SUPPORT VARIETIES FOR TRANSPORTER CATEGORY ALGEBRAS

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Abstract. Let $G$ be a finite group. Over any finite $G$-poset $P$ we may define a transporter category as the corresponding Grothendieck construction. The classifying space of the transporter category is the Borel construction on the $G$-space $BP$, while the $k$-category algebra of the transporter category is the (Gorenstein) skew group algebra on the $G$-algebra $kP$.

We introduce a support variety theory for the category algebras of transporter categories. It extends Carlson’s support variety theory on group cohomology rings to equivariant cohomology rings. In the mean time it provides a class of (usually non selfinjective) algebras to which Snashall-Solberg’s (Hochschild) support variety theory applies. Various properties will be developed. Particularly we establish a Quillen stratification for modules.

1. Introduction

Let $G$ be a finite group and $P$ a finite $G$-poset. Throughout this paper, we assume $k$ is an algebraically closed field of characteristic $p$, dividing the order of $G$. We are interested in a finite category $G \ltimes P$, which is the Grothendieck construction on the $G$-poset $P$ and which we will call a transporter category in this paper. When $G = \{e\}$ is trivial, $\{e\} \ltimes P \cong P$ and when $P = \bullet$ is trivial, $G \ltimes \bullet \cong G$. A transporter category $G \ltimes P$ is the algebraic or categorical predecessor of the Borel construction $EG \times_G BP$ on the finite $G$-CW-complex $BP$, in the sense that $B(G \ltimes P) \simeq EG \times_G BP$. Our interests in transporter categories come from the fact that the equivariant cohomology ring $H^*_G(BP, k) = H^*(EG \times_G BP, k)$ is Noetherian. Through an algebraic construction of the equivariant cohomology ring, we may introduce in a natural way modules over this ring and hence extend Carlson’s support variety theory for finite group algebras to one for finite transporter category algebras.

Let us recall some historical developments in support variety theory. Suppose $X$ is a compact $G$-space. Quillen [22, 23] proved that $H^*_G(X)$ is Noetherian. Following his notation, we put $H_G(X)$ to be $H^*_G(X)$ if $p = 2$ or $H^*_G(X)$, the even part of the equivariant cohomology ring, if $p \geq 3$. When $X = \bullet$ is just a point fixed by $G$, the equivariant ring reduces to the group cohomology ring and we shall write $H_G = H_G(\bullet)$ and $H = H_G(\bullet)$. Quillen’s work began with the observation that the graded ring $H_G(X)$ is commutative Noetherian. It enabled him to define a homogeneous affine variety $V_G, X$ as the maximal ideal spectrum MaxSpec$H_G(X)$, and subsequently described it in terms of $V_E = V_{E, \bullet} = \text{MaxSpec}H_E, \text{where } E \text{ runs over the set of all elementary abelian } p\text{-subgroups of } G \text{ such that } X^E \neq \emptyset$. This is what we nowadays refer to as the Quillen stratification. Restricting to the special case of $X = \bullet$, based on the fact that $\text{Ext}^*_{kG}(M, M)$ is finitely generated over $H^*_G \cong \text{Ext}^*_{kG}(k, k)$, Carlson [11] extended Quillen’s work by attaching to every finitely generated $kG$-module $M$ a subvariety of $V_G = V_G(\bullet)$, denoted by $V_G(M) = \text{MaxSpec}H_G/I_G(M)$, called the (cohomological) support variety of $M$, where $I_G(M)$ is the kernel of the following map

$$
\phi_M = - \otimes_k M : H^*_G \cong \text{Ext}^*_{kG}(k, k) \to \text{Ext}^*_{kG}(M, M).
$$

Especially since $\phi_M$ is the identity, $V_G = V_G(k)$. Following Carlson’s construction, Avrunin and Scott [5] quickly generalized the Quillen stratification from $V_G$ to $V_G(M)$. By showing that support varieties are well-behaved with respect to module operations, gradually Benson, Carlson and many others have extended the support variety theory to various contexts. Among these, we would like to mention Benson-Scott theory [4], which has been extensively used to study modules over $k$-categories.

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others developed a remarkable theory, being a significant progress in group representations and cohomology. Since then, some other analogous support variety theories have been introduced for restricted Lie algebras [16], for finite group schemes [6, 17], for complete intersections [4] and for certain finite-dimensional algebras [20, 14, 24].

Quillen’s work on equivariant cohomology rings has not been fully exploited, partially because there existed no suitable modules which $H^n_G(X)$ (hence $H_G(X)$) acts on or maps to, as in Carlson’s theory. In this article we attempt to use category algebras to solve the problem: if $X = BP$ comes from a finite $G$-poset, then we consider the category algebra $k(G \rtimes \mathcal{P})$ of the transporter category $G \rtimes \mathcal{P}$, based on which we will generalize Carlson’s theory. In fact, let $k$ be the trivial $(G \rtimes \mathcal{P})$-module (see Section 2.2). Then $\text{Ext}^*_k(G \rtimes \mathcal{P})(k, k)$ is a graded commutative ring and there exists a natural ring isomorphism

$$\text{Ext}^*_k(G \rtimes \mathcal{P})(k, k) \cong H^*(EG \times_G BP, k) = H^*_G(BP, k).$$

We shall call the above ring the ordinary cohomology ring of $k(G \rtimes \mathcal{P})$ (instead of the equivariant cohomology ring), as opposed to the Hochschild cohomology ring of $k(G \rtimes \mathcal{P})$. Then we define $V_{G \rtimes \mathcal{P}} = V_{G, BP} = \text{MaxSpec}H_G(BP)$. The virtue of having an entirely algebraic construction of the equivariant cohomology theory is that it allows us to consider

$$\text{Ext}^*_k(G \rtimes \mathcal{P})(\mathfrak{M}, \mathfrak{N})$$

for any finitely generated $\mathfrak{M}, \mathfrak{N} \in k(G \rtimes \mathcal{P})$-mod, and moreover construct a map

$$\phi_{\mathfrak{M}} = -\otimes: \text{Ext}^*_k(G \rtimes \mathcal{P})(k, k) \to \text{Ext}^*_k(G \rtimes \mathcal{P})(\mathfrak{M}, \mathfrak{N}).$$

Here $\otimes$ is the tensor product in the closed symmetric monoidal category $(k(G \rtimes \mathcal{P})$-mod, $\otimes, k$). Note that $k$ serves as the tensor identity. Since we have shown in [29] that $\text{Ext}^*_k(G \rtimes \mathcal{P})(\mathfrak{M}, \mathfrak{N})$ is finitely generated over the ordinary cohomology ring, we may define the support variety of $\mathfrak{M} \in k(G \rtimes \mathcal{P})$-mod as $V_{G \rtimes \mathcal{P}}(\mathfrak{M}) = \text{MaxSpec}H_G(BP)/I_{G \rtimes \mathcal{P}}(\mathfrak{M})$, where $I_{G \rtimes \mathcal{P}}(\mathfrak{M})$ is the kernel of $\phi_{\mathfrak{M}}$. Especially $V_{G \rtimes \mathcal{P}} = V_{G \rtimes \mathcal{P}}(k)$. When $\mathcal{P} = \bullet$, the is exactly Carlson’s construction because $k(G \rtimes \bullet) \cong kG$, $k$ becomes the trivial $kG$-module $k$ and $\otimes$ reduces to $\otimes_k$ under the circumstance.

As a surprising consequence of our investigations of transporter category algebras, we assert that Snashall-Solberg’s (Hochschild) support variety theory (for Gorenstein algebras) applies to every block of a finite transporter category algebra. Furthermore, our support variety theory is closely related with Snashall-Solberg’s as what happens in the case of group algebras and their blocks. A notable point is that the block algebras of a transporter category algebra are usually non-selfinjective and non-commutative, opposing to the cases of (selfinjective) Hopf algebras [11, 6, 17] and of commutative Gorenstein algebras [4] considered by others.

This paper is organized as follows. Section 2 recalls the definitions of the transporter category, the category algebra and the category cohomology. Various necessary constructions are recorded for the convenience of the reader. Here we show a transporter category algebra is Gorenstein and the ordinary cohomology ring of such an algebra is identified with an equivariant cohomology ring. Then in Section 3, we define the support variety for a module over a transporter category algebra. To motivate the reader, we describe the relevant works of Carlson, Linckelmann and Snashall-Solberg, before we develop some standard properties. Sections 4 and 5 contain various properties of support varieties, including the Quillen stratification for modules, as well as results related with sub-transporter categories and tensor products.

2. Preliminaries

In this section, we recall the definition of a transporter category and some background in category algebras. Throughout this article we will only consider finite categories, in the sense that they have finitely many morphisms. Thus a group $G$, or a $G$-poset $\mathcal{P}$, is always finite.

The morphisms in a poset are customarily denoted by $\leq$. The dimension of a poset $\mathcal{P}$, $\dim \mathcal{P}$, is defined to be the maximal integer $n$ such that a chain of non-isomorphisms $x_0 < x_1 < \cdots < x_n$ exists in $\mathcal{P}$. 
Any $G$-set is usually regarded as a $G$-poset with trivial relations. One the other hand, since in a $G$-poset $\mathcal{P}$, both $\text{Ob} \mathcal{P}$ and $\text{Mor} \mathcal{P}$ are naturally $G$-sets, we shall use terminologies for $G$-sets in our situation without further comments.

### 2.1. Transporter categories as Grothendieck constructions

We deem a group as a category with one object, usually denoted by $\bullet$. The identity of a group $G$ is always named $e$. We say a poset $\mathcal{P}$ is a $G$-poset if there exists a functor $F$ from $G$ to $s\mathcal{Cats}$, the category of small categories, such that $F(\bullet) = \mathcal{P}$. It is equivalent to saying that we have a group homomorphism $G \to \text{Aut}(\mathcal{P})$.

The Grothendieck construction on $F$ will be called a transporter category.

**Definition 2.1.1.** Let $G$ be a group and $\mathcal{P}$ a $G$-poset. The transporter category $G \ltimes \mathcal{P}$ has the same objects as $\mathcal{P}$, that is, $\text{Ob}(G \ltimes \mathcal{P}) = \text{Ob} \mathcal{P}$. For $x, y \in \text{Ob}(G \ltimes \mathcal{P})$, a morphism from $x$ to $y$ is a pair $(g, gx \leq y)$ for some $g \in G$.

If $(g, gx \leq y)$ and $(h, hy \leq z)$ are two morphisms in $G \ltimes \mathcal{P}$, then their composite is easily seen to be $(hg, (hg)x \leq z) = (h, hy \leq z) \circ (g, gx \leq y)$.

**Remark 2.1.2.** For instance, $\text{Ob}(G \ltimes \mathcal{P}) \equiv \text{Ob}(T_{G/H})$ where $T_{G/H}$ is identified with the transporter category $\text{tr}_{G/H}$.

**Example 2.1.3.** Let $G$ be a group and $\mathcal{P}$ a $G$-poset. From the definition one can easily see that there is a natural embedding $\iota : \mathcal{P} \hookrightarrow G \ltimes \mathcal{P}$ via $(x \leq y) \mapsto (e, x \leq y)$. On the other hand, the transporter category admits a natural functor $\pi : G \ltimes \mathcal{P} \to G$, given by $x \mapsto \bullet$ and $(g, gx \leq y) \mapsto g$. Thus we always have a sequence of functors $\mathcal{P} \hookrightarrow G \ltimes \mathcal{P} \xrightarrow{\pi} G$ such that $\pi \circ \iota(\mathcal{P})$ is the trivial subgroup or subcategory of $G$. Topologically it is well known that $BG \simeq \text{hocolim}_G BP \simeq EG \times_G BP$. Passing to classifying spaces, we obtain a fibrations sequence

$$BP \xrightarrow{B\iota} EG \times_G BP \xrightarrow{B\pi} BG.$$ Forming the transporter category over a $G$-poset eliminates the $G$-action, and thus is the algebraic analogy of introducing the Borel construction over a $G$-space. This is the first instance explaining why a transporter category has anything to do with the equivariant cohomology theory.

**Example 2.1.4.** Let $G$ be a finite group and $\mathcal{H}$ a subgroup. We consider the set of left cosets $G/H$ which can be regarded as a $G$-poset: $G$ acts via left multiplication. The transporter category $G \ltimes (G/H)$ is a connected groupoid whose skeleton is isomorphic to $H$. In this way one can recover all subgroups of $G$, up to category equivalences. Making Grothendieck constructions on transitive $G$-sets reveals the isotropy groups.

