Topological classification of function spaces with the Fell topology II✩

Zhongqiang Yang ∗, Yanmei Zheng, Jiyang Chen

Department of Mathematics, Shantou University, Shantou, Guangdong, 515063, PR China

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**ABSTRACT**

For a Tychonoff space X, let ↓C_F(X) denote the collection of the hypographs of all continuous maps from X to [0, 1] with the Fell topology. We show that for a Tychonoff k-space X, ↓C_F(X) is homeomorphic to c_0 if and only if ↓C_F(X) is metrizable and not Baire if and only if X is a weakly locally compact and hemicompact ω-space without dense set of isolated points, where Q = [−1, 1]^ω is the Hilbert cube, Σ = {(x_n) ∈ Q : sup |x_n| < 1} and c_0 = {(x_n) ∈ Σ : lim x_n = 0} are the subspaces of it.

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All spaces are assumed to be Tychonoff topological spaces.

1. Introduction and the Main Theorem

For a space X, let USC(X) and C(X) be the sets of all upper semi-continuous maps and of all continuous maps from X to I = [0, 1], respectively. For every f ∈ USC(X), define

\[ \downarrow f = \{(x, t) ∈ X × I : t \leq f(x)\}. \]

The closed set \( \downarrow f \) in the product space \( X × I \) is called the hypograph of f. Let

\[ \downarrow USC(X) = \{\downarrow f : f ∈ USC(X)\}, \quad \downarrow C(X) = \{\downarrow f : f ∈ C(X)\}. \]

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* Corresponding author.

E-mail addresses: zqyang@stu.edu.cn (Z. Yang), ymzheng@foxmail.com (Y. Zheng), 317205104@qq.com (J. Chen).

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Then $\downarrow C(X) \subset \downarrow USC(X) \subset \text{Cld}(X \times I)$, where $\text{Cld}(X \times I)$ consists of all non-empty closed sets in the product space $X \times I$. For a space $Y$, we recall that the Fell topology on the family $C(Y)$ of all non-empty closed sets in $Y$ is generated by

$$\{K^*, U^- : K \text{ and } U \text{ are compact and open, respectively in } Y\}$$

as a subbase, where

$$K^* = \{F \in C(Y) : F \cap K = \emptyset\}, \quad U^- = \{F \in C(Y) : F \cap U \neq \emptyset\}.$$

We use $\text{Cld}_F(Y)$ to denote $C(Y)$ with the Fell topology. And we use $\downarrow \text{USC}_F(X)$ and $\downarrow C_F(X)$ to denote the sets $\downarrow \text{USC}(X)$ and $\downarrow C(X)$ with the subspace topology of the space $\text{Cld}_F(X \times I)$. Similarly, we can define the space $\uparrow C_F(X)$. The topologies on $\downarrow C_F(X)$ and $\uparrow C_F(X)$ are called the Fell hypograph topology and Fell hypograph topology in the set $C(X)$, respectively. Note that the compact-open topology on the function $C(X)$ is the least upper bound of those two topologies on $C(X)$.

As usual, $\text{cly}$ and $\text{inty}$ denote the closure operator and interior operator in a space $Y$. We shall omit the subscript if no confusion occurs. Let $\mathbb{N}$ and $\mathbb{Q}$ denote the sets of all natural numbers and all rational numbers, respectively.

Let $Q = [-1, 1]^\mathbb{N}$ be the Hilbert cube. Define its two subspaces as follows

$$\Sigma = \{(x_n) \in Q : \sup |x_n| < 1\} \quad \text{and} \quad c_0 = \{(x_n) \in \Sigma : \lim_{n \to \infty} x_n = 0\}.$$

For two spaces $X$, $Y$ and their subspaces $A$, $B$, respectively, the symbol $(X, A) \approx (Y, B)$ means that there exists a homeomorphism $h : X \to Y$ such that $h(A) = B$. As the same as, the symbol $X \approx Y$ means that $X$ and $Y$ are homeomorphic.

In [14–19], the authors gave the topological classification for all metrizable function spaces $\downarrow C_F(X)$ under the condition that $X$ is metrizable. That is,

**Theorem 1.** For a metrizable space $X$, the function space $\downarrow C_F(X)$ is metrizable if and only if $X$ is locally compact and separable. Moreover, then

$$\downarrow \text{USC}_F(X), \downarrow C_F(X) \approx \begin{cases} (I^{1|X|}, I^{1|X|}) & \text{if } X \text{ is discrete;} \\ (Q, c_0) & \text{if } \text{cl } X_0 \neq X; \\ (Q, c_0 \cup (Q \setminus \Sigma)) & \text{otherwise}, \end{cases}$$

where $X_0$ is the set of all isolated points in $X$.

But, there exist many non-metrizable spaces $X$ such that $\downarrow C_F(X)$ are metrizable. See [7,13,17]. Hence, in [17], we put the following problem:

**Main Problem.** Give the topological classification of metrizable function spaces $\downarrow C_F(X)$ for all Tychonoff spaces $X$.

In fact, for non-metrizable spaces $X$, the problem on the topological classification of metrizable function spaces $\downarrow C_F(X)$ is much more complicated than the one for metrizable spaces $X$.

In this present paper, we give a partial answer for the above problem. Some concepts occurred in the Main Theorem will be defined in the next section. Our main result is
Main Theorem. For a Tychonoff k-space \( X \), the following statements are equivalent:

(a) \( \downarrow C_F(X) \approx c_0 \);
(b) \( \downarrow C_F(X) \) is metrizable but not Baire;
(c) \( X \) is a weakly locally compact and hemicompact \( \aleph_0 \)-space without dense set of isolated points.

Remark 1. In \([17]\), the authors gave examples to show that there exist non-metrizable k-spaces \( X \) such that \( \downarrow C_F(X) \) are metrizable. Hence, the above theorem is not a corollary of Theorem 1. On the other hand, there exist non-k-spaces \( X \) such that \( \downarrow C_F(X) \) are metrizable, see \([17]\) or Example 2 in the present paper.

2. Preliminaries

In this section, we at first give some concepts and facts on general topology and infinite-dimension topology. For more information, we refer to \([5,9,10]\). Secondly, we shall recall some results of \([7]\) and \([17]\).

For a metric space \((M, d)\), a closed subset \( A \) is called a Z-set if for \( \varepsilon > 0 \) there is a continuous map \( f : M \to M \setminus A \) with \( d(f(x), x) < \varepsilon \) for every \( x \in M \). A \( Z_\alpha \)-set in a space is a subset of the space which is a countable union of Z-sets in the space. A space is called a \( Z_\alpha \)-space if it is a \( Z_\alpha \)-set in itself. A Z-embedding is an embedding with a Z-set image.

A subset \( A \) of a space \( Y \) is called homotopy dense if there exists a homotopy \( H : Y \times I \to Y \) such that \( H(y, 0) = y \) and \( H(y, t) \in A \) for every \( y \in Y \) and \( t \in (0, 1] \).

Let \( \mathcal{M}_0 \) denote the class of compacta, and for a topological class \( \mathcal{C} \), let \( (\mathcal{M}_0, \mathcal{C}) \) denote the class of pairs \((Z, C)\) such that \( Z \in \mathcal{M}_0 \) and \( C \in \mathcal{C} \) with \( Z \supset C \).

