathbb{N}})) \setminus K = C \setminus K$.

In fact, if $y \in Y_k \cap Y_{k+1}$, then $\delta(y) = \frac{1}{2^k}$. It is not hard to verify that $\Psi_k(y) = \Psi_{k+1}(y)$. For every $y \in Y \setminus K$, $\Psi(y) \in \downarrow\mathrm{C}(S_{\mathbb{N}})$ if and only if it is continuous at m_∞ if and only if $\lim_{n\to\infty} g(y)(n) = 1$ if and only if $y \in C$. Hence $\downarrow\Psi^{-1}(\downarrow\mathrm{C}(S_{\mathbb{N}})) \setminus K = C \setminus K$.

Secondly, for every $y \in Y$, $\rho(\downarrow\Psi(y), \downarrow\Phi(y)) \leq 2\delta(y) < \varepsilon$.

In fact, for every $\frac{1}{2^j} \geq \delta(y)$ and $i \in \mathbb{N}$,

$$|\Psi(y)(a_{i,j}) - h(y)(a_{i,j})| = 0 \leq \delta(y).$$

Moreover, there exists $m_i \in \mathbb{N} \cup \{\infty\}$ such that $\frac{1}{2^{m_i}} \leq \delta(y)$ and

$$\Psi(y)(a_{i,m_i}) \geq \sup\{h(y)(a_{i,p}) : \frac{1}{2^p} \leq \delta(y)\}.$$

Thus $\downarrow h(y) \subset B_d(\downarrow\Psi(y)), \delta(y))$. Similarly, for every $i \in \mathbb{N}$,

$$\sup\{\Psi(y)(a_{i,j}) : \frac{1}{2^j} \leq \delta(y)\} \leq \sup\{h(y)(a_{i,j}) : \frac{1}{2^j} \leq \delta(y)\} + \delta(y).$$

Hence $\downarrow\Psi(y) \subset B_d(\downarrow h(y), \delta(y))$. Then

$$\rho(\downarrow\Psi(y), \downarrow h(y)) \leq d_H(\downarrow\Psi(y), \downarrow h(y)) \leq \delta(y)$$

for every $y \in Y$. It follows from $\rho(\downarrow h(y), \downarrow\Phi(y)) \leq \delta(y)$ that

$$\rho(\downarrow\Psi(y), \downarrow\Phi(y)) \leq 2\delta(y) < \varepsilon$$

for every $y \in Y$

Finally, $\downarrow\Psi$ is a Z-embedding.

In fact, similarly to [11, Lemma 9], by the continuity of Φ, L, g, δ, B, the map $\downarrow\Psi : Y \to A(S_{\mathbb{N}})$ is continuous. By the definition of Ψ and g being embedding, using the same method as the proof of [11, Lemma 9], we can prove that Ψ is one-to-one. For every $y \in Y$, $\Psi(y)$ either is in a Z-set $\Phi(K)$ in $A(S_{\mathbb{N}})$ or there exists a point $a_{i,j} \in M$ such that $\Psi(y)(a_{i,j}) = 0$. It follows from [8, Lemma 5], $\Psi(Y)$ is a Z-set in $A(S_{\mathbb{N}})$. Hence $\downarrow\Psi$ is a Z-embedding. □

Proposition 2. *For the space X, we have that $\downarrow\mathrm{C}_F(X)$ is strongly $\mathcal{F}_{\sigma\delta}$-universal in $A(X)$ if X is also a non-discrete k-space and* $\mathrm{cl}\, X_0 = X$.

Proof. Case A. $|X \setminus X_0| = 1$. Then, by Proposition 1, we know that either X is metrizable or $X \approx S_{\mathbb{N}}$. If the former holds, by Theorem 1, $(A(X) = \downarrow\mathrm{USC}_F(X), \downarrow\mathrm{C}_F(X)) \approx (\downarrow\mathrm{USC}_F(S), \downarrow\mathrm{C}_F(S))$, where S is the sequencing sequence with its limit. Using [15, Proposition 1], we have that $\downarrow\mathrm{C}_F(S)$ is strongly $\mathcal{F}_{\sigma\delta}$-universal in $\downarrow\mathrm{USC}_F(S)$. It follows that $\downarrow\mathrm{C}_F(X)$ is strongly $\mathcal{F}_{\sigma\delta}$-universal in $A(X)$. If the latter holds, using Lemma 7, $\downarrow\mathrm{C}_F(X)$ is strongly $\mathcal{F}_{\sigma\delta}$-universal in $A(X)$.