For an arbitrary $G$-poset $\mathcal{P}$ and $x \in \text{Ob} \mathcal{P}$, we have a category equivalence $G \ltimes [x] \simeq G_x$, see Remark 2.1.2. For instance, $G$ acts on $\mathcal{P} = S_p$, the poset of non-identity $p$-subgroup of $G$, by conjugation. Then $G_x = N_G(x)$ for every $x \in \text{Ob}(G \ltimes S_p) = \text{Ob} S_p$. The isotropy group of $x$, $G_x$, is frequently identified with the transporter category $G_x \ltimes x \simeq G_x \times \bullet$.

In the upcoming Section 2.2 we will see that a category equivalence $\mathcal{D} \to \mathcal{C}$ induces a Morita equivalence between their category algebras, $k\mathcal{D} \simeq k\mathcal{C}$, as well as a homotopy equivalence $BD \simeq BC$ (see [25]). It means there is no essential difference between $H$ and $G \ltimes (G/H)$ as far as we concern. Hence it makes sense if we deem transporter categories as generalized subgroups for a fixed finite group.
2.2. Category algebras and their representations. We recall some facts about category algebras. The reader is referred to [25, 27, 30] for further details. Let $C$ be a finite category and $k$ a field. One can define the category algebra $kC$, which, as a vector space, has a basis the set of all morphisms in $C$. We then define a product on the base elements and extend it linearly to $kC$. The product $\alpha \cdot \beta$ of any two base elements $\alpha, \beta \in \text{Mor}C$ is defined to be $\alpha \beta$, if they are composable, or zero otherwise. It is a finite-dimensional associative algebra with identity $1 = \sum_{x \in \text{Ob}C} 1x$. The category algebra $kC$ possesses a distinguished module $\mathcal{B}$, called the trivial module. It can be defined as $\mathcal{B} = k \cdot \text{Ob}C$. If $\alpha$ is a base element of $kC$ and $x \in \text{Ob}C$, we ask $\alpha \cdot x = y$ if $\alpha \in \text{Hom}(x, y)$. Otherwise we set $\alpha \cdot x = 0$.

When $C$ is a group, $kC$ is exactly the group algebra and $\mathcal{B} = k$. All modules we consider here are finitely generated left modules, unless otherwise specified. As a convention, throughout this article, the $kG$-modules are usually written as $M, N$ etc, while the modules of a (non-group) category algebra $kC$ are denoted by $\mathcal{M}, \mathcal{N}$ etc., except some distinguished modules, namely $\mathcal{B}$ and, in the special case of $C = G \rtimes \mathcal{P}$, $\kappa_M$ which are obtained from $kG$-modules (to be defined shortly).

A $k$-representation of $C$ is a covariant functor from $C$ to $\text{Vect}_k$, the category of finite dimensional $k$-vector spaces. All representations of $C$ form the functor category $\text{Vect}_k^C$. By a theorem of B. Mitchell (see [25]), the finitely generated left $kC$-modules are the same as the $k$-representations of $C$, in the sense that there exists a natural equivalence

$$\text{Vect}_k^C \simeq kC\text{-mod}.$$  

It is often helpful to utilize the underlying functor structure of a module. For instance, upon the preceding category equivalence we can alternatively define the trivial module $\mathcal{B}$ as a constant functor taking $k$ as its value at every object of $C$. Meanwhile since $\text{ Vect}_k^C$ is a symmetric monoidal category, $\text{ Vect}_k^C$ inherits this structure. It means there exists an (internal) tensor product, or the pointwise tensor product, written as $\otimes$, such that for any two $kC$-modules $\mathcal{M}, \mathcal{N}$, $(\mathcal{M} \otimes \mathcal{N})(x) := \mathcal{M}(x) \otimes \mathcal{N}(x)$. Let $\alpha \in \text{Mor}C$ be a base element of $kC$. Then $\alpha$ acts on $\mathcal{M} \otimes \mathcal{N}$ via $\alpha \otimes \alpha$. Obviously $\mathcal{B}$ is the tensor identity with respect to $\otimes$ and $\mathcal{M} \otimes \mathcal{N} \cong \mathcal{N} \otimes \mathcal{M}$. The category $kC$-mod has function objects, also called internal hom $\text{ hom}_C$ [27], in the sense that, for $\mathcal{L}, \mathcal{M}, \mathcal{N} \in kC$-mod,

$$\text{ Hom}_C(\mathcal{L} \otimes \mathcal{M}, \mathcal{N}) \cong \text{ Hom}_C(\mathcal{L}, \text{ Hom}_C(\mathcal{M}, \mathcal{N})) .$$

We record the basic tools for comparing category algebras and their modules. When $\tau : D \to C$ is a functor between two finite categories, there are adjoint functors for comparing their representations. The functor $\tau$ usually does not induce an algebra homomorphism from $kD$ to $kC$. However it does induce an exact functor, called the restriction along $\tau$,

$$\text{Res}_D^C : kC\text{-mod} \to kD\text{-mod}. $$

If we regard a $kC$-module as a functor, then its restriction is the precomposition with $\tau$. If we consider the functor $\pi : G \rtimes \mathcal{P} \to G$, then any $kG$-module $M$ restricts to a $k(G \rtimes \mathcal{P})$-module, written as $\kappa_M = \text{Res}_{G \rtimes \mathcal{P}}^G M$, with only one exception $\kappa_k = k$. It is known that $\text{Hom}(\kappa_M, \kappa_N) \cong \text{Hom}_{kG\rtimes \mathcal{P}}(M, N)$ for any two $M, N \in kG$-mod. In this paper, for the sake of simplicity, if $D \to C$ is a functor and $\mathcal{M}$ is a $kC$-module, when it will not cause confusions, we sometimes neglect $\text{Res}_D^C$ and regard $\mathcal{M}$ (instead of $\text{Res}_D^C\mathcal{M}$) as a $kD$-module.

The functor $\text{Res}_D^C$ is equipped with two adjoints: the left and right Kan extensions along $\tau$

$$L\text{K}_D^C, R\text{K}_D^C : kD\text{-mod} \to kC\text{-mod}. $$

The definition of the left and right Kan extensions depend on the so-called over-categories and under-categories, respectively. Despite their seemingly abstract definitions, they are quite computable and thus play an important role in category representations and cohomology, see [26, 28, 29], as well as Sections 2.5, 3.4 and 4.1.

Note that our notations for the restriction and Kan extensions are slightly different from earlier articles. The reason is that in this place we feel it is necessary to emphasize the two categories involved in order to make the notations more indicative.
2.3. EI categories. When a category $\mathcal{C}$ is an EI category, that is, every endomorphism is an isomorphism, there exists a natural partial order on the set of isomorphism classes of objects in $\mathcal{C}$ [25]. Indeed the partial order is given by $[x] \leq [y]$ if and only if $\text{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$. Groups, posets and transporter categories are all EI categories. The upshot is that this partial order allows us to give a filtration of any $k\mathcal{C}$-module $\mathcal{M}$. Let us call $x \in \text{Ob} \mathcal{C}$ an $\mathcal{M}$-object if $\mathcal{M}(x) \neq \emptyset$. Assume $x$ is minimal among all $\mathcal{M}$-object, then we can define a submodule $\mathcal{M}_x$ such that $\mathcal{M}_x(y) = \mathcal{M}(y)$ unless $y \equiv x$ in $\mathcal{C}$, in which case $\mathcal{M}_x(y) = 0$. Then we have a short exact sequence

$$0 \to \mathcal{M}_x \to \mathcal{M} \to \mathcal{M}/\mathcal{M}_x \to 0.$$ 

Denote by $\mathcal{M}_x = \mathcal{M}/\mathcal{M}_x$. This is an example of the so-called atomic modules of $k\mathcal{C}$. A $k\mathcal{C}$-module $\mathcal{M}$ is called atomic if as a functor $\mathcal{M}$ takes zero values on all but one isomorphism classes of objects. From the above analysis we see that every $\mathcal{M}$ admits a filtration with atomic modules as composition factors. Obviously from any module $\mathcal{M}$ and an $\mathcal{M}$-object $x$ we may define an atomic module by brutal truncation (the restriction along the inclusion $[x] \to \mathcal{C}$). Abusing terminology, we always write such modules as $\mathcal{M}_x$.

Assume $\mathcal{C}$ is finite EI. Then we can characterize projective and injective $k\mathcal{C}$-modules. Recall from [25] that each indecomposable left (resp. right) projective $k\mathcal{C}$-module, up to isomorphism, is a direct summand of $k\text{Hom}_\mathcal{C}(x, -) = k^\mathcal{C} \cdot 1_x$ (resp. $k\text{Hom}_\mathcal{C}(-, x) = 1_x \cdot k\mathcal{C}$) for some $x \in \text{Ob} \mathcal{C}$. The (indecomposable) right (resp. left) injective modules can be obtained as $k$-duality.

Lemma 2.3.1. (1) If $\mathcal{P}$ is a projective (or injective) $k(G \rtimes \mathcal{P})$-module, then $\mathcal{P}_x$ is a projective (k(G $\rtimes [x]$))-module, for all $\mathcal{P}$-objects $x$.

(2) A $k(G \rtimes \mathcal{P})$-module $\mathcal{M}$ is of finite projective (equivalently, finite injective) dimension if and only if $\mathcal{M}_x$ is a projective $k(G \rtimes [x])$-module, for all $\mathcal{M}$-objects $x$. Under the circumstance, both proj.dim $\mathcal{M}$ and inj.dim $\mathcal{M}$ are bounded by dim $\mathcal{P}$.

(3) The transporter category algebra $k(G \rtimes \mathcal{P})$ is Gorenstein, which means, as both the left and the right regular module, it has finite injective dimension.

Proof. These are direct consequences of the fact that if $\text{Hom}_{G \rtimes \mathcal{P}}(x, y) \neq \emptyset$, then $k\text{Hom}_{G \rtimes \mathcal{P}}(x, y)$ is both free $kG_{\mathcal{P}}$- and $kG_{\mathcal{P}}$-module, along with the characterizations of projectives and injectives. We only prove (2).

Let $\mathcal{P}_\ast \to \mathcal{M} \to 0$ be a finite projective resolution. Then its restriction to $G \rtimes [x]$ is a finite projective resolution of $\mathcal{M}_x$. Since $G \rtimes [x] \simeq G_x$, their category algebras are Morita equivalent and thus $k(G \rtimes [x])$ is selfinjective. Then the finite projective resolution $(\mathcal{P}_\ast)_x \to \mathcal{M}_x \to 0$ splits and $\mathcal{M}_x$ is a projective (and injective) $k(G \rtimes [x])$-module. On the other hand, assume $\mathcal{M}$ satisfies the property that $\mathcal{M}_x$ is projective (or zero) for every object $x$. Take the projective cover $\mathcal{P} \to \mathcal{M}$. We immediately know that $\mathcal{P}_\ast \to \mathcal{M}_\ast$ splits as a $k(G \rtimes [x])$-map. Let $[y_1], \cdots, [y_n]$ be the isomorphism classes of all minimal objects among $\mathcal{M}$-objects. Then $(\bigoplus_{i=1}^n \mathcal{P}_i)_\ast$ is the projective cover of $\mathcal{M}_y$ which implies $\bigoplus_{i=1}^n \mathcal{P}_i \simeq \mathcal{M}_y$. Hence if we examine the kernel $\mathcal{M}'$ of $\mathcal{P} \to \mathcal{M}$, it has the property that $\mathcal{M}'_y$ is a projective $k(G \rtimes [y])$-module for all objects $y$ such that $[y] > [y_i]$ for some $1 \leq i \leq n$, and zero else. Repeat the same process for $\mathcal{M}'$, eventually we will obtain the finite projective resolution of $\mathcal{M}$. It implies proj.dim $\mathcal{M} \leq \dim \mathcal{P}$.

As to the injective dimension, we consider the right $k(G \rtimes \mathcal{P})$-module $\mathcal{M}^\wedge$ (the $k$-dual of $\mathcal{M}$). It satisfies $\mathcal{M}^\wedge(x) = \mathcal{M}(x)^\wedge$. Similar to the left module situation, it has a finite projective resolution. When we dualize it, it becomes an injective resolution of $\mathcal{M}$. $\square$

Since $G \rtimes [x] \simeq G_x$ (see Example 2.1.4.), in the first two statements, we may replace $\mathcal{P}_x$, $\mathcal{M}_x$, $k(G \rtimes [x])$ by $\mathcal{P}(x)$, $\mathcal{M}(x)$, $kG_x$, respectively.