Definition 1. Let \((M, d)\) be a copy of the Hilbert cube \( Q \). We say that a subspace \( Z \) of \( M \) is strongly \( \mathcal{C} \)-universal in \( M \) provided that for every \((Y, C) \in (\mathcal{M}_0, \mathcal{C})\), every continuous map \( f : Y \to M \), every closed subset \( K \) of \( Y \) such that \( f|K : K \to M \) is a Z-embedding and for every \( \varepsilon > 0 \), there is a Z-embedding \( g : Y \to M \) such that \( g|K = f|K \), \( g^{-1}(Z) \setminus K = C \setminus K \) and \( d(g(y), f(y)) \leq \varepsilon \) for \( y \in Y \).

Definition 2. Let \( M \) be a copy of the Hilbert cube \( Q \). We say that a subspace \( Z \) of \( M \) is called a \( \mathcal{C} \)-absorber in \( M \) if

(i) \( Z \in \mathcal{C} \);
(ii) \( Z \) is contained in a \( Z_\alpha \)-set of \( M \);
(iii) \( Z \) is strongly \( \mathcal{C} \)-universal in \( M \).

The following result is a main tool to show that two spaces are homeomorphic.

Theorem 2. ([11]) If \( Y \) and \( Z \) are \( \mathcal{C} \)-absorbers in a copy \( M \) of the Hilbert cube \( Q \), then \((M, Y) \approx (M, Z)\).

A metrizable space \( Y \) is called an absolute \( F_{\sigma\delta} \)-space if it is an \( F_{\sigma\delta} \)-set in any metrizable space which contains \( Y \) as a subspace. Let \( F_{\sigma\delta} \) be the class of all absolute \( F_{\sigma\delta} \)-spaces. In \([4]\), cf. \([15]\), it was proved that \( c_0 \) is an \( F_{\sigma\delta} \)-absorber in \( Q \).

A space \( X \) is called a k-space provided for a subset \( A \) of \( X \), \( A \) is closed in \( X \) if and only if \( A \cap C \) is closed in \( C \) for every compact subspace \( C \) of \( X \). It is well-known that a space is a k-space if and only if it is a quotient space of a locally compact space, see \([5, \text{Theorem 3.3.18}]\).

A space \( X \) is called weakly locally compact if for every compact set \( K \) in \( X \) there exists an open set \( U \) in \( X \) such that \( K \subset \text{cl } U \) and \( \text{cl } U \) is compact. Trivially, every locally compact space is weakly locally compact, but the converse is not true. Indeed, the quotient space of the disjoint sum of infinitely many real lines by collapsing the set of all 0’s to a point is weakly locally compact but not locally compact.
A space $X$ is said to be hemicompact if there exists a countable family $\mathcal{K}$ of compact sets in $X$ such that every compact set in $X$ is included in some element of $\mathcal{K}$.

A space is called an $\aleph_0$-space if there exists a countable $k$-network, where a family $\mathcal{K}$ of sets in a space $X$ is called a $k$-network if, for every compact set $C$ and every open set $U$ in $X$ with $U \supset C$, there exists $K \in \mathcal{K}$ such that $C \subset K \subset U$. A space $(X, T)$ is called submetrizable if $T$ is stronger than a metrizable topology in the set $X$. The reviewer of this paper pointed out that, for a hemicompact space $X$, $X$ is an $\aleph_0$-space if and only if it is submetrizable. Professor S. Lin helped the authors to affirm this fact (which follows from [6, Corollary 2.9 and Theorem 2.13]).

A continuous map $r : Y \to X$ is called compact-covering if for every compact set $C$ in $X$, there exists a compact set $D$ in $Y$ such that $r(D) = C$, or, equivalently, $r(D) \supset C$. Note that a space is a $\aleph_0$-space if and only if it is an images of a separable metrizable space under compact-covering map, see [8]. A continuous map $r : Y \to X$ is called weakly open if for every non-empty open set $U$ in $Y$, $\text{int} \, r(U)$ is non-empty in $X$.

In [7], cf. [17], McCoy and Ntantu gave a sufficient and necessary condition for $\downarrow C_F(X)$ being metrizable:

**Theorem 3.** The space $\downarrow C_F(X)$ is metrizable if and only if $X$ is a weakly locally compact and hemicompact $\aleph_0$-space.

Moreover, by [17], if $X$ is a weakly locally compact and hemicompact $\aleph_0$-space, then we can constructed a countable family $\{K_n : n = 1, 2, \ldots\}$ of sets in $X$ satisfying the following properties: For every $n = 1, 2, \ldots$,

(i) $K_n$ is a compact metrizable set in $X$;
(ii) $K_n \subseteq K_{n+1}$;
(iii) every compact set in $X$ is included in some $K_n$;
(iv) $\text{cl}_{X} \text{int}_{X} K_n = K_n$, that is, $K_n$ is regular closed in $X$. Hence every non-empty open set in $K_n$ has a non-empty interior in the whole space $X$.

Furthermore, if $\text{cl} \, X_0 \neq X$, where $X_0$ is the set of all isolated points in $X$, then there exists a non-isolated point $x_{\infty} \in X$ and an open neighborhood $W$ of $x_{\infty}$ in $X$ such that $\text{cl} \, W$ is compact and $\text{cl} \, W \cap X_0 = \emptyset$, see [13]. Hence, without loss of generality, we assume

(v) $\text{cl} \, W \subseteq K_1$ if $\text{cl} \, X_0 \neq X$.

Let

$$M = \bigoplus_{n=1}^{\infty} K_n \times \{n\}.$$  

Then $M$ is a locally compact separable metrizable space. Define a natural homeomorphism $j_n : K_n \times \{n\} \to K_n$ as $j_n(x, n) = x$ and let $r = \bigoplus_{n=1}^{\infty} j_n : M \to X$. Then $r$ is compact-covering and weakly open. Define $r^# : \downarrow C_F(X) \to \downarrow C_F(M)$ by

$$r^#(\downarrow f) = \downarrow (f \circ r)$$  \hspace{1cm} (1)

for $f \in C(X)$. Let

$$M^Q = \text{cl}_{\mathcal{I}_{\text{USC}}(M)} r^#(\downarrow C_F(X)).$$

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1 In literatures, a family $\mathcal{K}$ of sets in a space $X$ is called a $k$-network if, for every compact set $C$ and every open set $U$ in $X$ with $U \supset C$, there exists a finite subfamily $\mathcal{K}_0$ of $\mathcal{K}$ such that $C \subset \bigcup \mathcal{K}_0 \subset U$. Such difference does not make any trouble.
In [17], the following theorem has been proved:

**Theorem 4 (The Embedding Theorem).** For every weakly locally compact and hemicompact infinite $\aleph_0$-space $X$, we have that

1. the map $r^\#: \downarrow C_F(X) \rightarrow \downarrow C_F(M)$ is an embedding;
2. $M^Q \approx Q$;
3. $r^\#(\downarrow C_F(X))$ is a homotopy dense set in $M^Q$.