Case B. $|X \setminus X_0| \geq 2$. In [9, Theorem 7], the authors proved that $\downarrow\mathrm{C}_n(X)$ is strongly $\mathcal{F}_{\sigma\delta}$-universal in $B_n(X)$. Using the same method, we can show that $\downarrow\mathrm{C}_F(X)$ is strongly $\mathcal{F}_{\sigma\delta}$-universal in $A(X)$. But, we have been informed by the reviewer of this present paper that there are two gaps in Proof of [9, Theorem 7]. In here, we only amend the two gaps in the following:

The first gap is at the beginning of Step two of Part I, in Proof of Theorem 7. Here, we stated in the paper that "Using Lemma 5 to $\Psi_1 : Y \setminus K \to \mathrm{C}(M), (u,i)$ for $i \geq n+1$ and $\frac{\delta(y)}{2}$, there exists a continuous map $\gamma_0 : Y \setminus K \to (0,\mu)$ such that

$$\Psi_1(y)(m) < \frac{\delta(y)}{2} \text{ for every } m \in B_{d_w}((u,i),\gamma_0(y)).\text{"}$$

The problem is that for different points (u,i), using Lemma 5, we only obtained different continuous maps γ_0's rather than a single one. For this problem, we could fix it by using Lemma 5 to choose a continuous map γ_0 for the specified point $(u, n+1)$ and space $K_{n+1} \times \{n+1\}$ such that

$$\Psi_1(y)(m) < \frac{\delta(y)}{2} \text{ for every } m \in B_{d_w}((u,n+1),\gamma_0(y)).$$

This γ_0 could work for the points (u,i) and the spaces $K_i \times \{i\}$ for all $i \geq n+1$ due to the following facts:

(1) In Step two we only change the value of $\Psi_1(y)$ on some of the isolated points $\{r^{-1}(u_i)\}$;

(2) $\Psi_1(y) \in r^{\#}(\mathrm{C}(X))$ for $y \in Y \setminus K$, and

(3) the sequence of isolated points $\{u_i\}$'s are all located in K_{n+1}, and therefore it holds that $\{(u_i,k)\}_i \subset K_k \times \{k\}$ for $k \geq n+1$.

The second gap is that the map $\Psi_2(y)$, which was defined on the top of page 128, has some continuity problems. We believe this mistake could be fixed by the following adjustments. In fact, the following changes of the functions g and Ψ_2 could solve this problem.

At first, we construct an imbedding $g_1 : Y \to c_1$ satisfying that for every pair of distinct $y_1, y_2 \in Y$ and every $N \in \mathbb{N}$, there exists $k \geq N$, such that $g_1(y_1)(k) \neq g_1(y_2)(k)$. In fact, let $j : Y \to Q_u$ be an imbedding. Then $J : Y \to Q_u^{\mathbb{N}}$, where $J(y) = (j(y), j(y), \cdots) \in Q_u^{\mathbb{N}} \approx Q_u$, is also an imbedding. Trivially, we can define two homeomorphisms $\phi : Q_u^{\mathbb{N}} \to Q_u$ and $\psi : Q_u \to [0,1] \times [\frac{1}{2},1] \times [\frac{2}{3},1] \times \cdots$ such that $g_1 = \psi \circ \phi \circ J : Y \to [0,1] \times [\frac{1}{2},1] \times [\frac{2}{3},1] \times \cdots \subset c_1$ satisfies our requirements.

Secondly, let $g_2 : Y \to Q_u$ be a Z-imbedding with $g_2^{-1}(c_1) = C$ which is the same as the original g in [9].

Thirdly, we define a new Z-imbedding $g : Y \to Q_u$ as follows

$$g(y)(k) = \begin{cases} g_1(y)(\frac{1+k}{2}) & k \text{ is odd,} \\ g_2(y)(\frac{k}{2}) & k \text{ is even.} \end{cases}$$

It is easy to see that g is an imbedding and $g^{-1}(c_1) = C$. Moreover, for every pair of distinct points $y_1, y_2 \in Y$ and every $N \in \mathbb{N}$, there exists $k \geq N$, such that $g(y_1)(k) \neq g(y_2)(k)$.