It is helpful to give the following characterization of a transporter category algebra as a skew group algebra. Recall from [3, Chapt. III, Section 4] that if a $k$-algebra $A$ is a $G$-algebra, then we may define the skew group algebra $A[G]$ to be the $k$-vector space $A \otimes_k kG$ equipped with a multiplication rule determined by

$$(a_1 \otimes g_1) \ast (a_2 \otimes g_2) = a_1(g_1 \cdot a_2) \otimes g_1g_2,$$

where $a_1, a_2 \in A$ and $g_1, g_2 \in G$. Here $g_1 \cdot a_2$ means the image of $a_2$ under the action of $g_1$. For instance we immediately have $kG \cong (k\bullet)[G]$. The reader is referred to [2] and [3] for further
details and some known results about skew group algebras. Proposition 2.2 of \[2\] says that \(A(G)\) is Gorenstein if and only if \(A\) is. Since it is easy to verify that \(k\mathcal{P}\) is Gorenstein, with the following result we have another proof of \(k(G \rtimes \mathcal{P})\) being Gorenstein.

**Lemma 2.3.2.** There is an algebra isomorphism \(k(G \rtimes \mathcal{P}) \cong k\mathcal{P}[G]\).

**Proof.** The isomorphism is given by \((gx \leq y, g) \mapsto (gx \leq y) \otimes g\) with inverse \((x \leq y) \otimes h \mapsto (h(h^{-1}x) \leq y, h)\) for \(x, y \in \text{Ob} \mathcal{P}\) and \(g, h \in G\). \(\square\)

The modular representation theory of \(k(G \rtimes \mathcal{P})\) will be studied in another place.

### 2.4 Category cohomology and spectrum.

For any two \(k\mathcal{C}\)-modules it makes sense to consider \(\text{Ext}^i_{k\mathcal{C}}(\mathcal{M}, \mathcal{N}) = \bigoplus_{j \geq 0} \text{Ext}^i_{k\mathcal{C}}(\mathcal{M}, \mathcal{N})\). Furthermore if \(\mathcal{M}'\) and \(\mathcal{N}'\) are two other modules, the tensor product \(\otimes\) induces a *cup product* as follows

\[
\cup : \text{Ext}^i_{k\mathcal{C}}(\mathcal{M}, \mathcal{N}) \otimes \text{Ext}^j_{k\mathcal{C}}(\mathcal{M}', \mathcal{N}') \to \text{Ext}^{i+j}_{k\mathcal{C}}(\mathcal{M} \otimes \mathcal{M}', \mathcal{N} \otimes \mathcal{N}')
\]

In particular \(\text{Ext}^*_{k\mathcal{C}}(\mathcal{K}, k)\) is a graded commutative ring and we have a natural isomorphism \(\text{Ext}^*_{k\mathcal{C}}(\mathcal{K}, k) \cong H^*(BC, k)\) [27]. This ring is called the *ordinary cohomology ring* of \(k\mathcal{C}\) and it acts on \(\text{Ext}^*_{k\mathcal{C}}(\mathcal{M}, \mathcal{N})\) via the cup product. For any \(k\mathcal{C}\)-module \(\mathcal{M}\), the Yoneda splice provides a ring structure on \(\text{Ext}^*_{k\mathcal{C}}(\mathcal{M}, \mathcal{M})\). When \(\mathcal{M} = k\), the cup product and Yoneda splice give the same ring structure on \(\text{Ext}^*_{k\mathcal{C}}(k, k)\). There exists a ring homomorphism whose image lies in the center of the graded ring \(\text{Ext}^*_{k\mathcal{C}}(\mathcal{M}, \mathcal{M})\).

Moreover given a short exact sequence \(0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0\) the resulting connecting homomorphism is a morphism of \(\text{Ext}^*_{k\mathcal{C}}(\mathcal{K}, k)\)-modules.

In summary, finite category cohomology behaves very much like the special case of finite group cohomology, except the finite generation property. The ordinary cohomology ring of a category algebra is usually far from finitely generated, but it is so when \(\mathcal{C} = G \rtimes \mathcal{P}\) is finite as it is isomorphic to the equivariant cohomology ring \(H^*_G(B\mathcal{P}, k)\), see Section 2.5 and [22, 28].

The functor \(\text{Res}^\tau_G\) introduced earlier leads to a restriction on cohomology

\[
\text{res}^\tau_G : \text{Ext}^*_{k\mathcal{C}}(\mathcal{K}, k) \to \text{Ext}^*_{k\mathcal{D}}(\mathcal{K}, k).
\]

It coincides with the restriction \(H^*(BC, k) \to H^*(BD, k)\), induced by the continuous map \(B\tau : BD \to BC\) between two classifying spaces, see [27].

From now on, we assume \(k\mathcal{C}\) is algebraically closed. Throughout this paper let us denote by

\[
H(\mathcal{C}) = \text{Ext}^*_{k\mathcal{C}}(\mathcal{K}, k) = \begin{cases} 
\text{Ext}^*_{k\mathcal{C}}(\mathcal{K}, k), & \text{if the characteristic of } k = 2; \\
\text{Ext}^*_{k\mathcal{C}}(\mathcal{K}, k), & \text{if the characteristic of } k > 2.
\end{cases}
\]

This graded ring is commutative. If \(\text{Ext}^*_{k\mathcal{C}}(\mathcal{K}, k)\) is Noetherian, then we can consider the *maximal ideal spectrum*, a homogeneous affine variety (see [7]),

\[
V_C = \text{MaxSpec} H(\mathcal{C})
\]

Under the circumstance we will call \(V_C\) the *variety* of \(\mathcal{C}\).

Assume both \(\text{Ext}^*_{k\mathcal{C}}(\mathcal{K}, k)\) and \(\text{Ext}^*_{k\mathcal{D}}(\mathcal{K}, k)\) are Noetherian and there exists a functor \(\tau : \mathcal{D} \to \mathcal{C}\). Since the preimage of a maximal ideal is still a maximal ideal, there exists a map between two varieties

\[
\iota_D := (\text{res}^\tau_G)^{-1} : V_D \to V_C.
\]

These varieties and their subvarieties are our main subjects and thus it is helpful if we can handle the restriction. In various interesting cases the map is well understood. As an application of Lemma 2.3.1 we obtain the following result.
Lemma 2.4.1. Consider a transporter category \( G \times \mathcal{P} \) and an object \( x \in \text{Ob}(G \times \mathcal{P}) \). Then the inclusions \( G_x \times x \to G \times [x] \to G \times \mathcal{P} \) induce two restrictions which fit into the following commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^*_k(G \times \mathcal{P})(\mathcal{M}, \mathcal{N}) & \to & \text{Ext}^*_kG_{[x]}(\mathcal{M}_x, \mathcal{N}_x) \\
\downarrow & & \downarrow \cong \\
\text{Ext}^*_k(G \times \mathcal{P})(\mathcal{M}, \mathcal{N}) & \to & \text{Ext}^*_kG_{x}(\mathcal{M}(x), \mathcal{N}(x))
\end{array}
\]

for any two modules \( \mathcal{M}, \mathcal{N} \in k(G \times \mathcal{P})\text{-mod} \).

**Proof.** Let \( \mathcal{M} \to \mathcal{M} \to 0 \) be a projective resolution. Then \( \mathcal{M}_x \to \mathcal{M}_x \to 0 \) remains a projective resolution of the \( kG \times [x] \)-module \( \mathcal{M}_x \), by Lemma 2.3.1 (1). Hence the functor

\[ \text{res}^*_{G \times \mathcal{P}} : k(G \times \mathcal{P}) \to k(G \times [x]) \]

induces a restriction

\[ \text{res}^*_{G \times \mathcal{P}} : \text{Ext}^*_k(G \times \mathcal{P})(\mathcal{M}, \mathcal{N}) \to \text{Ext}^*_kG_{[x]}(\mathcal{M}_x, \mathcal{N}_x). \]

Similarly we have a map \( \text{res}^*_{G \times \mathcal{P}} \). The commutative diagram follows directly from the natural isomorphism \( \text{Ext}^*_kG_{[x]}(\mathcal{M}_x, \mathcal{N}_x) \cong \text{Ext}^*_kG_x(\mathcal{M}(x), \mathcal{N}(x)) \). \( \square \)

Because the two restrictions \( \text{res}^*_{G \times \mathcal{P}} : \text{Ext}^*_k(G \times \mathcal{P})(\mathcal{M}, \mathcal{N}) \to \text{Ext}^*_kG_{[x]}(\mathcal{M}_x, \mathcal{N}_x) \) and \( \text{res}^*_{G \times [x]} : \text{Ext}^*_k(G \times \mathcal{P})(\mathcal{M}, \mathcal{N}) \to \text{Ext}^*_kG_x(\mathcal{M}(x), \mathcal{N}(x)) \) are ring homomorphisms with respect to the Yoneda splice, there are two maps

\[ \iota^*_{G \times \mathcal{P}} : V_{G \times \mathcal{P}} \to V_{G \times \mathcal{P}}, \]

induced by \( \text{res}_x \) for \( \mathcal{M} = \mathcal{N} = k \), and similarly

\[ \iota^*_{G \times [x]} : V_{G \times [x]} \to V_{G \times \mathcal{P}}. \]

Note that \( V_{G \times \mathcal{P}} = V_{G \times x} \).

2.5. Bar resolution and equivariant cohomology. One concept that we will refer to is the bar resolution \( \mathfrak{B}^C \), a combinatorially constructed projective resolution, of \( k \in kC\text{-mod} \). Given a functor \( \tau : \mathcal{D} \to \mathcal{C} \), one can define \( C_\ast(\tau/\ast, k) \), a complex of projective \( k\mathcal{C}\)-modules, such that, for each \( x \in \text{Ob}\mathcal{C}, \tau/x \) is a finite category (the category over \( x \) or just an overcategory) and \( C_\ast(\tau/x, k) \) is the (normalized) chain complex resulting from the simplicial \( k\)-vector spaces coming from the nerve of \( \tau/x \) [26]. Since overcategories are used in the proof of a couple of results in the next section, we recall its definition. The objects of \( \tau/x \) are pairs \((d, \alpha)\) where \( d \in \text{Ob}\mathcal{D} \) and \( \alpha : \tau(d) \to x \) is a morphism in \( \mathcal{C} \). A morphism from \((d, \alpha)\) to \((d', \alpha')\) is a morphism \( f : d \to d' \) in \( \mathcal{D} \), satisfying \( \alpha = \tau(f)\alpha' \). For each finite category \( \mathcal{D} \) we define \( \mathfrak{B}^\mathcal{D}_\ast = C_\ast(\text{Id}_\mathcal{D} / \ast, k) \). It is known that

\[ LK^C_{\mathcal{D}} \mathfrak{B}^\mathcal{D}_\ast = C_\ast(\tau/-, k). \]

As an example, for a group \( G \) the only overcategory \( \text{Id}_G / \ast \) is the Cayley graph and thus \( \mathfrak{B}^*_G \) is the bar resolution of \( k \) in group cohomology. For the convenience of the reader we recall from [28] how we obtain equivariant cohomology from category cohomology. Since we can explicitly calculate the unique overcategory as \( \pi/\ast \cong \text{Id}_G / \ast \times \mathcal{P} \), it follows that

\[ LK^G_{G \times \mathcal{P}} \mathfrak{B}^G_{G \times \mathcal{P}} \cong C_\ast(\pi/\ast, k) \cong B^G \otimes C_\ast(\mathcal{P}, k) \]

Thus we have the following chain isomorphisms and homotopy, for any \( M \in kG\text{-mod} \),

\[ \text{Hom}_{k}(G \times \mathcal{P}) \mathfrak{B}^G_{G \times \mathcal{P}, \kappa_M} \cong \text{Hom}_{kG}(LK^G_{G \times \mathcal{P}} \mathfrak{B}^G_{G \times \mathcal{P}, M}) \cong \text{Hom}_{kG}(\mathfrak{B}^G_{G \times \mathcal{P}, k}, \text{Hom}_{kG}(C_\ast(\mathcal{P}, k), M)). \]

Note that \( C_\ast(\mathcal{P}, k) \) is a finite complex and consists in each dimension of a permutation \( kG\)-module. From [10, VII.7] we see immediately \( \text{Ext}^*_{k(G \times \mathcal{P})}(\mathfrak{B}^G_{G \times \mathcal{P}}, \kappa_M) \cong H^G(\mathcal{P}, M) \) for any \( M \in kG\text{-mod} \).
In [28] we also established the following isomorphism
\[ \text{Ext}_k(G \times P)(\kappa_M, \mathfrak{M}) \cong \text{Ext}_k(G \times P)(k, \text{Hom}(\kappa_M, \mathfrak{M})). \]
Since \( \text{Hom}(\kappa_M, \kappa_N) \cong \kappa_{\text{Hom}(M,N)} \), the \( \text{Ext}_k(G \times P)(k, \mathfrak{M}) \)-action on
\[ \text{Ext}_k(G \times P)(\kappa_M, \kappa_M) \cong \text{Ext}_k(G \times P)(k, \text{Hom}(\kappa_M, \kappa_M)) \]
can be obtained from the canonical map \( k \to \text{Hom}(\kappa_M, \kappa_M) \cong \kappa_{\text{End}_k(M)} \), induced by \( 1_k \to \text{Id}_M \).
This is analogous to the group case.