In the remainder of this paper, we always assume that $X$ is a weakly locally compact and hemicompact infinite $\aleph_0$-space and fix the above symbols, for example, the sequence $\{K_n\}$ and the metrizable space $M$.

### 3. Revisions of the Embedding Theorem

To prove the Main Theorem, in this section, we give an extension of Theorem 4. At first, we note that the map $r^\#: \downarrow \text{USC}_F(X) \rightarrow \downarrow \text{USC}_F(M)$ is not continuous though we can define a map $r^\#: \downarrow \text{USC}(X) \rightarrow \downarrow \text{USC}(M)$ using Formula (1). See Example 1 below.

**Example 1.** Let $M = \mathbb{I} \times \{1\} \oplus \mathbb{I} \times \{2\}$ and $X$ be the quotient space of $M$ by adjoining $(0,1), (0,2)$ to a point $x_0$, then the natural map $r: M \rightarrow X$ is a weakly open and compact-covering map. Define an upper semi-continuous map $f: X \rightarrow \mathbb{I}$ by

$$f(x) = \begin{cases} 
 1 & \text{if } x = x_0; \\
 0 & \text{otherwise.}
\end{cases}$$

Then $r^\#: \downarrow \text{USC}_F(X) \rightarrow \downarrow \text{USC}_F(M)$ is not continuous at $\downarrow f$.

In fact, define an upper semi-continuous map $g: M \rightarrow \mathbb{I}$ by

$$g(m) = \begin{cases} 
 1 & \text{if } m = (0,1); \\
 0 & \text{otherwise.}
\end{cases}$$

For every natural number $n$, define a continuous map $f_n: X \rightarrow \mathbb{I}$ by

$$f_n(y,i) = \begin{cases} 
 ny & \text{if } 0 < y \leq \frac{1}{n}, \quad i = 1; \\
 2 - ny & \text{if } \frac{1}{n} \leq y \leq \frac{2}{n}, \quad i = 1; \\
 0 & \text{otherwise.}
\end{cases}$$

Then $\lim f_n = \downarrow f$ in $\downarrow \text{USC}_F(X)$. But, $\lim r^\#(\downarrow f_n) = \downarrow g \neq \downarrow r^\#(\downarrow f)$ in $\downarrow \text{USC}_F(M)$. In fact, no upper semi-continuous map $h \in \text{USC}(X)$ satisfies $g = h \circ r$ since $g(0,1) \neq g(0,2)$. That is, $\downarrow g \notin r^\#(\downarrow \text{USC}_F(X))$.

Secondly, we give a generalization of [18, Lemma 2.2].

**Lemma 1.** Let $Y$ be a Tychonoff space and $U$ an open set in $Y$. Define

$$\mathcal{A} = \{f \in \text{USC}(Y): f \text{ is continuous at every point in } \text{cl} U \setminus U\}$$

and a map $k: \mathcal{A} \rightarrow \text{USC}(\text{cl} U)$ as

$$k(f) = f|\text{cl} U.$$
Then

(1) the map \( \downarrow k : \downarrow A \to \downarrow \text{USC}_F(\text{cl} U) \) is continuous;
(2) \( \downarrow k : \downarrow A \to \downarrow k(\downarrow A) \) is open if \( \text{cl} U \) is compact.

**Proof.** The proof of (1) is similar to the one of [18, Lemma 2.2] in which it was proved that, for every open set \( U \) in \( Y \), if \( \text{cl} U \) is compact, then \( \downarrow f \mapsto \downarrow f|\text{cl} U \) is open and continuous from \( \downarrow C_F(Y) \) onto \( \downarrow C_F(\text{cl} U) \). In fact, in the proof of [18, Lemma 2.2], we only use the continuity of maps at the points in \( \text{cl} U \setminus U \). Moreover, \( \downarrow k : \downarrow A \to \downarrow k(\downarrow A) \) is open if \( \text{cl} U \) is compact, see [18, Lemma 2.2]. □

Now we are in a position to give the main result in this section.

**Proposition 1.** Let \( X \) be a weakly locally compact and hemicompact \( K_0 \)-space. Set

\[
\mathcal{K} = \{ f \in \text{USC}(X) : f \text{ is continuous on } \bigcup_{n=1}^{\infty} K_n \setminus \text{int} K_n \}.
\]

Then \( r^\# : \downarrow \mathcal{K} \to \downarrow \text{USC}_F(M) \) is an embedding. Hence \( r^\#(\downarrow \mathcal{K}) \subset M^Q \) and \( \downarrow \mathcal{K} \) is separable metrizable.

**Proof.** It is trivial that \( r^\# : \downarrow \mathcal{K} \to \downarrow \text{USC}_F(M) \) is one-to-one. We have defined a natural homeomorphism \( j_n : K_n \times \{ n \} \to K_n \) as \( j_n(x, n) = x \). It follows from \( K_n = \text{cl}_X \text{int}_X K_n \) and Lemma 1 that \( \downarrow f \mapsto \downarrow (f|K_n \circ j_n) \) is continuous from \( \downarrow \mathcal{K} \) to \( \downarrow \text{USC}_F(K_n \times \{ n \}) \) for every \( n \). Hence the map \( \downarrow f \mapsto (\downarrow (f|K_1 \circ j_1), \downarrow (f|K_2 \circ j_2), \ldots) \) is continuous from \( \downarrow \mathcal{K} \) to \( \prod_{n=1}^{\infty} \text{USC}_F(K_n \times \{ n \}) \). It is not hard to verify that the map

\[
(\downarrow g_1, \downarrow g_2, \ldots) \mapsto \bigcup_{n=1}^{\infty} \downarrow g_n
\]

is a homeomorphism from \( \prod_{n=1}^{\infty} \text{USC}_F(K_n \times \{ n \}) \) onto \( \text{USC}_F(M) \). Thus, \( r^\# : \downarrow \mathcal{K} \to r^\#(\downarrow \mathcal{K}) \) is continuous since it is the composition of the above two maps. Also, \( r^\# : \downarrow \mathcal{K} \to r^\#(\downarrow \mathcal{K}) \) is open. To this end, let \( V_i (i \leq n) \) be open sets and \( C \) a compact set in \( X \times I \). We show that \( r^\#(\downarrow \mathcal{K} \cap \bigcap_{i=1}^{n} V_i^{-} \cap C^*) \) is in \( r^\#(\downarrow \mathcal{K}) \). For every \( f \in \downarrow \mathcal{K} \cap \bigcap_{i=1}^{n} V_i^{-} \cap C^* \), there exists \( t \in \mathbb{N} \) such that \( V_i \cap \downarrow f|K_i \neq \emptyset \) for every \( i \leq n \) and \( C \subset K_t \times I \).