Fourthly, we can use the aforementioned map g to give a new definition of $\Psi_2(y)$ as follows. For $y \in Y \setminus K$ and $(x,i) \in M$, put

$$\Psi_2(y)(x,i) = \begin{cases} \Psi_1(y)(x,i) & x \in X \setminus \{u_k : k \geq n(y)\}, \\ \Psi_1(y)(x,i)\frac{2d_w(x,u)-\gamma(y)}{\gamma(y)} + (1 - \frac{2d_w(x,u)-\gamma(y)}{\gamma(y)})g(y)(k)\frac{\delta(y)}{4} & x = u_k \text{ and } n(y) \leq k < m(y), \\ g(y)(k)\frac{\delta(y)}{4} & x = u_k \text{ and } k \geq m(y). \end{cases}$$

Then this new constructed map $\Psi_2(y)$ could work well as what we had expected. □

5. Proof of The Main Theorem

At first, we have.

Lemma 8. *For the space X, we have that $\downarrow\mathrm{C}_F(X)$ is an AR.*

Proof. It follows from Lemma 4 and [1, 1.2.1 Proposition] that $\downarrow\mathrm{C}_F(X)$ is an AR. □

Lemma 9. *Suppose the space X is also a k-space and $\mathrm{cl}\,X_0 = X$, $x_\infty \in X \setminus X_0$ and $\{x_k\} \subset X_0$ with $x_k \to x_\infty$. Let*

$$S = \{x_n : n = 1, 2, \cdots, \infty\} \quad \textit{and} \quad S_0 = S \setminus \{x_\infty\},$$

$$Y = \begin{cases} X \setminus S_0 & \textit{if } x_\infty \in \mathrm{cl}(X \setminus S), \\ X \setminus S & \textit{if } x_\infty \notin \mathrm{cl}(X \setminus S). \end{cases}$$

Then Y is a weakly locally compact and hemicompact $\aleph_0$-space and a k-space. Thus, $\downarrow\mathrm{C}_F(Y)$ is an AR.

Proof. We have only to verify that Y is weakly locally compact since Y is closed in X in the both cases.

Case A. If $x_\infty \notin \mathrm{cl}(X \setminus S)$, then, Y is also open in X and hence is weakly locally compact.

Case B. Otherwise. Then $Y \setminus \{x_\infty\}$ is not closed in X. At first, we show that there exists a sequence in $Y \cap X_0$ converging to x_∞. By [12, Lemma 3], there exists a k-network $\{K_n\}$ for X consisting of compact regular closed sets. Since X is a k-space, there exists n such that $(Y \setminus \{x_\infty\}) \cap K_n$ is not closed in K_n. Note that K_n is metrizable and regular closed in X, $\mathrm{cl}\,X_0 = X$ and $x_k \to x_\infty$. It follows that there exists a sequence $\{z_k\} \subset Y \cap X_0 \cap K_n$ such that $z_k \to x_\infty$. Now, let C be a compact set in Y. Then there exists an open set U in X such that $C \subset \mathrm{cl}\,U$ and $\mathrm{cl}\,U$ is compact. Then

$$V = (U \cap Y) \cup \{z_k\}$$

is open in Y and

$$C \subset \mathrm{cl}_Y V = \mathrm{cl}_X V \subset \mathrm{cl}\,U \cup \{z_k : k = 1, 2, \cdots\} \cup \{x_\infty\}.$$

Moreover, $\mathrm{cl}\,U \cup \{z_k : k = 1, 2, \cdots\} \cup \{x_\infty\}$ is compact and hence $\mathrm{cl}_Y V$ is compact. This shows that Y is weakly locally compact. □

We need a technical tool.

Lemma 10. *Suppose A is a regular closed subspace of a space Y. Then the map $\downarrow f \mapsto \downarrow f|A$ is continuous from $\downarrow\mathrm{C}_F(Y)$ to $\downarrow\mathrm{C}_F(A)$. Conversely, the map $(\downarrow f, \downarrow g) \mapsto \downarrow(f \cup g)$ is also continuous from*

$$\mathcal{A} = \{(\downarrow f, \downarrow g) \in \downarrow\mathrm{C}(A) \times \downarrow\mathrm{C}(B) : f|A \cap B = g|A \cap B\} \subset \downarrow\mathrm{C}_F(A) \times \downarrow\mathrm{C}_F(B)$$

to $\downarrow\mathrm{C}_F(Y)$ if Y is a union of two closed subspaces A, B of it. Therefore, the map $\downarrow f \mapsto (\downarrow f|A, \downarrow f|B)$ is an imbedding from $\downarrow\mathrm{C}_F(Y)$ to $\downarrow\mathrm{C}_F(A) \times \downarrow\mathrm{C}_F(B)$ if Y is a union of two regular closed subspaces A, B of it.