### 3. Support varieties for modules

Let \( k \) be an algebraically closed field. Suppose \( \mathcal{C} = G \times \mathcal{P} \) is a transporter category. We have learned that \( \text{Ext}_k(G \times P)(k, k) \) is Noetherian, over which \( \text{Ext}_k(G \times P)(\mathfrak{M}, \mathfrak{N}) \) is a finitely generated module. The action factors through the natural ring homomorphism
\[ -\otimes \mathfrak{M} : \text{Ext}_k(G \times P)(k, k) \to \text{Ext}_k(G \times P)(\mathfrak{M}, \mathfrak{M}), \]
and
\[ -\otimes \mathfrak{N} : \text{Ext}_k(G \times P)(k, k) \to \text{Ext}_k(G \times P)(\mathfrak{N}, \mathfrak{N}). \]

Based on these, we are about to develop a support variety theory. Since a transporter category \( G \times \mathcal{P} \) is intimately related with \( G \), we will see our theory is a generalization of Carlson’s theory. Standard references for Carlson’s theory include [7, Chapter 5] and [15, Chapters 8, 9, 10].

#### 3.1. Basic definitions

For convenience, we shall assume \( G \times \mathcal{P} \) is connected, unless otherwise specified. It is equivalent to saying that \( k \in k(G \times \mathcal{P}) \)-mod is indecomposable or that
\[ \text{Ext}_k^0(G \times P)(k, k) = H^0(B(G \times \mathcal{P}), k) = k. \]
It does not mean that \( \mathcal{P} \) is connected, see Example 2.1.4.

**Definition 3.1.1.** Given a transporter category \( G \times \mathcal{P} \) and modules \( \mathfrak{M}, \mathfrak{N} \in k(G \times \mathcal{P}) \)-mod, we write \( I_{G \times P}(\mathfrak{M}) \) for the kernel of the map
\[ -\otimes \mathfrak{M} : \text{Ext}_k(G \times P)(k, k) \to \text{Ext}_k(G \times P)(\mathfrak{M}, \mathfrak{M}), \]
and \( V_{G \times P}(\mathfrak{M}) \), the support variety of \( \mathfrak{M} \), for the subvariety \( \text{MaxSpec}(H(G \times \mathcal{P})/I_{G \times P}(\mathfrak{M})) \) of \( V_{G \times P} = V_{G \times P}(k) \).

Since both \( \text{Ext}_k^*(G \times P)(\mathfrak{M}, \mathfrak{M}) \) and \( \text{Ext}_k^*(G \times P)(\mathfrak{N}, \mathfrak{N}) \) act on \( \text{Ext}_k^*(G \times P)(\mathfrak{M}, \mathfrak{N}) \) via Yoneda splice, we further define \( I_{G \times P}(\mathfrak{M}, \mathfrak{N}) \) as the annihilator of \( \text{Ext}_k(G \times P)(k, k) \) on \( \text{Ext}_k(G \times P)(\mathfrak{M}, \mathfrak{N}) \). Then we set \( V_{G \times P}(\mathfrak{M}, \mathfrak{N}) = \text{MaxSpec}(H(G \times \mathcal{P})/I_{G \times P}(\mathfrak{M}, \mathfrak{N})). \)

We say a subvariety of \( V_{G \times P} \) is *trivial* if it is \( \mathfrak{m} = \text{Ext}_k^*(G \times P)(k, k) \), the positive part of \( \text{Ext}_k(G \times P)(k, k) \).

Since \( I_{G \times P}(\mathfrak{M}, \mathfrak{N}) = I_{G \times P}(\mathfrak{M}) \), we get that \( V_{G \times P}(\mathfrak{M}, \mathfrak{N}) = V_{G \times P}(\mathfrak{M}) \), and that
\[ I_{G \times P}(\mathfrak{M}, \mathfrak{N}) \supset I_{G \times P}(\mathfrak{M}) \cap I_{G \times P}(\mathfrak{N}). \]
The latter implies
\[ V_{G \times P}(\mathfrak{M}, \mathfrak{N}) \subset V_{G \times P}(\mathfrak{M}) \cap V_{G \times P}(\mathfrak{N}). \]

Let \( \mathcal{P} \) be a \( G \)-poset and \( \mathcal{Q} \) an \( H \)-poset. Suppose there exists a group homomorphism \( \phi : H \to G \) as well as a functor \( \theta : \mathcal{Q} \to \mathcal{P} \) such that \( \phi(h)\theta = \theta \circ h \) for all \( h \in H \). For convenience we record such a map as \( (\phi, \theta) : (H, \mathcal{Q}) \to (G, \mathcal{P}) \). They induce a functor
\[ \Theta : H \times \mathcal{Q} \to G \times \mathcal{P}, \]
which in turn gives rise to a restriction map
\[ \text{res}_{H \times \mathcal{Q}}^{G \times \mathcal{P}} : \text{Ext}_k^*(G \times P)(k, k) \to \text{Ext}_k^*(H \times \mathcal{Q})(k, k) \]
and a map between varieties
\[ i_{H \times \mathcal{Q}}^{G \times \mathcal{P}} : V_{H \times \mathcal{Q}} \to V_{G \times \mathcal{P}}. \]
For instance, Lemma 2.4.1 dealt with \( (i, i) : (G_\times x) \to (G, \mathcal{P}) \) and \( (\text{Id}_G, i) : (G, [x]) \to (G, \mathcal{P}) \), where \( i \) stands for the inclusion.
**Example 3.1.2.** Suppose $G$ acts trivially on $\mathcal{P}$. Then $(\text{Id}_G, pt) : (G, \mathcal{P}) \to (G, \bullet)$ induce $\pi : G \times \mathcal{P} \to G$ and

$$\text{res}_{G \times \mathcal{P}}^G : \text{Ext}^*_G(k, k) \to \text{Ext}^*_{k(G \times \mathcal{P})}(k, k)$$

Since $\text{Ext}^*_G(k, k) \cong \text{Ext}^*_{kG}(k, k) \otimes \text{Ext}^*_{\mathcal{P}}(k, k)$ by the Künneth formula, and $\text{Ext}^*_G(k, k) = H^*(B\mathcal{P}, k)$ is finite-dimensional, $V_{G \times \mathcal{P}} = \prod_n V_G$, where $n$ is the number of connected components of $\mathcal{P}$. The restriction induces a natural map $V_{G \times \mathcal{P}} \to V_G$. With the assumption that $G \times \mathcal{P}$ is connected, we actually have $n = 1$ because $G \times \mathcal{P}$ is connected if and only if $\mathcal{P}$ is.

At this point, it seems to be a good idea to compare our theory with those of Carlson, Linckelmann and Snashall-Solberg. By putting our approach into the right context, we can better understand the ideas and see what properties we may expect. Afterwards, we will present various results concerning support varieties.

### 3.2. Carlson’s theory

When $\mathcal{P} = \bullet$ our theory is just the theory of Carlson. However, combining recent works in group and category cohomology, Carlson’s theory can be recovered in a more subtle way. To be explicit, if $\kappa_M \in k(G \rtimes \mathcal{P})$-mod for some $M \in kG$-mod, then we have a commutative diagram [29]

$$\begin{array}{ccc}
\text{Ext}^*_G(k, k) & \overset{\otimes \pi}{\longrightarrow} & \text{Ext}^*_G(M, M) \\
\text{res}_{G \times \mathcal{P}}^G \downarrow & & \downarrow \text{res}_{G \times \mathcal{P}}^G \\
\text{Ext}^*_G(k, k) \otimes \kappa_M & \longrightarrow & \text{Ext}^*_G(M, \kappa_M)
\end{array}$$

One can quickly deduce that the restriction map induces

$$\text{Ext}^*_G(k, k)/I_G(M) \to \text{Ext}^*_G(k, k) \otimes \kappa_M/I_G(k, \kappa_M),$$

and hence a finite (usually not surjective) map

$$V_{G \times \mathcal{P}}(\kappa_M) \to V_G(M).$$

If the Euler characteristic $\chi(\mathcal{P}, k)$ is invertible in $k$, then by using the Becker-Gottlieb transfer map [29], both vertical maps are injective. Furthermore, if we let $\mathcal{P} = \mathcal{S}_p$, the poset of non-identity $p$-subgroups of $G$, the left $\text{res}_{G \times \mathcal{S}_p}^G$ becomes an algebra isomorphism (see [10, Chap. X, Section 7] and [13, Part I, Sections 7 and 8] for instance). Hence we get

$$I_G(M) \cong I_{G \times \mathcal{S}_p}(\kappa_M) \quad \text{and} \quad V_G(M) \cong V_{G \times \mathcal{S}_p}(\kappa_M).$$

It means that various properties of $V_G(M)$ can be rewritten for $V_{G \times \mathcal{S}_p}(\kappa_M)$. As an example we have a tensor product formula

$$V_{G \times \mathcal{S}_p}(\kappa_M \otimes \kappa_N) = V_{G \times \mathcal{S}_p}(\kappa_M \otimes \kappa_N) = V_G(M \otimes N) = V_G(M \cap N) = V_{G \times \mathcal{S}_p}(\kappa_M) \cap V_{G \times \mathcal{S}_p}(\kappa_N).$$

Here the third equality comes from [7, Theorem 5.7.1]. One can similarly deduce other properties for $V_{G \times \mathcal{S}_p}(\kappa_M)$ but we shall leave it to the reader as they are just reformulations of known results for $V_G(M)$. Our interests really lie in $V_{G \times \mathcal{S}_p}(\mathfrak{M})$, or more generally $V_{G \times \mathcal{P}}(\mathfrak{M})$, for modules $\mathfrak{M} \neq \kappa_M$ for any $M \in kG$-mod.

In the above arguments, $\mathcal{S}_p$ may be replaced by various $G$-subposets which are $G$-homotopy equivalent to it, see [7, Section 6.6]. A typical example is $\mathcal{E}_p$, the $G$-poset of all elementary abelian $p$-subgroups of $G$. However we want to emphasize that most of our results are established without specifying a poset, see for instance Sections 4 and 5.

### 3.3. Varieties in blocks.

A group algebra can be written as a direct product of (indecomposable) block algebras $\prod_i b_i$. (Here for convenience we denote by $b$, instead of $kGb$, a block algebra.) Each indecomposable $kG$-module belongs to exactly one of these blocks in the sense that all but one block act as zero on it. The block that $k$ belongs to is called the principal block, denoted by $b_0$.

In [19], Linckelmann introduced to each block algebra $b$ a Noetherian graded commutative ring $H^*(b)$, called the block cohomology ring. Then he showed that there exists a natural injective homomorphism

$$H^*(b) \to \text{HH}^*(b),$$
and thus $H^*(b)$ acts on $\text{Ext}_A^*(M,N)$ via $HH^*(b) = \text{Ext}_A^*(b,b)$ (this action is explained in Section 3.4), if $M \in b\text{-mod}$. Particularly if $b_0$ is the principal block of a group algebra $kG$, then $H^*(b_0)$ is isomorphic to $H^*(G,k)$ and the above injection coincides with the composite of two canonical maps

$$H^*(b_0) \cong H^*(G,k) \rightarrow HH^*(kG) \rightarrow HH^*(b_0).$$

Based on these, he was able to define support varieties for modules of a block algebra as above in a natural way [20], being a refinement of Carlson’s theory. Most significantly Linckelmann’s work brought Hochschild cohomology into the theory of support varieties, which was taken up by Snashall and Solberg to develop a new support variety theory using Hochschild cohomology rings, see Section 3.4. Recently [21] Linckelmann has demonstrated that, for a block algebra $b$, the inclusion $H^*(b) \rightarrow HH^*(b)$ induces an isomorphism upon quotient out nilpotent elements. It implies that the two support variety theories, of Lincklemann based on the block cohomology ring and of Snashall-Solberg defined over the Hochschild cohomology ring of a block of a finite group, are identical. See [8] as well.