For every \( i \leq n \), let

\[
U_i = (V_i \times \{ l \}) \cap (K_i \times \{ l \} \times I) \quad \text{and} \quad K = (C \times \{ l \}) \cap (K_t \times \{ l \} \times I).
\]

Then \( U_i \) and \( K \) are open and compact in \( K_t \times \{ l \} \times I \), respectively. Hence they are open and compact in \( M \times I \), respectively. Moreover, it is trivial to verify that

\[
r^\#(\downarrow f) \in r^\#(\downarrow \mathcal{K} \cap \bigcap_{i=1}^{n} U_i^{-} \cap K^*) \subset r^\#(\downarrow \mathcal{K} \cap \bigcap_{i=1}^{n} V_i^{-} \cap C^*).
\]

This shows that \( r^\#(\downarrow \mathcal{K} \cap \bigcap_{i=1}^{n} V_i^{-} \cap C^*) \) is open in \( r^\#(\downarrow \mathcal{K}) \). Thus, \( r^\# : \downarrow \mathcal{K} \to \downarrow \text{USC}_F(M) \) is an embedding. Notice that \( \downarrow C_F(X) \) is dense in \( \downarrow \text{USC}_F(X) \). It follows that \( r^\#(\downarrow \mathcal{K}) \subset M^Q \). Hence \( \downarrow \mathcal{K} \) is separable metrizable. □

**Remark 2.** By Example 1, the set \( \mathcal{K} \) in Proposition 1 cannot be replaced by \( \text{USC}(X) \).
At last, we define an admissible metric in the space $\downarrow K$. It is well-known that, for the locally compact and separable metrizable space $M$, there exists an admissible metric $d$ on $M$ such that a set in $M$ is compact if and only if it is closed and bounded in $(M,d)$, see [12]. Then

$$d((m,t),(m',t')) = \max\{d(m,m'),|t-t'|\}$$

is an admissible metric in the product space $M \times I$. Fix a point $m_0$ in $M$. For every $f \in USC(M)$, define

$$f^*(m) = e^{-d(m,m_0)}f(m)$$

for every $m \in M$. Then $f^* \in USC(M)$. Using it, in [18], an admissible metric on $\downarrow USC_F(M)$ was defined as follows

$$d'_H(\downarrow f, \downarrow g) = d_H(\downarrow f^*, \downarrow g^*) < +\infty$$

for every pair $f, g \in USC(M)$, where $d_H$ is the Hausdorff metric in the hyperspace $\text{Cld}(M \times I)$. Therefore, we can define an admissible metric $\rho_H$ on $\downarrow K$ as follows

$$\rho_H(\downarrow f, \downarrow g) = d'_H(r^*(\downarrow f), r^*(\downarrow g)) = d'_H(\downarrow (f \circ r), \downarrow (g \circ r))$$

for $f, g \in K$.

4. Properties of function spaces of $k$-spaces

In this section, we show that $\downarrow C_F(X)$ is in $F_{\sigma\delta}$ if $X$ is also a $k$-space.

We know that $M_0$ and $X_0$ are the sets of all isolated points in $M$ and $X$, respectively. We have the following lemma.

**Lemma 2.** The formula $M_0 = r^{-1}(X_0)$ holds and $X_0$ is dense in $X$ if and only if $M_0$ is dense in $M$.

**Proof.** Since the map $r$ is weakly open, we have that $M_0 \subset r^{-1}(X_0)$. Conversely, for every $x_0 \in X_0$, we have $r^{-1}(x_0) = \{(x_0, n) : x_0 \in K_n, n \in \mathbb{N}\}$. Moreover, for $n \in \mathbb{N}$, if $x_0 \in K_n$, then $(K_n \times \{n\}) \cap r^{-1}(x_0) = \{(x_0, n)\}$ is open in $K_n \times \{n\}$ and hence it is open in $M$. That is, $r^{-1}(x_0) \subset M_0$.

It follows from the continuity of $r$ that $X_0$ is dense in $X$ if $M_0$ is dense in $M$. On the other hand, if $M_0$ is not dense in $M$, then there exist $l \in \mathbb{N}$ and a non-empty open set $U$ in $K_l$ such that $(U \times \{l\}) \cap M_0 = \emptyset$. Then, $\text{int cl} \ r(U)$ is a non-empty open set and $\text{int cl} \ r(U) \cap X_0 = \emptyset$. Hence $X_0$ is not dense in $X$. □

We give our main result in this section.

**Proposition 2.** The following statements are equivalent:

(a) $r^*(\downarrow C_F(X))$ is closed in $\downarrow C_F(M)$;
(b) $r : M \rightarrow X$ is a quotient map;
(c) $X$ is a $k$-space.

Hence $r^*(\downarrow C_F(X))$ is an absolute $F_{\sigma\delta}$-space if $X$ is a $k$-space.

**Proof.** (b) ⇒ (c). If $r : M \rightarrow X$ is a quotient map, then $X$ is a quotient space of a locally compact space $M$. Hence, $X$ is a $k$-space.
(c) ⇒ (b). Assume that $X$ is a $k$-space. We show that $r : M \to X$ is a quotient map. To this end, for $F \subset X$, if $r^{-1}(F)$ is closed in $M$, we verify that $F$ is closed in $X$. Since $r^{-1}(F)$ is closed in $M$, $F \cap K_n = j_n(r^{-1}(F) \cap (K_n \times \{n\}))$ is closed in $K_n$ for every $n = 1, 2, \cdots$. For every compact set $A$, there exists $n$ such that $A \subset K_n$. Thus, $A \cap F = A \cap K_n \cap F$ is closed in $A$. By the definition of $k$-space, $F$ is closed in $X$.

(b) ⇒ (a). Assume that $r : M \to X$ is a quotient map. Then

$$r^\#(\downarrow C_F(X)) = \{ (\downarrow f_1, \downarrow f_2, \downarrow f_3, \cdots) \in \prod_{n=1}^{\infty} \downarrow C_F(K_n \times \{n\}) : f_{n+1}(x, n + 1) = f_n(x, n) \text{ for } n \in \mathbb{N} \text{ and } x \in K_n \}. $$

Now we claim that $r^\#(\downarrow C_F(X))$ is closed in $\downarrow C_F(M)$. To this end, we show that, for every

$$\downarrow f = (\downarrow f_1, \downarrow f_2, \downarrow f_3, \cdots) \in \downarrow C_F(M) \setminus r^\#(\downarrow C_F(X)), $$

there exists a neighborhood $V$ of $\downarrow f$ in $\downarrow C_F(M)$ such that $V \cap r^\#(\downarrow C_F(X)) = \emptyset$. Choose $l \in \mathbb{N}$ and $x_0 \in K_l$ such that $f_{l+1}(x_0, l + 1) \neq f_l(x_0, l)$.

Case 1: $f_{l+1}(x_0, l + 1) > f_l(x_0, l)$. Choose a point $a \in I$ such that $f_{l+1}(x_0, l + 1) > a > f_l(x_0, l)$. By the continuity of $f_l$ and $f_{l+1}|K_l \times \{l + 1\}$, there exists a neighborhood $U$ of $x_0$ in $K_l$ such that

$$f_l(x, l) < a < f_{l+1}(x, l + 1) $$

for $x \in \text{cl}_{K_l} U$. It is clear that $\text{cl}_{K_l} U$ is compact and $\downarrow f_l \in (\text{cl}_{K_l} U \times \{l\} \times [a, 1])^*$. By the property of $\{K_n : n = 1, 2, 3, \cdots\}$, we have that $\text{int}_{K_{l+1}} U \supset \text{int}_X U \neq \emptyset$. Then $\downarrow f_{l+1} \in \text{int}_{K_{l+1}} U \times \{l + 1\} \times (a, 1]$. Let

$$V = \{ (\downarrow g_1, \downarrow g_2, \cdots) \in \prod_{n=1}^{\infty} \downarrow C_F(K_n \times \{n\}) : \downarrow g_l \in (\text{cl}_{K_l} U \times \{l\} \times [a, 1])^* \text{ and } \downarrow g_{l+1} \in (\text{int}_{K_{l+1}} U \times \{l + 1\} \times (a, 1])^- \}. $$

It is easy to verify that $V$ is as required.