Proof. In [13, Lemma 2.2], the authors have proved that the first statement in the present lemma holds in the case that A is a compact regular closed subspace. That proof is valid for here. In fact, the compactness of A was used in [13, Lemma 2.2] so that the map $\downarrow f \mapsto \downarrow f|A$ is a surjection from $\downarrow\mathrm{C}_F(Y)$ to $\downarrow\mathrm{C}_F(A)$.

Note that $\cup : \mathrm{Cld}_F(Y\times \mathbf{I})\times \mathrm{Cld}_F(Y\times \mathbf{I}) \to \mathrm{Cld}_F(Y\times \mathbf{I})$ is continuous. It follows that the map $(\downarrow f, \downarrow g) \mapsto \downarrow(f\cup g) = \downarrow f \cup \downarrow g$ is also continuous from $\mathcal{A}$ to $\downarrow\mathrm{C}_F(Y)$ if Y is a union of two closed subspaces A, B of it.

The third statement is a combination of the first two ones. □

Lemma 11. *Suppose the space X is also a non-discrete k-space and $\mathrm{cl}\, X_0 = X$. We have that $\downarrow\mathrm{C}_F(X)$ satisfies SDAP.*

Proof. By Lemma 6, we can choose a point $x_\infty \in X \setminus X_0$ and a sequence $\{x_n\}$ in X_0 such that $x_n \to x_\infty$. Define S, S_0 and Y as the same as in Lemma 9. Then, by Lemma 9, $\downarrow\mathrm{C}_F(Y)$ is an AR. It follows from Theorem 1 that

$$(Q\setminus\Sigma)\cup c_0 \approx \downarrow\mathrm{C}_F(S) \approx \downarrow\mathrm{C}_F(\bigoplus_{i\in\mathbb{N}} S_i) \approx \prod_{i\in\mathbb{N}} \downarrow\mathrm{C}_F(S_i) \approx ((Q\setminus\Sigma)\cup c_0)^{\mathbb{N}},$$

where $S_i \approx S$, which shows that $(Q\setminus\Sigma)\cup c_0$ satisfies SDAP.

If we identify $f \in \mathrm{C}(X)$ with the pair $(f|S, f|Y)$, by Lemma 10, $\downarrow\mathrm{C}_F(X)$ can be as a subspace of $\downarrow\mathrm{C}_F(S)\times\downarrow\mathrm{C}_F(Y)$. Moreover, we verify that $\downarrow\mathrm{C}_F(X)$ is homotopy dense in $\downarrow\mathrm{C}_F(S)\times\downarrow\mathrm{C}_F(Y)$. In fact, if $x_\infty \notin \mathrm{cl}(X\setminus S)$, then $\downarrow\mathrm{C}_F(X) = \downarrow\mathrm{C}_F(S)\times\downarrow\mathrm{C}_F(Y)$. If $x_\infty \in \mathrm{cl}(X\setminus S)$, then $\downarrow\mathrm{C}(X) \supset \downarrow\mathrm{C}_0(S)\times\downarrow\mathrm{C}_0(Y)$ and, by Lemmas 6 and 9, $\downarrow\mathrm{C}_0(S)$, $\downarrow\mathrm{C}_0(Y)$ are homotopy dense subspaces of the spaces $\downarrow\mathrm{C}_F(S)$, $\downarrow\mathrm{C}_F(Y)$, respectively.

Hence, using results in [1, P25], we have that $\downarrow\mathrm{C}_F(X)$ satisfies SDAP. □

Lemma 12. *Suppose the space X is also a non-discrete k-space and $\mathrm{cl}\, X_0 = X$. Then $\downarrow\mathrm{C}_F(X)$ is an $\mathcal{F}_{\sigma\delta}$-coabsorbing AR.*

Proof. (i). The space $\downarrow\mathrm{C}_F(X)$ is a strongly $\mathcal{F}_{\sigma\delta}$-universal AR satisfying SDAP. Using Lemmas 1, 8 and 11 and Proposition 2, $\downarrow\mathrm{C}_F(X)$ is a strongly-$\mathcal{F}_{\sigma\delta}$-universal AR satisfying SDAP.