3.4. **Snashall-Solberg’s theory.** Snashall and Solberg [24] developed a support variety theory for certain finite-dimensional algebras using Hochschild cohomology rings. Let $A$ be a finite-dimensional algebra, and $M,N$ two finitely generated $A$-modules. Then there exists a natural action of the Hochschild cohomology ring on Ext groups via the following homomorphism

$$\phi_M = - \otimes_A M : \text{Ext}_A^*(A,A) \rightarrow \text{Ext}_A^*(M,M).$$

Based on Yoneda splice, one can introduce an action on $\text{Ext}_A^*(M,N)$ for any two $A$-modules.

For technical reasons, now let $A$ be an indecomposable algebra. Consequently $Z(A)$ becomes a commutative local algebra. Let $HH(A) = \text{Ext}_A^*(A,A)$ be defined in the same fashion as $H(C)$ in Section 2.4. Assume

(Fg.1) there is a graded Noetherian subalgebra $H \subset \text{Ext}_A^*(A,A)$ with $H^0 = \text{Ext}_A^0(A,A) = Z(A)$; and

(Fg.2) for any $M,N \in A\text{-mod}$, $\text{Ext}_A^*(M,N)$ is finitely generated over $H$.

Under the above assumptions, Snashall-Solberg considered the maximal ideal spectrum $V_H = \text{MaxSpec}H$. Since $H$ acts on $\text{Ext}_A^*(M,N)$ for any two $A$-modules, assuming $I_H(M,N)$ is the annihilator they defined a subvariety by

$$V_H(M,N) = \text{MaxSpec}(H/I_H(M,N)).$$

Write $I_H(M) = I_H(M,M)$. Then the (Hochschild) support variety of $M \in A\text{-mod}$ is given by

$$V_H(M) = \text{MaxSpec}(H/I_H(M)).$$

They showed that $V_H(M) = V_H(M,A/\text{Rad}A) = V_H(A/\text{Rad}A,M)$ and $V_H(A/\text{Rad}A) = V_H$. Various satisfactory properties were obtained in [24, 14], see Theorem 3.4.4 below for a summary. For future reference, we record the following definition.

**Definition 3.4.1.** A subvariety of $V_H$ is called **trivial** if it is $(\text{Rad}H^0,H^+)$, where $H^+$ consists of all the positive degree elements in $H$.

Unfortunately the above conditions (Fg.1) and (Fg.2) are not met by all finite-dimensional algebras, see [26]. Indeed they put restrictions on the algebras that one may consider. For example, two necessary conditions are that the algebra $A$ has to be Gorenstein [14], and that $\text{Ext}_A^*(A,A)$ itself must be Noetherian.

Although many algebras do not satisfy (Fg.1) and (Fg.2), we can show that Snashall-Solberg theory works for block algebras of a transporter category algebra $k(G \rtimes \mathcal{P})$. From [28] we know that $\text{Ext}^*_k(G \rtimes \mathcal{P})(\mathfrak{r},\overline{\mathfrak{r}})$ is a Noetherian graded commutative ring such that $\text{Ext}^*_k(G \rtimes \mathcal{P})(\mathfrak{r},\mathfrak{r})$ is finitely generated over it, for any pair of $\mathfrak{r},\overline{\mathfrak{r}} \in k(G \rtimes \mathcal{P})\text{-mod}$. It also was showed that $\text{Ext}^*_k(G \rtimes \mathcal{P})(k(G \rtimes \mathcal{P}),k(G \rtimes \mathcal{P}))$ is Noetherian. We shall prove that $\text{Ext}^*_k(G \rtimes \mathcal{P})(\mathfrak{r},\overline{\mathfrak{r}})$- and $\text{Ext}^*_k(G \rtimes \mathcal{P})(k(G \rtimes \mathcal{P}),k(G \rtimes \mathcal{P}))$-actions on $\text{Ext}^*_k(G \rtimes \mathcal{P})(\mathfrak{r},\overline{\mathfrak{r}})$ are compatible and hence it implies that $\text{Ext}^*_k(G \rtimes \mathcal{P})(\mathfrak{r},\overline{\mathfrak{r}})$ is finitely generated over the Hochschild cohomology ring as well.
Theorem 3.4.2. Let $\mathcal{M} \in k \mathcal{C}\text{-mod}$. We have a commutative diagram

\[
\begin{array}{c}
\text{Ext}^*_F(k, k) \\
\downarrow_{\cong} \\
\text{Ext}^*_K(k, k) \\
\end{array} \cong \begin{array}{c}
\text{Ext}^*_K(k, k) \\
\downarrow_{\otimes k \mathcal{M}} \\
\text{Ext}^*_K(k, k) \\
\end{array}
\]

with the left vertical map an injective algebra homomorphism.

This actually generalizes [26, Theorem A]. Let us first recall some other necessary constructions from [26]. For any category $\mathcal{C}$ there is a category of factorizations in $\mathcal{C}$, written as $F(\mathcal{C})$. The objects are the morphisms in $\mathcal{C}$. When a morphism $\alpha \in \text{Mor} \mathcal{C}$ is regarded as an object in $F(\mathcal{C})$, we will denote it by $[\alpha]$ to distinguish their roles. If $[\alpha], [\beta]$ are two objects in $F(\mathcal{C})$, then a morphism $\alpha \rightarrow [\beta]$ is a pair $(\mu, \gamma)$, $\mu, \gamma \in \text{Mor} \mathcal{C}$, such that $\beta = \mu \circ \gamma$ (that is, $\alpha$ is a factor of $\beta$). When $\mathcal{C}$ is a group, $F(\mathcal{C})$ plays the role of the diagonal subgroup $\Delta G \subset G \times G$. Indeed, there is a category equivalence $\Delta G \simeq F(G)$.

Given a morphism $\alpha$ in $\mathcal{C}$, we denote by $t(\alpha)$ and $s(\alpha)$ the target and source of $\alpha$. They induce two functors $t : F(\mathcal{C}) \rightarrow \mathcal{C}$ and $\nabla = (t, s) : F(\mathcal{C}) \rightarrow \mathcal{C}^e = \mathcal{C} \times \mathcal{C}^{op}$, fitting into the following commutative diagram

\[
\begin{array}{c}
F(\mathcal{C}) \\
\downarrow^{\nabla} \\
\mathcal{C} \\
\end{array} \cong \begin{array}{c}
\mathcal{C}^e \\
\downarrow^{p} \\
\mathcal{C}^{e} \\
\end{array}
\]

where $p$ is the projection. By definition $t$ and $s$ send $[\alpha]$ to the target and source of $\alpha$, respectively. In [26], we investigated the left Kan extensions $L K_F^C : k F(\mathcal{C})\text{-mod} \rightarrow k \mathcal{C}\text{-mod}$ and $L K_F^C : k F(\mathcal{C})\text{-mod} \rightarrow k \mathcal{C}^e\text{-mod}$, proving $L K_F^C [\alpha] \cong k$ and $L K_F^C \alpha \cong k \mathcal{C}$. Furthermore $L K_F^C$ induces an isomorphism $\text{Ext}^*_F(k, k) \rightarrow \text{Ext}^*_K(k, k)$, while $L K_F^C$ induces an injective algebra homomorphism $\text{Ext}^*_F(k, k) \rightarrow \text{Ext}^*_K(k, k)$. Especially $F(\mathcal{C})$ is connected if and only if $\mathcal{C}$ is.

At last $L K_F^C : k \mathcal{C}\text{-mod} \rightarrow k \mathcal{C}\text{-mod}$ is explicitly expressed as $L K_F^C \cong \otimes k \mathcal{M}$, where $k \mathcal{M}$ is the trivial $k \mathcal{C}\text{-module}$.

When $\mathcal{M} = k$, the lower horizontal map becomes the split surjection in [26, Theorem A].

**Proof.** Consider the bar resolution $\mathcal{B}^n_F(\mathcal{C}) \rightarrow k \rightarrow 0$, and a map $f : \mathcal{B}^n_F(\mathcal{C}) = C_n(\text{Id}_F(\mathcal{C}) / f, k) \rightarrow k$ representing an element $\xi \in \text{Ext}^n_F(k, k)$. We need to prove $L K_F^C \xi \otimes k \mathcal{M} = L K_F^C \xi \otimes \mathcal{M} \in \text{Ext}^n_F(k, k)$. We do it by explicit calculations.

Firstly, $L K_F^C \xi f : L K_F^C(\mathcal{B}^n_F(\mathcal{C})) = C_n(\nabla / / f, k) \rightarrow L K_F^C k = k \mathcal{C}$ is given on each $(x, y) \in \text{Ob} \mathcal{C}^e$ as

$$(L K_F^C(\mathcal{B}^n_F(\mathcal{C}))(x, y) : C_n(\nabla / / (x, y), k) \rightarrow k \mathcal{C}(x, y) = k \text{Hom}_{\mathcal{C}}(y, x)$$

by

$$([\alpha], (\beta, \gamma, \gamma)) \mapsto f(\alpha) \beta \alpha \gamma\gamma$$

Here we denote by $([\alpha], (\beta, \gamma, \gamma)) = ([\alpha_0], (\beta_0, \gamma_0)) \rightarrow \cdots \rightarrow ([\alpha_n], (\beta_n, \gamma_n))$ a base element of $C_n(\nabla / / (x, y), k)$, where $(\beta, \gamma) : \nabla([\alpha]) \rightarrow (x, y)$ is a morphism in $\mathcal{C}$. From $([\alpha], (\beta, \gamma, \gamma))$ we can extract a base element of $C_n(\text{Id}_F(\mathcal{C}) / [\beta_0, \alpha_0, \gamma_0], k)$, written as $[\alpha] = [\alpha_0] \rightarrow \cdots \rightarrow [\alpha_n]$, so our definition makes sense. Note that $\beta_0 = \cdots = \beta_0 \alpha_0 \gamma_0$.

Secondly, in a similar fashion, $L K_F^C \xi f : L K_F^C(\mathcal{B}^n_F(\mathcal{C})) = C_n(t / / f, k) \rightarrow L K_F^C k = k \mathcal{C}$ is given on each $x \in \text{Ob} \mathcal{C}$ as

$$(L K_F^C(\mathcal{B}^n_F(\mathcal{C}))(x) : C_n(t / / x, k) \rightarrow k \mathcal{C}(x) = k$$

by

$$([\alpha], (\mu, \mu)) \mapsto f(\alpha).$$

Here we denote by $([\alpha], (\mu, \mu)) = ([\alpha_0], \mu_0) \rightarrow \cdots \rightarrow ([\alpha_n], \mu_n)$ a base element of $C_n(t / / x, k) [\alpha]$ the encoded base element of $C_n(\text{Id}_F(\mathcal{C}) / [\mu_0, \alpha_0], k)$.
Thirdly, $LK^c_{F(C)}\xi \otimes_{kC} M$ is represented by $LK^c_{F(C)}f \otimes_{kC} Id_M$, while $LK^c_{F(C)}\xi \otimes_{kC} M$ is represented by some $(LK^c_{F(C)}f \otimes_{kC} Id_M) \circ \Phi$, provided that $P_n \to M_n \to 0$ is a projective resolution and $\Phi : P_n \to LK^c_{F(C)}B^C \otimes_{kC} M$ is a lifting of the identity map of $M$. Let us take $P_n = LK^c_{F(C)}B^C \otimes_{kC} M$ and construct $\Phi_n$, which has to be unique up to homotopy. To this end, for any $x \in \text{Ob} C$ and $n \geq 0$ we define

$$\Phi_n : 1_x \cdot LK^c_{F(C)}B^C \otimes_{kC} M \to 1_x \cdot LK^c_{F(C)}B^C \otimes_{kC} M(x)$$

by

$$([\alpha_*], (\beta_*), \gamma_*) \otimes (m_w) \mapsto ([\alpha_*], \beta_*) \otimes (\beta_* \alpha_* \gamma_*) \cdot m_y.$$ 

Here we write each element in $M = \bigoplus_{w \in \text{Ob} C} M(w)$ as $\sum_w m_w$, in which $m_w \in M(w)$. Directly from the definition one can verify $LK^c_{F(C)}f \otimes_{kC} Id_M = (LK^c_{F(C)}f \otimes_{kC} Id_M) \circ \Phi$. It means $LK^c_{F(C)}\xi \otimes_{kC} M = LK^c_{F(C)}\xi \otimes_{kC} M \in \text{Ext}^n_{kC}(M, M)$. 