Case 2: $f_{l+1}(x_0, l + 1) < f_l(x_0, l)$. It has a similar proof to Case 1.

(a) ⇒ (b). Assume that $r : M \to X$ is not a quotient map. There exists $V \subset X$ such that $r^{-1}(V)$ is open in $M$ but $V$ is not open in $X$. Choose $x_0 \in V$ such that $U \setminus V \neq \emptyset$ for every neighborhood $U$ of $x_0$ in $X$. Now we will construct $\downarrow g = (\downarrow g_1, \downarrow g_2, \cdots) \in \downarrow C_F(M) \setminus r^\#(\downarrow C_F(X))$ such that $W \cap r^\#(\downarrow C_F(X)) \neq \emptyset$ for every neighborhood $W$ of $\downarrow g$ in $\downarrow C_F(M)$. This shows that $r^\#(\downarrow C_F(X))$ is not closed in $\downarrow C_F(M)$.

We at first inductively construct $g = (g_1, g_2, \cdots) \in \prod_{n=1}^{\infty} C(K_n \times \{n\}) = C(M)$ such that

(i) $g_n(x_0, n) = 0$ if $x_0 \not\in K_n$;
(ii) $g_n(x, n) = 1$ if $x \in K_n \setminus V$;
(iii) $g_{n+1}(x, n + 1) = g_n(x, n)$ for every $x \in K_n$.

Note that $K_n \setminus V$ is closed in $K_n$ and $x_0 \not\in K_n \setminus V$ for every $n$. Thus, we can define $g_1 : K_1 \times \{1\} \to I$ such that (i) and (ii) hold for $n = 1$. Suppose that $g_n : K_n \times \{n\} \to I$ has been defined and satisfies (i)–(iii). Let

$$A_1 = K_n \times \{n + 1\}, \quad A_2 = (K_{n+1} \times \{n + 1\}) \cap r^{-1}(x_0),$$
$$A_3 = (K_{n+1} \times \{n + 1\}) \cap r^{-1}(X \setminus V) \quad \text{and} \quad A = A_1 \cup A_2 \cup A_3.$$
Then they are closed in \( K_{n+1} \times \{n+1\} \). We define \( g_{n+1}^0 : A \to I \) as follows

\[
g_{n+1}^0(x, n + 1) = \begin{cases} 
g_n(x, n) & \text{if } (x, n + 1) \in A_1; \\
0 & \text{if } (x, n + 1) \in A_2; \\
1 & \text{if } (x, n + 1) \in A_3.
\end{cases}
\]

Then, by \( g_n \in C(K_n \times \{n\}) \) satisfying (i) and (ii), \( g_{n+1}^0 : A \to I \) is well-defined and continuous. Thus, there exists a continuous extension \( g_{n+1} : K_{n+1} \times \{n+1\} \to I \). Then it satisfies (i)–(iii).

Therefore, we define \( g = (g_1, g_2, \ldots) \in C(M) \). By (iii), there exists a unique map \( f : X \to I \) such that \( g = f \circ r \). But if \( f : X \to I \) is not continuous at \( x_0 \). In fact, otherwise, since \( f(x_0) = 0 \), there exists a neighborhood \( U \) of \( x_0 \) such that \( f(x) < 1 \) for \( x \in U \). Take \( x_1 \in U \setminus V \) and \( l \in \mathbb{N} \) such that \( x_1 \in K_l \). Then

\[
1 = g(x_1, l) = f \circ r(x_1, l) = f(x_1) < 1.
\]

This is impossible. So \( \downarrow g \in \downarrow C_F(M) \setminus r^\#(\downarrow C_F(X)) \).

Secondly, let \( W = U_1^2 \cap U_2^2 \cap \cdots \cap U_n^2 \cap K^* \cap \downarrow C_F(M) \) be a neighborhood of \( \downarrow g \) in \( \downarrow C_F(M) \), where \( U_1, U_2, \ldots, U_n \) are open in \( M \times I \) and \( K \) is compact in \( M \times I \). Take \( (x_i, n_i, t_i) \in \downarrow g \cap U_i \) for each \( i = 1, 2, \ldots, n \).

Let \( p : M \times I \to M \) be the projection to \( M \) and \( A = \{(x_i, n_i) : i = 1, 2, \ldots, n\} \cup p(K) \). Then \( A \) is compact in \( M \). Hence \( r[A : A \to r(A)] \) is a quotient map. It follows from the continuity of \( g[A \text{ that } f[r(A) : r(A) \to I] \) is continuous. We can continuously extend \( f[r(A)] \) to \( \bar{f} : X \to I \) since \( r(A) \) is compact. It is easy to see that \( \bar{f} \circ r \in W \cap r^\#(\downarrow C_F(X)) \).

We completed the proof of the equivalence of the statements (a)–(c).

In [18], it was proved that \( \downarrow C_F(M) \) is an absolute \( F_{\sigma\delta} \)-space. Hence, \( r^\#(\downarrow C_F(X)) \) is also an absolute \( F_{\sigma\delta} \)-space if \( X \) is a \( k \)-space since \( r^\#(\downarrow C_F(X)) \) is a closed set of \( \downarrow C_F(M) \) in this case. \( \square \)

5. A condition for function spaces being Baire

In this section, we prove that \( \downarrow C_F(X) \) is a \( Z_2 \)-space if \( \text{cl} \{X_0 \neq X \). This shows that \( \downarrow C_F(X) \) is not Baire in this case. Conversely, we will prove that \( \downarrow C_F(X) \) is Baire if \( X \) is a \( k \)-space and \( \text{cl} X_0 = X \).

**Lemma 3.** For every \( d \in X \setminus X_0 \) and every \( a \in (0, 1] \), the set \( F = \{\downarrow f \in \downarrow C_F(X) : f(d) \geq a\} \) is a \( Z \)-set in \( \downarrow C_F(X) \).

**Proof.** Trivially, \( F \) is a closed set in \( \downarrow C_F(X) \). Moreover, we will claim that \( F \) is a \( Z \)-set in \( \downarrow C_F(X) \). For every \( \varepsilon > 0 \), we can choose \( i_0 \in \mathbb{N} \) such that \( d \in \bigcup_{i=1}^{i_0} K_i \) and \( e^{-d(m, m_0)} < \varepsilon \) for every \( m \in M \setminus \bigcup_{i=1}^{i_0} K_i \times \{i\} \).