(ii) Every Z-set in $\downarrow\mathrm{C}_F(X)$ is in $\mathcal{F}_{\sigma\delta}$. By Lemma 5, $\downarrow\mathrm{C}_F(X) \in \mathcal{F}_{\sigma\delta}$. It follows that every Z-set in $\downarrow\mathrm{C}_F(X)$ is in $\mathcal{F}_{\sigma\delta}$.

(iii) The space $\downarrow\mathrm{C}_F(X)$ is a co-Z_σ-space. Lemma 6 shows that the subspace $\downarrow\mathrm{C}_0(X)$ of $r^{\#}(\downarrow\mathrm{C}(X)) \approx \downarrow\mathrm{C}_F(X)$ is a homotopy dense absolute G_δ-subset. □

Proof of The Main Theorem. It follows immediately from Lemmas 2 and 12 and Theorem 1. □

As a corollary of The Main Theorem, we have the following result.

Theorem 5. *The space $c_0 \cup (Q\setminus\Sigma)$ is an $\mathcal{F}_{\sigma\delta}$-coabsorbing.*

6. Remarks and open problems

In [12], it was proved that there exist many non-k-spaces X such that the function spaces $\downarrow\mathrm{C}_F(X)$ are metrizable. In fact, for every $p \in \mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$, we can define a strictly coarser Tychonoff topology $\mathcal{T}_p$ on the set $S_\mathbb{N}$ than the topology $\mathcal{T}$ of the sequential fan space $S_\mathbb{N}$ such that $\downarrow\mathrm{C}_F(S_\mathbb{N}, \mathcal{T}_p)$ is a subspace of the separable metrizable space $\downarrow\mathrm{C}_F(S_\mathbb{N}, \mathcal{T})$. Moreover, it is not hard to verify that $\downarrow\mathrm{C}_F(S_\mathbb{N}, \mathcal{T}_p) \neq \downarrow\mathrm{C}_F(S_\mathbb{N}, \mathcal{T}_q)$ if $p \neq q$. Hence

$$|\{\downarrow\mathrm{C}_F(S_\mathbb{N}, \mathcal{T}_p) : p \in \mathbb{N}^*\}| = 2^c.$$

Using Lavrentieff's Theorem (cf. [2]), it can be proved[2] that, for every subspace Y_0 of a separable metrizable space Y,

$$|\{Z \subset Y : Z \approx Y_0\}| \leq c.$$

It follows that there are 2^c topologically different function spaces $\downarrow\mathrm{C}_F(S_N, \mathcal{T}_p)$ when p runs in $\mathbb{N}^*$. On the other hand, every metrizable function space $\downarrow\mathrm{C}_F(X)$ is separable, see [7]. Therefore, we have

Theorem 6. *There are exactly 2^c topologically different metrizable function spaces $\downarrow\mathrm{C}_F(X)$.*

In [12], an open problem giving a topological classification of metrizable function spaces $\downarrow\mathrm{C}_F(X)$ was put. Theorem 6 shows that it seem to be impossible to completely solve this problem. But, we can consider the following problems:

Open Problem 1. To give a relation between spaces $\downarrow\mathrm{C}_F(S_{\mathbb{N}}, \mathcal{T}_p)$ and $\mathrm{C}(\mathbb{N} \cup \{p\})$ with the pointwise converging topology.

Open Problem 2. If $\downarrow\mathrm{C}_F(X) \in \mathcal{F}_{\sigma\delta}$, then X is a k-space. Is it true?

If the answer of the above problem is no, then we can consider the following problem:

Open Problem 3. Is it true that $\downarrow\mathrm{C}_F(X) \approx c_0$ or $\downarrow\mathrm{C}_F(X) \approx (Q \setminus \Sigma) \cup c_0$ if $\downarrow\mathrm{C}_F(X) \in \mathcal{F}_{\sigma\delta}$ and X is non-discrete?

Trivially, if X is discrete, then $\downarrow\mathrm{C}_F(X)$ does not satisfy SDAP. Thus, the following is interesting:

Open Problem 4. Does $\downarrow\mathrm{C}_F(X)$ satisfy SDAP if $\downarrow\mathrm{C}_F(X)$ is metrizable and X is non-discrete?

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