To consider Snashall-Solberg’s theory, we want the algebra in question to be indecomposable. Suppose $k(G \rtimes \mathcal{P}) = \bigoplus_{b \in B} b$ is a decomposition into (indecomposable) block algebras, see for instance [1, Section 13]. Since $k(G \rtimes \mathcal{P})$ is Gorenstein, so are its blocks. Let $b$ be a block of $k(G \rtimes \mathcal{P})$. The above theorem implies that $\text{Ext}^n_{kC}(M, M)$ is a finitely generated module over the Noetherian ring $\text{Ext}^n_{kC}(b, b)$, if $M, \mathcal{M} \in b$-mod. It means Snashall-Solberg’s theory works perfectly for blocks of finite transporter category algebras.

Here we record some standard properties from Snashall-Solberg’s theory. For convenience, we write $V_b(M, \mathcal{M}) = V_{\mathcal{HH}(b)}(M, \mathcal{M})$ if $b$ is a block of $k(G \rtimes \mathcal{P})$ and $M, \mathcal{M} \in b$-mod. To be consistent, write $V_b(M) = V_b(\mathcal{M}, \mathcal{M})$. Some terminologies are recalled first.

**Definition 3.4.3.** Let $A$ be a finite-dimensional algebra and $P_* : M \to 0$ to the minimal projective resolution of $M$. Then the complexity of $M$ is

$$c_A(M) = \min \{s \in \mathbb{N} \mid r \in \mathbb{R} \text{ such that } \dim_k P_n \leq r n^{s-1}, \text{ for } n \geq 0\}$$

Let $(-)^\wedge = \text{Hom}_k(-, \mathcal{O}) : \mathcal{C}-\text{mod} \to k\text{C-}\text{mod} \to k\text{C-}\text{mod}$ be the $k$-dual functor. Recall that $BC \simeq BC^{\text{op}}$ and thus $\text{Ext}^n_{kC}(\mathcal{O}, \mathcal{O}) \simeq \text{Ext}^n_{kC}(\mathcal{O}, \mathcal{O})$. It implies that $\text{Ext}^n_{kC}(\mathcal{O}, \mathcal{O})$ also acts on $\text{Ext}^n_{kC}(\mathcal{O}, \mathcal{O})$, for any $\mathcal{M}, \mathcal{N} \in k(G \rtimes \mathcal{P})\text{-mod}$. Some of these constructions pass to every block algebra of $kC$.

The following statements are taken from Snashall-Solberg [24] and Erdmann-Holloway-Snashall-Solberg-Taillefer [14]. Note that a block of a transporter category algebra is Gorenstein, but in general neither selfinjective nor of finite global dimension.

**Theorem 3.4.4.** Let $G$ be a finite group, $\mathcal{P}$ a finite $G$-poset, and $k$ an algebraically closed field of characteristic $p$ dividing the order of $G$. Suppose $b$ is a block of $k(G \rtimes \mathcal{P})$ and $M, \mathcal{M}$ are two finitely generated modules of $b$. Then

1. $V_{\mathcal{M}}(\mathcal{M}) = V_{\mathcal{M}_1}(\mathcal{M}_1, \mathcal{S}) = \bigcup_{\mathcal{S}} V_{\mathcal{M}}(\mathcal{S}, \mathcal{M})$, where $\mathcal{S}$ runs over the set of all simple $b$-modules.

2. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence, then $V_{\mathcal{M}}(M_1) \subset V_{\mathcal{M}}(M_2) \cup V_{\mathcal{M}}(M_3)$ for $\{i, j, l\} = \{1, 2, 3\}$.

3. $V_{\mathcal{M}}(M_1 \oplus M_2) = V_{\mathcal{M}_1}(M_1) \cup V_{\mathcal{M}_2}(M_2)$.

4. $V_{\mathcal{M}}(\mathcal{M}^n) = V_{\mathcal{M}}(\Omega^n(\mathcal{M}))$ for any integer $n$ such that $\Omega^n(\mathcal{M}) \neq 0$.

5. $V_{\mathcal{M}}(\mathcal{M}, \mathcal{N}) = V_{\mathcal{M}^\mathcal{O}}(\mathcal{O} \mathcal{M}, \mathcal{O} \mathcal{N})$. Particularly $V_{\mathcal{M}}(\mathcal{M}) = V_{\mathcal{M}^\mathcal{O}}(\mathcal{O} \mathcal{M})$.

6. $\dim V_{\mathcal{M}}(\mathcal{M}) = c_0(\mathcal{M})$.

7. $V_{\mathcal{M}}(\mathcal{M})$ is trivial if and only if $\mathcal{M}$ has finite projective dimension if and only if $\mathcal{M}$ has finite injective dimension.

8. Let $\mathcal{A}$ be a homogeneous ideal in $\mathcal{HH}(b) = \text{Ext}^n_{kC}(b, b)$. Then there exists a module $\mathcal{M}_\mathcal{A} \in b$-mod such that $V_{\mathcal{M}_\mathcal{A}}(\mathcal{M}_\mathcal{A}) = V_{\mathcal{A}}$.

9. If $V_{\mathcal{M}}(\mathcal{M}) \cap V_{\mathcal{M}}(\mathcal{N})$ is trivial, then $V_{\mathcal{M}}(\mathcal{M}, \mathcal{N}) = 0$ for $i > \text{inj.dim}$.

10. If $V_{\mathcal{M}}(\mathcal{M}) = V_1 \cup V_2$ for some homogeneous non-trivial varieties $V_1$ and $V_2$ with $V_1 \cap V_2$ trivial, then $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $V_1(\mathcal{M}_1) = V_1$ and $V_2(\mathcal{M}_2) = V_2$.
Proof. The first five properties follow directly from the definition of support variety for a module of Snashall-Solberg, and can be found in [24]. The rest come from [14, Theorem 2.5, Theorem 4.4, Proposition 7.2, Theorem 7.3].

4. Standard properties of $V_{G \times P}(\mathcal{M})$

Snashall-Solberg’s theory on a block algebra of a transporter category algebra is quite satisfactory in many ways. However, Hochschild cohomology rings do not behave well comparing with ordinary cohomology rings. For example, since an algebra homomorphism does not necessarily induce a homomorphism between their Hochschild cohomology rings, certain important properties in Carlson’s theory are not expected to exist in Snashall-Solberg’s theory. This is one of the reasons why we believe $V_{G \times P}(\mathcal{M})$ has various advantages over $V_b(\mathcal{M})$ which we try to demonstrate in the rest of this paper.

4.1. Principal block. Let $k (G \rtimes P)$ be a transporter category algebra. Remember that we assume $G \rtimes P$ is connected, which is equivalent to saying that $k$ is indecomposable. We pay special attention to a special block of the transporter category algebra, closely related to our support variety theory.

Definition 4.1.1. Assume $C$ is a finite connected category. We call a block of $kC$ the principal block if the (indecomposable) trivial module $k$ belongs to it, and consequently name the block $b_0$.

Since one can take a minimal projective resolution of $k$ consisting of projective modules belonging to $b_0$, $\text{Ext}^*_{kC}(k, k) = \text{Ext}^*_{b_0}(k, k)$ is an invariant of the principal block, comparable to the group case, see for example [19].

Return to transporter category algebras. We claim

(i) $\text{Ext}^*_{k(G \rtimes P)}(k, k)$ is a (usually proper) subring of $\text{Ext}^*_{b_0}(b_0, b_0)$;

(ii) Snashall-Solberg’s theory is valid for the subring

$$\mathbb{H} := \langle Z(b_0), \text{Ext}^*_{k(G \rtimes P)}(k, k) \rangle \hookrightarrow \text{Ext}^*_{b_0}(b_0, b_0);$$

(iii) $\text{Ext}^*_{k(G \rtimes P)}(k, k) \hookrightarrow \langle Z(b_0), \text{Ext}^*_{k(G \rtimes P)}(k, k) \rangle$ induces an isomorphism after quotient out nilpotent elements.

The ordinary cohomology ring is known to be finitely generated [22, 27], and thus so are the rings $\langle Z(b_0), \text{Ext}^*_{k(G \rtimes P)}(k, k) \rangle$ and $\mathbb{H}$. Assuming these claims, along with Theorem 3.4.2, it guarantees that Snashall-Solberg’s theory can be implemented to $\mathbb{H} \subset \text{HH}(b_0)$ for the principal block of $k (G \rtimes P)$.

Claim (ii) comes from Theorem 3.4.2. Claim (iii) is easy to verify since the commutative local algebra $Z(b_0)$ quotients out the unique maximal ideal, that is, its nilradical, is exactly $k$ for $k$ is algebraically closed. To establish Claim (i), we have the following proposition.

Proposition 4.1.2. Let $C$ be a finite connected category and $b_0$ the principal block of $kC$. The injective homomorphism

$$\text{Ext}^*_{kC}(k, k) \hookrightarrow \text{Ext}^*_{b_0}(b_0, b_0);$$

induces an injective homomorphism

$$\text{Ext}^*_{kC}(k, k) \hookrightarrow \text{Ext}^*_{b_0}(b_0, b_0).$$

Proof. As a $kC^e$-module, $kC = \bigoplus b_i$. Since $LK_{kC}^e kC \cong k$, and $LK_{kC}^e b_i$ preserves direct sums, we see $LK_{kC}^e b_0 = b_0 \otimes_{kC} k = k$, and $LK_{kC}^e b_i = 0$ if $b_i$ is not principal. It implies that the split surjection in [26]

$$\text{Ext}^*_{kC}(kC, kC) \twoheadrightarrow \text{Ext}^*_{kC}(k, k)$$

induced by $LK_{kC}^e$, restricts to a split surjection $\text{Ext}^*_{b_0}(b_0, b_0) \twoheadrightarrow \text{Ext}^*_{kC}(k, k)$. Hence we are done.

From the preceding discussions, we get

$$V_{G \times P}(\mathcal{M}) = V_b(\mathcal{M})$$
for all $\mathfrak{M} \in b_0$-mod. Note that, for any $\mathfrak{M} \in b_0$-mod, there exists a finite surjective map $V_{b_0}(\mathfrak{M}) \rightarrow V_{G \times P}(\mathfrak{M})$.

We do not know yet when it becomes an isomorphism.

4.2. Standard properties. A subvariety of $V_{G \times P}$ is trivial if it is the ideal consisting of all positive degree elements of $\text{Ext}^k_{k(G \times P)}(\mathfrak{M}, \mathfrak{M})$.

**Theorem 4.2.1.** Let $G$ be a finite group, $P$ a finite $G$-poset, and $k$ an algebraically closed field of characteristic $p$ dividing the order of $G$. Suppose $\mathfrak{M}, \mathfrak{N}$ are two finitely generated modules of $k(G \times P)$. Then

1. $V_{G \times P}(\mathfrak{M}) = \bigcup_{\mathfrak{S}} V_{G \times P}(\mathfrak{M}, \mathfrak{S}) = \bigcup_{\mathfrak{M}} V_{G \times P}(\mathfrak{S}, \mathfrak{M})$, where $\mathfrak{S}$ runs over the set of all simple $k(G \times P)$-modules.

2. If $0 \rightarrow \mathfrak{M}_1 \rightarrow \mathfrak{M}_2 \rightarrow \mathfrak{M}_3 \rightarrow 0$ is an exact sequence, then $V_{G \times P}(\mathfrak{M}_i) \subseteq V_{G \times P}(\mathfrak{M}) \cup V_{G \times P}(\mathfrak{M}_l)$ for $\{i, j, l\} = \{1, 2, 3\}$.

3. $V_{G \times P}(\mathfrak{M}_1 \oplus \mathfrak{M}_2) = V_{G \times P}(\mathfrak{M}_1) \cup V_{G \times P}(\mathfrak{M}_2)$.

4. $V_{G \times P}(\mathfrak{N}) = V_{G \times P}(\Omega^n(\mathfrak{N}))$ for any integer $n$ such that $\Omega^n(\mathfrak{M}) \neq 0$.

5. $V_{G \times P}(\mathfrak{M}, \mathfrak{N}) = V_{G \times P}(\mathfrak{M}, \mathfrak{N}^\lor)$. Particularly $V_{G \times P}(\mathfrak{M}) = V_{G \times P}(\mathfrak{M}, \mathfrak{M}^\lor)$.

6. $\dim V_{G \times P}(\mathfrak{M}) = c_k(G \times P)$.

7. $V_{G \times P}(\mathfrak{M})$ is trivial if and only if $\mathfrak{M}$ has finite projective dimension if and only if $\mathfrak{M}$ has finite injective dimension.