Let \( C = \Theta_{i=1}^{i_0} K_i \times \{i\} \). Then \( C \) is a compact and open set in \( M \), and \( r^{-1}(d) \cap C \) is finite and every point in it is a non-isolated point in \( C \). Notice that \( (C, d) \) is a compact metric space. Thus, by the proof of [14, Lemma 4], there exists a continuous map \( \zeta_0 : \downarrow C_F(C) \to (0, +\infty) \) such that

\[
d_H \left( \downarrow g \cap (C \setminus B(r^{-1}(d) \cap C, \zeta_0(\downarrow g)) \times I) \right) < \varepsilon \tag{2}
\]

for every \( g \in C(C) \). Note that \( \downarrow f \mapsto \downarrow r^\#(f) | C \) is continuous from \( \downarrow C_F(X) \) to \( \downarrow C_F(C) \). We can define a continuous map \( \zeta : \downarrow C_F(X) \to (0, +\infty) \) by

\[
\zeta(\downarrow f) = \min \{\zeta_0(\downarrow r^\#(f) | C), \min \{d(m, l) : m \in C, l \in M \setminus C\}\}.
\]

Moreover, for every \( f \in C(X) \), we can define \( g_f \in \text{USC}(M) \) as
\[
g_f(m) = \begin{cases} 
 r^*(f)(m) & \text{if } m \in M \setminus B(r^{-1}(d), \zeta(\downarrow f)); \\
 0 & \text{if } m \in B(r^{-1}(d), \zeta(\downarrow f)). 
\end{cases}
\]

It follows from the definition \(d'_H\), Formula (2) and the choose of \(C\), we have

\[
d'_H(\downarrow r^*(f), \downarrow g_f) < \varepsilon. \tag{3}
\]

Using the continuous map \(\zeta : \downarrow C_F(X) \to (0, +\infty)\), we show that there exists a map \(\varphi : C(X) \to C(X)\) such that the map \(\downarrow \varphi : \downarrow C_F(X) \to \downarrow C_F(X)\) is continuous and also for every \(f \in C(X)\),

(i) \(\rho_H(\downarrow f, \downarrow \varphi(\downarrow f)) = d'_H(\downarrow f \circ r, \downarrow \varphi(f) \circ r) < \varepsilon\);

(ii) \(\varphi(f)(d) = 0\).

Hence, \(F\) is a \(Z\)-set in \(\downarrow C_F(X)\).

For every \(f \in C(X)\), let

\[
K_f = r(C \setminus B(r^{-1}(d) \cap C, \zeta(\downarrow f))), \quad \text{and} \quad A = \{(\downarrow f, x) \in \downarrow C_F(X) \times \beta X : x \in K_f\},
\]

where \(\beta X\) is the Čech–Stone compactification of \(X\). Then \(A\) is closed in \(\downarrow C_F(X) \times \beta X\). In fact, if \((\downarrow f, x) \in \downarrow C_F(X) \times \beta X \setminus A\), then \(x \notin K_f\). Since \(K_f\) is compact, there exists a pair of disjoint open sets \(U, V\) in \(\beta X\) such that \(x \in U\) and \(K_f \subset V\). It follows that

\[
C \setminus B(r^{-1}(d) \cap C, \zeta(\downarrow f)) \subset r^{-1}(V).
\]

By the continuity of \(\zeta\) and \(r\), there exists a neighborhood \(W\) of \(\downarrow f\) in \(\downarrow C_F(X)\) such that

\[
C \setminus B(r^{-1}(d) \cap C, \zeta(\downarrow g)) \subset r^{-1}(V)
\]

for every \(\downarrow g \in W\). Therefore, \(W \times U\) is a neighborhood of \((\downarrow f, x)\) in \(\downarrow C_F(X) \times \beta X\) and \((W \times U) \cap A = \emptyset\).

Since \(\downarrow C_F(X)\) is metrizable and \(\beta X\) is compact, \(\downarrow C_F(X) \times \beta X\) is normal, see [11]. For a pair of disjoint closed sets \(A\) and \(\downarrow C_F(X) \times \{d\}\) in this space, there exists a continuous map \(\alpha : \downarrow C_F(X) \times \beta X \to I\) such that

\[
\alpha(\downarrow C_F(X) \times \{d\}) \subset \{0\} \quad \text{and} \quad \alpha(A) \subset \{1\}.
\]

Now we can define \(\varphi : C(X) \to C(X)\) by

\[
\varphi(f)(x) = \alpha(\downarrow f, x) \cdot f(x)
\]

for every \(f \in C(X)\) and \(x \in X\). Then \(\varphi(f)(d) = 0\) and \(\downarrow \varphi : \downarrow C_F(X) \to \downarrow C_F(X)\) is continuous by [15, Lemma 2.12]. Moreover, it is easy to see that

\[
\downarrow g_f \cap C \subset \downarrow \varphi(f) \circ r \cap C \subset \downarrow f \circ r \cap C.
\]

It follows from Formula (3) that

\[
\rho_H(\downarrow f, \downarrow \varphi(\downarrow f)) = d'_H(\downarrow f \circ r, \downarrow \varphi(f) \circ r) < \varepsilon
\]

for every \(f \in C(X)\). Therefore, \(\varphi : C(X) \to C(X)\) is as required. \(\square\)
Lemma 4. The space $\downarrow C_F(X)$ is a $Z_\sigma$-space if $\text{cl} \ X_0 \neq X$. Therefore, $\downarrow C_F(X)$ is included in a $Z_\sigma$-set in $M^Q$ if $\text{cl} \ X_0 \neq X$.

Proof. Since $X$ is a continuous image of a separable metrizable space, $X$ is hereditary separable. Thus, there exists a countable dense $D$ in $X \setminus X_0$. It follows from the above lemma that

$$\mathcal{F} = \{F_{d,s} : d \in D, s \in (0,1) \cap \mathbb{Q}\}$$

is a countable family of $Z$-sets in $\downarrow C_F(X)$, where

$$F_{d,s} = \{\downarrow f \in \downarrow C_F(X) : f(d) \geq s\}.$$

Now let

$$E = \text{cl}_{\downarrow C_F(X)}(\downarrow C_F(X) \setminus \bigcup \mathcal{F}).$$

Then, it is not hard to verify that $f(x) = 0$ for every $\downarrow f \in E$ and $x \in X \setminus \text{cl} \ X_0$. By our assumption, $X \setminus \text{cl} \ X_0$ is not empty. It follows that $E$ is a $Z$-set in $\downarrow C_F(X)$, see [14, Lemma 5]. Hence $\downarrow C_F(X)$ is a $Z_\sigma$-space if $\text{cl} \ X_0 \neq X$. Since the closure of a $Z$-set in a homotopy dense subspace is a $Z$-set in the whole space, $\downarrow C_F(X)$ is included in a $Z_\sigma$-set in $M^Q$ if $\text{cl} \ X_0 \neq X$. □

The following proposition is the main result in this section.

Proposition 3. For a weakly locally compact and hemicompact $\aleph_0$-space and $k$-space $X$, $\downarrow C_F(X)$ is a Baire space if and only if the set of isolated points is dense in $X$.