8. Let $a$ be a homogeneous ideal in $\text{Ext}^k_{k(G \times P)}(\mathfrak{M}, \mathfrak{M})$. Then there exists a module $\mathfrak{M}_a \in k(G \times P)$-mod such that $V_{G \times P}(\mathfrak{M}_a) = V(a)$.

9a. If $V_{G \times P}(\kappa_{\text{Hom}_{(M,N)}}) = 0$ for $i > \dim P$.

9b. If $\mathfrak{M} \in b_0$-mod and $V_{G \times P}(\mathfrak{M}) \cap V_{G \times P}(\mathfrak{M}) = 0$ for $i > \dim P$.

10a. If $\chi(\mathfrak{P}) \equiv 1 (\text{mod } p)$ and $V_{G \times P}(\mathfrak{M}) = \mathfrak{V}_1 \cup \mathfrak{V}_2$ for some homogeneous non-trivial varieties $\mathfrak{V}_1$ and $\mathfrak{V}_2$ with $\mathfrak{V}_1 \cap \mathfrak{V}_2$ trivial, then $\kappa_M = \kappa_{\mathfrak{M}_1} \oplus \kappa_{\mathfrak{M}_2}$ with $V_{G \times P}(\kappa_M) = \mathfrak{V}_1$ and $V_{G \times P}(\kappa_{\mathfrak{M}_2}) = \mathfrak{V}_2$.

10b. If $\mathfrak{M} \in b_0$-mod such that $V_{G \times P}(\mathfrak{M}) = \mathfrak{V}_1 \cup \mathfrak{V}_2$ for some homogeneous non-trivial varieties $\mathfrak{V}_1$ and $\mathfrak{V}_2$ with $\mathfrak{V}_1 \cap \mathfrak{V}_2$ trivial, then $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$ with $V_{G \times P}(\mathfrak{M}_1) = \mathfrak{V}_1$ and $V_{G \times P}(\mathfrak{M}_2) = \mathfrak{V}_2$.

**Proof.** The first five properties follow directly from the definition of $V_{G \times P}(\mathfrak{M}, \mathfrak{N})$. The proofs are entirely analogous to the group case, see [7, 15].

The proof of (6) is exactly the same as that for groups, see for example [7, Proposition 5.7.2]. As for (7), one direction is straightforward. Also by Lemma 2.3.1 (2), it is equivalent to saying that $\mathfrak{M}$ is of finite injective dimension. Now let us assume $V_{G \times P}(\mathfrak{M})$ is trivial. Then by (6) $c_k(G \times P)(\mathfrak{M}) = 0$. It forces the minimal projective resolution of $\mathfrak{M}$ to be finite. Part (8) follows from $V_{G \times P}(\mathfrak{M}) = V_{G}(\mathfrak{M})$ if $\mathfrak{M} \in b_0$-mod. Let $a$ be a homogeneous ideal in $\text{Ext}^k_{k(G \times P)}(\mathfrak{M}, \mathfrak{M})$. Then we can define a homogeneous ideal $a' = (\text{Rad} Z(b_0), a)' \in \mathbb{H}$. From Snashall-Solberg’s theory [14, Theorem 4.4], the general form of Theorem 3.4.4 (8) for $b_0$, there exists a $b_0$-module $\mathfrak{M}_a$ such that $V_{G}(\mathfrak{M}_a) = V(a')$. But $V(a')$ is identified with $V(a)$ under the isomorphism $V_{G} \rightarrow V_{G \times P}$, and $V_{G \times P}(\mathfrak{M}_a) = V_{G}(\mathfrak{M}_a)$.

Since (9b) and (10b) are Theorem 3.4.4 (9) and (10) specialized to the principal block, we shall prove only (9a) and (10a).

To prove (9a), we notice that $\text{Ext}^k_{k(G \times P)}(\kappa_M, \kappa_N) \cong \text{Ext}^k_{k(G \times P)}(\kappa_{\text{Hom}(M,N)})$ (see Section 2.5). From (7), the assumption implies that $\kappa_{\text{Hom}(M,N)}$ has finite injective dimension. Hence the statement follows from Lemma 2.3.1.

As to (10a), we recall from Section 3.2 that if $\chi(\mathfrak{P}) \equiv 1 (\text{mod } p)$ then there exists a finite surjective map $V_{G \times P}(\kappa_M) \rightarrow V_G(M)$ for $M \in kG$-mod. Under the circumstance, let $V_i'$ be the images of $V_i$, for $i = 1, 2$, we see that $V_G(M) = V_1' \cup V_2'$ with $V_1' \cap V_2'$ trivial. By [7, Theorem 5.12.1], $M = M_1 \oplus M_2$ satisfying $V_G(M_1) = V_1'$ and $V_G(M_2) = V_2'$. Then $\kappa_M = \kappa_{M_1} \oplus \kappa_{M_2}$ and hence $V_{G \times P}(\kappa_M) = V_{G \times P}(\kappa_{M_1}) \cup V_{G \times P}(\kappa_{M_2})$. Moreover since the preceding map between varieties restricts to finite surjective maps $V_{G \times P}(\kappa_{M_i}) \rightarrow V_G(M_i)$ for $i = 1, 2$. It implies $V_i = V_{G \times P}(\kappa_{M_i})$ for $i = 1, 2$. □
4.3. Consequences of module filtrations. Recall that since \( G \times P \) is a finite EI-category, every \( k(G \times P) \)-module \( \mathcal{M} \) is constructed from atomic modules \( \mathcal{M}_x \), where \( \mathcal{M}_x(y) \cong \mathcal{M}(x) \) if \( y \cong x \) or zero otherwise. The following result says that only \( \mathcal{M} \)-objects contribute to the variety \( V_{G \times P}(\mathcal{M}) \). Moreover the non-isomorphisms do not play a big role.

**Proposition 4.3.1.** We have \( V_{G \times P}(\mathcal{M}) = \bigcup_{x} \mathcal{M}_x \), where \( x \) runs over the set of isomorphism classes of \( \mathcal{M} \)-objects. Particularly \( V_{G \times P}(\kappa_M) = \bigcup_{x} \mathcal{M}_x \), for any \( M \in kG\text{-mod}. \)

**Proof.** The containment \( V_{G \times P}(\mathcal{M}) \subset \bigcup_{x \in \text{Is}(G \times P)} V_{G \times P}(\mathcal{M}_x) \) follows from Theorem 4.2.1 (2). We establish the equality. Firstly we note that \( \text{Ext}^*_k(G \times P)(\mathcal{M}_x, \mathcal{M}_x) \cong \text{Ext}^*_k(G \times P)(\mathcal{M}_x, \mathcal{M}_x) \). Moreover we have a commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^*_k(G \times P)(\mathcal{M}_x, \mathcal{M}_x) & \xrightarrow{-\otimes \mathcal{M}} & \text{Ext}^*_k(G \times P)(\mathcal{M}, \mathcal{M}) \\
\text{Ext}^*_k(G \times P)(\mathcal{M}_x, \mathcal{M}_x) & \xrightarrow{-\otimes \mathcal{M}_x} & \text{Ext}^*_k(G \times P)(\mathcal{M}_x, \mathcal{M}_x)
\end{array}
\]

To establish the above commutative diagram, we can represent cohomology classes by \( n \)-fold extensions and notice that both restrictions and \( -\otimes \)-are exact.

The \( \text{Ext}^*_k(G \times P)(\mathcal{M}_x, \mathcal{M}_x) \)-action on \( \text{Ext}^*_k(G \times P)(\mathcal{M}_x, \mathcal{M}_x) \) factors through the action by \( \text{Ext}^*_k(G \times P)(\mathcal{M}_x, \mathcal{M}_x) \). Hence we have \( V_{G \times P}(\mathcal{M}_x) = I_{G \times P}(\mathcal{M}_x) \). Based on the same diagram we see that \( I_{G \times P}(\mathcal{M}) \) kills \( \text{Ext}^*_k(G \times P)(\mathcal{M}_x, \mathcal{M}_x) \). It means \( V_{G \times P}(\mathcal{M}_x) \subset V_{G \times P}(\mathcal{M}) \). \( \square \)

We remind the reader that since \( G \times P \) is equivalent to \( G_x \), \( V_{G \times P}(\mathcal{M}_x) \) can be identified with \( V_{G \times P}(\mathcal{M}(x)) \).

**Corollary 4.3.2.** We have \( V_{G \times P}(\mathcal{M}) = \bigcup_{x} \mathcal{M}_x \), where \( x \) runs over the set of isomorphism classes of \( \mathcal{M} \)-objects, and hence \( V_{G \times P}(\kappa_M) = \bigcup_{x} \mathcal{M}_x \mathcal{M}_x(M) \) for any \( M \in kG\text{-mod}. \)

If \( G \) acts trivially on a connected poset \( P \), then \( V_{G \times P}(\mathcal{M}) = \bigcup_{x} \mathcal{M}_x \mathcal{M}_x(M) \). Especially \( V_{G \times P}(\kappa_M) = V_{G}(M) \) for any \( M \in kG\text{-mod} \).

The reader may go back and have another look at Example 3.1.2. We note that if \( x \cong y \) in \( G \times P \), then there exists an element \( g \in G \) inducing an isomorphism by conjugation \( G_x \rightarrow G_y \). It implies that \( V_{G \times P}(\mathcal{M}(x)) \cong V_{G \times P}(\mathcal{M}(y)) \cong V_{G \times P}(\mathcal{M}_x) \) and

\[
i_{G \times P}(\mathcal{M}(x)) = \mathcal{M}_x \mathcal{M}_x(M(\mathcal{M}(y))) = \mathcal{M}_x \mathcal{M}_x(M(\mathcal{M}(x))) = \mathcal{M}_x \mathcal{M}_x(M).\]

**Corollary 4.3.3.** If \( \mathcal{M} \in k(G \times P)\text{-mod} \) and \( H \subset G \) is a subgroup, then \( i_{G \times P}(H \times P)(\mathcal{M}) \subset V_{G \times P}(\mathcal{M}) \).

**Proof.** By the preceding result, we have \( V_{H \times P}(\mathcal{M}) = \bigcup_{x} i_{H \times P}(H \times x)(\mathcal{M}(x)) \), and \( V_{G \times P}(\mathcal{M}) = \bigcup_{x} \mathcal{M}_x \mathcal{M}_x(M(\mathcal{M}(x))) \). By \( \mathcal{M} \) runs over the set of all \( \mathcal{M} \)-objects.

Since \( i_{H \times P}(H \times x)(\mathcal{M}(x)) = i_{G \times P}(G \times x)(\mathcal{M}(x)) \), it follows that \( i_{G \times P}(G \times x)(\mathcal{M}(x)) \) is known in Carlson’s theory, our claim follows. \( \square \)

In a similar fashion we can show if \( Q \) is a \( G \)-subposet of \( P \) then

\[
i_{G \times Q}(H \times Q)(\mathcal{M}) \subset V_{G \times P}(\mathcal{M})\]

for any \( \mathcal{M} \in k(G \times P)\text{-mod} \).
5. **Further properties**

In this section we shall deal with comparing varieties of categories. The main results are the generalized Quillen stratification and its consequences.

Bear in mind that, for the sake of simplicity, if \( D \to C \) is a functor and \( \mathcal{M} \) is a \( kC \)-module, when it will not cause confusions, we often regard \( \mathcal{M} \) (instead of writing \( \text{Res}_D^C(\mathcal{M}) \)) as a \( kD \)-module, even though its underlying vector space structure usually changes.

5.1. **Quillen stratification.** In [22, 23] Quillen worked with \( G \)-spaces and equivariant cohomology rings. Here we are interested in \( G \)-spaces which are classifying spaces of finite \( G \)-posets. In order to make a smooth transition from \( G \)-spaces and equivariant cohomology to \( G \)-posets and transporter category cohomology, we first recall some of the original constructions and then restrict to our case.

Given a \( G \)-space \( X \) and a \( H \)-space \( Y \), a morphism from \( (H,Y) \to (G,X) \) is a pair \((\phi,F)\) such that \( \phi : H \to G \) is a group homomorphism and \( \theta : Y \to X \) is a continuous map satisfying the condition that

\[ \phi(h)\theta(y) = \theta(hy), \quad \forall h \in H, y \in Y. \]

Such a morphism induces a continuous map \( \Theta : EH \times_H Y \to EG \times_G X \) and thus a restriction map between equivariant cohomology

\[ \text{res}_{H,Y}^G : H^*_G(X) \to H^*_H(Y). \]

One can compare these constructions with those introduced before Example 3.1.2.