Proof. For the “only if” part, by Lemma 4, if the set of isolated points is not dense in $X$, we have $\downarrow C_F(X)$ is a $Z_\sigma$-space, hence it is not a Baire space. For the “if” part, the proof is similar to the one of [18, Lemma 2.3] and hence we omit some details. Note that $M_0$ is the set of all isolated points of $M$. Let

$$\mathcal{A} = \{A \subset M_0 : A \text{ is finite}\}.$$

For $A \in \mathcal{A}$ and natural numbers $n$, $l$, define

$$U_{A,n,l} = \{\downarrow f \in M^Q : f(y) < \frac{1}{n} \text{ for } y \in K_l \times \{l\} \setminus A\}.$$

Then $U_{A,n,l} = M^Q \cap ((K_l \times \{l\} \setminus A) \times [\frac{1}{n},1])^*$ is open in $M^Q$. Define

$$U_{n,l} = \bigcup_{A \in \mathcal{A}} U_{A,n,l}.$$

Then $U_{n,l}$ is open and dense in $M^Q$. To this end, we only need to prove that it is dense in $r^\#(\downarrow C_F(X))$. For every $\downarrow f \in r^\#(\downarrow C_F(X))$ and $\varepsilon \in (0,1)$, there exists a compact set $C$ such that $e^{-d(y,m_0)} < \varepsilon$ for each $y \in M \setminus C$. Using Lemma 2 and the assumption that $\text{cl} \ X_0 = X$, we can choose $A \in \mathcal{A}$ such that

$$d_H(\{(a,f(a)) : a \in A\}, G(f) \cap (C \times I)) < \varepsilon,$$

where $G(f)$ is the graph of $f$. Define a continuous map $g : M \to I$ by
\[ g(y) = \begin{cases} f(y) & \text{if } y \in r^{-1}(r(A)); \\ 0 & \text{otherwise.} \end{cases} \]

It is easy to verify that \( \downarrow g \in r^\#(\downarrow C_F(X)) \) and \( d_H(\downarrow g, \downarrow f) < \varepsilon \). Moreover, we have \( \downarrow g \in U_{n,l} \) since \( A' = r^{-1}(r(A)) \cap (K_i \times \{l\}) \in \mathcal{A} \).

Note that \( M^Q \) is a compact metrizable space. To show that \( r^\#(\downarrow C_F(X)) \) is a Baire space, it suffices to verify that
\[ G = \bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} U_{n,l} \subset r^\#(\downarrow C_F(X)). \]

For every \( \downarrow f \in G \), we have \( f(y) = 0 \) for each \( y \in M \setminus M_0 \). It implies that \( f \in C(M) \). Since \( X \) is a \( k \)-space, using Lemma 2, we have that \( r^\#(\downarrow C_F(X)) \) is closed in \( \downarrow C_F(M) \). Thus \( \downarrow f \in \downarrow C_F(M) \cap M^Q = r^\#(\downarrow C_F(X)) \). Therefore, \( r^\#(\downarrow C_F(X)) \) is a Baire space. \( \square \)

**Remark 3.** The reviewer of this paper told the authors that the assumption that \( X \) is a \( k \)-space in Proposition 3 is essential. The following example is due to him or her.

**Example 2.** Let \( X = \{ -\frac{1}{n} : n \in \mathbb{N} \} \cup \{0\} \cup \mathbb{N} \). Define the topology on \( X \) in which all non-zero points are isolated points and a set \( A \) containing 0 is open if and only if \( A \ni -\frac{1}{n} \) for all but finitely many \( n \in \mathbb{N} \) and
\[ \lim_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} = 1. \]

Then it is trivial to verify that \( X \) is a weakly locally compact and hemicompact \( \aleph_0 \)-space and the set of isolated points are dense in \( X \). But, using the technique in [2], it can be proved that \( \downarrow C_F(X) \) is not Baire.

### 6. Proof of the Main Theorem

In this section, we will give a proof of the Main Theorem. We first give the last result we need.

**Proposition 4.** The subspace \( r^\#(\downarrow C_F(X)) \) of \( M^Q \) is strongly \( F_{\sigma \delta} \)-universal in \( M^Q \) if \( cl \ X_0 \neq X \).

**Proof.** Let \( Y \) be a compact metrizable space and \( C, K \) an \( F_{\sigma \delta} \)-subset and a closed set in \( Y \), respectively. And let \( \Phi : Y \to M^Q \) be a continuous map satisfying that \( \downarrow \Phi[K] : K \to M^Q \) is a Z-embedding. By [3, Lemma 1.1] and the fact \( M^Q \approx Q \), we can assume that \( \downarrow \Phi(K) \cap \downarrow \Phi(Y \setminus K) = \emptyset \). For every \( \varepsilon \in (0, 1) \), we will define a Z-embedding \( \Psi : Y \to M^Q \) such that \( \downarrow \Psi[K] = \downarrow \Phi[K], \downarrow \Psi^{-1}(r^\#(\downarrow C_F(X))) \setminus K = C \setminus K \) and \( d_H(\downarrow \Phi(y), \downarrow \Psi(y)) < \varepsilon \) for each \( y \in Y \).

Define a map \( \delta : Y \to [0, 1) \) by
\[ \delta(y) = \frac{1}{3} \min\{\varepsilon, d_H(\downarrow \Phi(y), \downarrow \Phi(K))\}. \]

Then \( \delta \) is continuous and \( \delta(y) = 0 \) if and only if \( y \in K \). It follows from [10, Proposition 4.1.7] and Theorem 4 that there exists a homotopy \( H : M^Q \times I \to M^Q \) such that
\[ H_0 = id_{M^Q}, \quad H_t(M^Q) \subset r^\#(\downarrow C_F(X)), \quad d_H(H_t(\downarrow f), \downarrow f) \leq t \]
for each \( \downarrow f \in M^Q \) and \( t \in (0, 1] \). Using them, we can define a continuous map \( \downarrow h : Y \to M^Q \) as
\[ \downarrow h(y) = H(\downarrow \Phi(y), \delta(y)). \]
Then $\downarrow h | K = \downarrow \Phi | K$, $\downarrow h | (Y \setminus K) \subset r^\#(\downarrow C_F(X))$ and $d'_H(\downarrow h(y), \downarrow \Phi(y)) \leq \delta(y)$. Now, we define $k : Y \setminus K \to C(X)$ by $k(y) = (r^\#)^{-1}(h(y))$ for every $y \in Y \setminus K$. Then $\downarrow k : Y \setminus K \to \downarrow C_F(X)$ is continuous.

Since $\text{cl} X_0 \neq X$, by our assumptions in Section 3, we have chosen a non-isolated point $x_{\infty}$ and an open neighborhood $W$ of $x_{\infty}$ in $X$ such that $X_0 \cap W = \emptyset$, $\text{cl} W$ is compact and $\text{cl} W \subset K_1$. Therefore the inherited metric $d$ on $\text{cl} W$ from $(K_1, d)$, where

$$d(x, y) = d((x, 1), (y, 1))$$

for $x, y \in K_1$, is an admissible metric on $\text{cl} W$. We can define a map $k_0 : Y \setminus K \to C(\text{cl} W)$ as follows $k_0(y) = k(y) | \text{cl} W$. Then $\downarrow k_0 : Y \setminus K \to \downarrow C_F(\text{cl} W)$ is continuous, see [18, Lemma 2.2] or Lemma 1. Using the proof of [19, Proposition 1], there exists a continuous injection $\downarrow \Omega_0 : Y \setminus K \to \downarrow \text{USC}_F(\text{cl} W)$ such that

1. $\downarrow \Omega_0(y) \in \downarrow C_F(\text{cl} W)$ if and only if $y \in C \setminus K$;
2. $\Omega_0(y) : \text{cl} W \to I$ is not continuous only at $x_{\infty}$ for $y \in Y \setminus (C \cup K)$;
3. $d'_H(\downarrow \Omega_0(y), \downarrow k_0(y)) < \delta(y)$ for $y \in Y \setminus K$;
4. $\Omega_0(y) | \text{cl} W \setminus W = k_0(y) | \text{cl} W \setminus W$ for $y \in Y \setminus K$;
5. for every $y \in Y \setminus K$, there exists $x \in W$ such that $\Omega_0(y)(x) = 0$.

For every $y \in Y \setminus K$, we extend $\Omega_0(y) : \text{cl} W \to I$ to $\Omega(y) : X \to I$ as follows

$$\Omega(y)(x) = \begin{cases} 
\Omega_0(y)(x) & \text{if } x \in \text{cl} W; \\
k(y)(x) & \text{if } x \in X \setminus \text{cl} W.
\end{cases}$$

Then $\downarrow \Omega : Y \setminus K \to \downarrow \text{USC}_F(X)$ is continuous and

$$d'_H(\downarrow r^\#(\Omega(y)), \downarrow r^\#(k(y))) < \delta(y)$$

for every $y \in Y \setminus K$. In fact, for every $y \in Y \setminus K$, by (iv) and the definition of $k_0$,

$$\downarrow \Omega(y) = \downarrow \Omega_0(y) \cup \downarrow k(y) | \text{cl}(X \setminus \text{cl} W).$$

It is well-known that $\cup : \text{Cld}_F(X \times I) \times \text{Cld}_F(X \times I) \to \text{Cld}_F(X \times I)$ is continuous. It follows from Lemma 1 that $\downarrow \Omega : Y \setminus K \to \downarrow \text{USC}_F(X)$ is continuous. Moreover, $\Omega(y) \in \mathcal{K}$, where $\mathcal{K}$ is the family defined in Proposition 1, for every $y \in Y \setminus K$. Note that the metric $d$ on $\text{cl} W$ inherits from $M$. It follows from (iii) that Formula (4) holds.

Now we define $\downarrow \Psi : Y \to M^Q$ as

$$\Psi(y) = \begin{cases} 
\Phi(y) & \text{if } y \in K; \\
r^\#(\Omega(y)) & \text{if } y \in Y \setminus K.
\end{cases}$$

We verify that $\downarrow \Psi : Y \to M^Q$ is as required.

At first, $\downarrow \Psi : Y \to M^Q$ is well-defined and $\downarrow \Psi^{-1}(r^\#(\downarrow C_F(X))) \setminus K = C \setminus K$. For every $y \in Y \setminus K$, by (i) and (ii), $\Omega(y)$ is not continuous at most at $x_{\infty}$. It follows from Proposition 1 that $\Psi(y) \in M^Q$ for $y \in Y \setminus K$. Moreover, for $y \in Y \setminus K$, $\Psi(y)$ is in $r^\#(\downarrow C_F(X))$ if and only if $y \in C \setminus K$. We are done.

Secondly, $\downarrow \Psi : Y \to M^Q$ is continuous. In fact, using Proposition 1, $\downarrow \Psi : Y \to M^Q$ is continuous in $Y \setminus K$ since $\downarrow \Omega : Y \setminus K \to \downarrow \text{USC}_F(X)$ is continuous. Moreover, for every $y \in K$ and $y_n \in Y \setminus K$, we have $\delta(y_n) \to 0$ if $y_n \to y$. Moreover, we have
\[d'_H(\downarrow \Phi(y), \downarrow \Phi(y_n)) = d'_H(\downarrow \Phi(y), \downarrow \Phi(y_n)) \]
\[\leq d'_H(\downarrow \Phi(y), \downarrow \Phi(y_n)) + d'_H(\downarrow \Phi(y_n), \downarrow h(y_n)) + d'_H(\downarrow h(y_n), \downarrow \Phi(y_n)) \]
\[\leq d'_H(\downarrow \Phi(y), \downarrow \Phi(y_n)) + \delta(y_n) + \delta(y_n).\]

It follows from the continuity of \(\downarrow \Phi : Y \to M^Q\) that \(d'_H(\downarrow \Phi(y), \downarrow \Phi(y_n)) \to 0\) if \(y_n \to y\). Hence, \(\downarrow \Phi : Y \to M^Q\) is continuous in \(K\).

Thirdly, \(d'_H(\downarrow \Phi(y), \downarrow \Phi(y)) < \varepsilon\) for every \(y \in Y\). For every \(y \in Y \setminus K\),
\[d'_H(\downarrow \Phi(y), \downarrow \Phi(y)) \leq d'_H(\downarrow \Phi(y), \downarrow h(y)) + d'_H(\downarrow h(y), \downarrow \Phi(y)) \leq \delta(y) + \delta(y) < \varepsilon.
\]

At last, \(\downarrow \Phi : Y \to M^Q\) is a \(Z\)-embedding. Since \(\Omega : Y \setminus K \to \downarrow \text{USC}_F(X)\) is an injection, \(\Psi : Y \setminus K \to M^Q\) is one-to-one. The above formula and the definition of \(\delta(y)\) show that \(\Psi(y) \notin \Phi(K) = \Psi(K)\) for every \(y \in Y \setminus K\). Therefore, \(\downarrow \Phi : Y \to M^Q\) is an embedding. For every \(y \in Y\), \(\Psi(y)\) is either in the \(Z\)-set \(\Psi(K)\) or there exists \(m \in M\) such that \(\Psi(y)(m) = 0\). This shows that \(\Psi(Y)\) is a \(Z\)-set by [14, Lemma 5].

Thus, \(\downarrow \Phi : Y \to M^Q\) is as required. \(\square\)

Now, we can give a proof for the Main Theorem.

**Proof of Main Theorem.** (a) \(\Rightarrow\) (b) is obvious since \(c_0\) is a \(Z_\sigma\)-space.

(b) \(\Rightarrow\) (c) follows directly from Proposition 3 and Theorem 3.

(c) \(\Rightarrow\) (a). Assume that (c) holds. Using Proposition 2, we have that \(r^\#(\downarrow C_F(X))\) is an absolute \(F_{\sigma\delta}\)-space. It follows Lemma 4 that \(r^\#(\downarrow C_F(X))\) is contained in a \(Z_\sigma\)-set of \(M^Q\). By Proposition 4, \(r^\#(\downarrow C_F(X))\) is strongly \(F_{\sigma\delta}\)-universal in \(M^Q\). Thus, \(r^\#(\downarrow C_F(X))\) is an \(F_{\sigma\delta}\)-absorber in \(M^Q\). Therefore, using Theorem 2, the facts \(M^Q \approx Q\) and \(c_0\) is an \(F_{\sigma\delta}\)-absorber in \(Q\), we have \((M^Q, r^\#(\downarrow C_F(X))) \approx (Q, c_0)\). In particular, (a) holds since \(\downarrow C_F(X) \approx r^\#(\downarrow C_F(X))\). \(\square\)

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**References**