In Quillen’s papers [22, 23], it is proved that if \( X \) is a \( G \)-space, either compact or paracompact with finite cohomological dimension, then there exists an \( F \)-isomorphism

\[ q = q_{G,X} : H^*_G(X) \to \lim_{\leftarrow A_p(X)} \mathbb{H}^*(E) \]

where \( A_p(X) \) is called the Quillen category and its objects will be named Quillen pairs for the \( G \)-space \( X \) in this article. More explicitly, the objects in \( A_p(X) \) are of the form \((E,C)\), where \( E \) is an elementary abelian \( p \)-subgroup of \( G \) and \( C \) is a (non-empty) connected component of \( \mathbb{X}^E \). A morphism from \((E',C') \to (E,C)\) is a pair \((\phi,i)\) with \( \phi = c_g : E' \to E \) for some element \( g \in G \) and \( i : gC \to C' \) an inclusion. Particularly if \( q \) can be chosen to be the identity element of \( G \), then we call \((E',C')\) a Quillen subpair of \((E,C)\), and write \((E',C') \leq (E,C)\). In this way, all Quillen pairs form a poset \( \mathcal{Q}_p \), with obvious \( G \)-action.

**Remark 5.1.1.** Another way to construct the Quillen category is to define it as a quotient category of the transporter category \( G \ltimes \mathcal{Q}_p \), where \( \mathcal{Q} \) is the \( G \)-poset of all Quillen pairs for \( X \), through

\[ \text{Hom}_{\mathcal{Q}_p}(((E',C'), (E,C))) = \text{Hom}_{G \times \mathcal{Q}_p}((E',C'), (E,C)) / C_G(E',C') \]

where \( C_G(E',C') = \{ g \in G \mid gC' = C', ghg^{-1} = h \text{ for all } h \in E \} \) is the centralizer of \((E',C')\). We also denote by \( N_G(E',C') = G(E',C') \) the normalizer of \((E',C')\) and \( W_G(E',C') = N_G(E',C') / C_G(E',C') \) the Weyl group of \((E',C')\).

Quillen’s map \( q \) is induced by \((E,\bullet) \to (E,C) \to (G,X)\), where \( \bullet \) is sent into \( C \). Since \( E \) acts trivially on \( C \), the choice of an embedding \( \bullet \to C \) does not matter. In fact in any case this map induces a surjective map \( H^*_E(C) \to H^*_E(\bullet) \) which becomes an isomorphism after quotient out nilpotents in both rings. Consequently \( \text{MaxSpec} H_E(C) = \text{MaxSpec} H_E(\bullet) \), that is, \( V_{E,C} = V_{E,\bullet} = V_E \).

Let \( V_{G,X} = \text{MaxSpec} H_G(X) \). The geometric version of Quillen’s map is

\[ V_{G,X} = \bigcup_{(E,C)} \iota_{E,C}^{G,X} V_{E,C} = \bigcup_{(E,C)} \iota_{E,C}^{G,X} \iota_{E,\bullet}^{E,C} V_{E,\bullet}. \]

where \( \iota_{E,C}^{G,X} : V_{E,C} \to V_{G,X} \) is induced by \( \text{res}_{E,C}^{G,X} \). Based on the observation that each morphism \((E',C') \to (E,C)\) in the category \( A_p(X) \) induces a ring homomorphism \( H^*_E \to H^*_E \), which is compatible with the two maps \( H^*_G(X) \to H^*_E \) and \( H^*_G(X) \to H^*_E \) coming from \((E',C') \to (G,X)\).
and \((E, C) \to (G, X)\), Quillen then continued to demonstrate that \(V_{G,X}\) is a disjoint union of some locally closed subvarieties \(V_{E,C}^{G,X^+}\) by examining more closely the following diagram

\[
\begin{array}{ccc}
V_{E'} & \xrightarrow{\iota_{E',C'}} & V_{E,C} = V_E \\
\downarrow{\iota_{E',C'^+}} & & \downarrow{\iota_{E,C}} \\
V_{G,X} & & \\
\end{array}
\]

Here the horizontal map is \(\iota_{E'}\) (corresponding to \(H^*_E \to H^*_E\)), not the senseless \(\iota_{E,C'}\), since a morphism in \(\mathbb{A}_p(X)\) is different from the morphisms introduced in the second paragraph of this section.

**Theorem 5.1.2 (Quillen).** The dimension of \(V_{G,X}\) equals the maximum \(p\)-rank of an elementary abelian \(p\)-subgroup from Quillen pairs.

We shall come back to Quillen’s results shortly after we sort out all terminologies.

Now we turn to the case of \(X = BP\) for a finite \(G\)-poset \(\mathcal{P}\). Let us remind the reader of several relevant results.

(a) Suppose a \(G\)-space \(X\) is either compact or paracompact with finite cohomological dimension. If \(H^*(X)\) is finite-dimensional then so is \(H^*(C)\) for all possible \(C\) from Quillen pairs, [22, Corollary 4.3].

(b) \((BP)^g = B(P^g)\) (fixed points) for any \(g \in G\), [12, Proposition 66.3];

(c) Let \(\mathcal{S}_p\) and \(\mathcal{E}_p\) be the \(G\)-posets of non-identity \(p\)-subgroups and of elementary abelian \(p\)-subgroups of \(G\). Then obviously any elementary abelian \(p\)-subgroup \(E\) has the property that \((BP_E)^g\) and \((BE_p)^E\) are non-empty.

When \(X = BP\) for a \(G\)-poset \(\mathcal{P}\) and \(M \in kG\)-mod, we know \((\text{Id}_G, pt) : (G, BP) \to (G, \bullet)\) induces the following restriction map

\[
\text{res} : \text{Ext}^1_{kG}(k, k) = H^*(G) \to H^*_G(BP) = \text{Ext}^*_G(G, \mathcal{P})(\underline{k}, \underline{k}),
\]

which is injective if the Euler characteristic \(\chi(\mathcal{P}) = \chi(BP)\) is invertible in the base field, by invoking the Becker-Gottlieb transfer [29]. A typical example of such \(G\)-posets is \(\mathcal{S}_p\), the poset of non-identity \(p\)-subgroups in \(G\), since Brown’s theorem [7, Corollary 6.7.4] says that \(\chi(\mathcal{S}_p) \equiv 1 \mod p\). There are several well-known subposets that are \(G\)-homotopy equivalent to it, and thus possess the same property.

Consider the canonical map \((\text{Id}_E, pt) : (E, X) \to (E, \bullet)\), where \(E\) is an elementary abelian \(p\)-group. A more involved result says that the restriction \(\text{res} : H^*_E(\bullet) \to H^*_E(X)\) being injective is equivalent to saying that \(X^E\) is non-empty, [18, IV.1, Corollary 1]. If \(X = BP\) such that \(\chi(\mathcal{P}) \equiv 1 \mod p\), by [9, Capter III, Theorem 4.3] we get \(\chi(\mathcal{P}^E) \equiv 1 \mod p\). This implies that \(BP^E \neq \emptyset\) and that \(\text{res}\) is injective, matching the observation we made using the Becker-Gottlieb transfer.

From now on we focus on the case \(X = BP\) for a finite \(G\)-poset \(\mathcal{P}\). By doing so, we restrict to a special case of Quillen in order to get rid of topology and unveil the underlying algebraic/categorical constructions. Most importantly we gain the freedom to work with varieties of modules.

Let \(E\) be an elementary abelian \(p\)-subgroup of \(G\). Because \(BP_E = B(\mathcal{P}^E)\), a Quillen pair for the \(G\)-space \(BP\) is of the form \((E, BC)\) in which \(C\) is a connected component of the poset \(\mathcal{P}^E\). Thus under the circumstance, we may write each Quillen pair as \((E, C)\) with \(C\) a connected subposet of \(\mathcal{P}^E\), and call \((E, C)\) a Quillen pair for the \(G\)-poset \(\mathcal{P}\).

Quillen stratification in our setting says (to be consistent with terminologies in Section 3.1 we choose to write \(i_{E,C}^{G,BP}\) for \(i_{E,B}^{G,BP}\) and accordingly \(V_{G,\mathcal{P}} \) for \(V_{G,BP}\) etc.)

\[
V_{G,\mathcal{P}} = \bigcup_{(E, C)} i_{E,C}^{G,\mathcal{P}} V_{E, C} = \bigcup_{(E, C)} i_{E,C}^{G,\mathcal{P},E, C} V_{E, C}.
\]

Note that \(E \prec C = E \times C\) is a subcategory of \(G \prec \mathcal{P}\), and \(V_{E, C} \cong V_{E, \bullet}\).
Remark 5.1.3. The theorem of Alperin-Avrunin-Evens [15, Theorem 8.3.1] says that
\[ V_G(M) = \bigcup_{E \subseteq G} \iota_E^G V_E(M), \]
where \(E\) runs over the set of all elementary abelian subgroups of \(G\) and \(M\) is a \(k\)-\(G\)-module. Since \(G_x\) is the isotropy group of \(x \in \text{Ob}\,\mathcal{P}\), we know \(x \in \text{Ob}\,\mathcal{P}^E\) if and only if \(E \subseteq G_x\). Thus if \((E, C)\) is a Quillen pair, any object \(x \in \text{Ob}\,\mathcal{C}\) satisfies the condition that \(E \subseteq G_x\). Thus
\[ \iota_{E \times x}^G V_{E \times x} \subseteq V_{G \times [x]} = V_{G \times x} \]
Based on the Alperin-Avrunin-Evens’ theorem, we can rewrite Proposition 4.3.1 and Corollary 4.3.2.
\[ V_{G \times \mathcal{P}}(\mathcal{M}) = \bigcup_{x \in C} \iota_{C}^{G \times \mathcal{P}} V_{G \times [x]}(\mathcal{M}) = \bigcup_{y \in G \times [y]} V_{G \times [y]}(\mathcal{M}(y)) = \bigcup_{y \in G \times [y]} V_{E \times [y]}(\mathcal{M}(y)) = \bigcup_{y \in \mathcal{E}} \iota_{\mathcal{E}}^{G \times \mathcal{P}} V_{E \times \mathcal{C}}(\mathcal{M}). \]
These shall be useful for us in the sequel. We note that if \(((E, C) \simeq (E', gC)) for some \(g \in C\), that is, two Quillen pairs are isomorphic in \(G \in \mathcal{P}\), then \(\iota_{E \times C}^{G \times \mathcal{P}} V_{E \times \mathcal{C}}(\mathcal{M}) = \iota_{E' \times C}^{G \times \mathcal{P}} V_{E' \times \mathcal{C}}(\mathcal{M})\).

Let \((E, C)\) be a Quillen pair and \(E'\) a subgroup of \(E\). Then we denote by \(C|E'\) the connected component of \(\mathcal{P}^E\) which contains \(C\), determining a unique Quillen subpair \(((E', C|E')) \subseteq (E, C)\). One can easily verify
\[ \iota_{E \times E' \times x}^E V_{E' \times \mathcal{C}}(\mathcal{M}) = \iota_{E \times E' \times E' \times x}^E V_{E' \times E'}(\mathcal{M}) = V_{E' \times x} \rightarrow V_{E \times x} = V_E, \]
for every \(x \in \text{Ob}\,\mathcal{C}\).

Lemma 5.1.4. For any Quillen pair \((E, C)\) and \(\mathcal{M} \in k(E \times C)\)-mod
\[ \iota_{E \times C}^E V_{E' \times \mathcal{C}}(\mathcal{M}) = \{ \iota_{E \times C}^E V_{E' \times \mathcal{C}}(\mathcal{M}) \} \cap V_{E \times C}(\mathcal{M}). \]

Proof. By Corollary 4.3.2, \(V_{E \times C}(\mathcal{M}) = \bigcup_{x \in \text{Ob}\,\mathcal{C}} \iota_{E \times x}^{E \times \mathcal{C}} V_{E \times x}(\mathcal{M}(x))\) because the isotropy subgroup of every \(x \in \mathcal{C}\) is exactly \(E\). Then
\[ \{ \iota_{E \times C}^E V_{E' \times \mathcal{C}}(\mathcal{M}(x)) \} \cap \bigcup_{E \times x} \iota_{E \times x}^{E \times \mathcal{C}} V_{E \times x}(\mathcal{M}(x)) = \bigcup_{E \times x} \{ \iota_{E \times x}^{E \times \mathcal{C}} V_{E' \times \mathcal{C}}(\mathcal{M}(x)) \} \]
for every \(x \in \text{Ob}\,\mathcal{C}\). The second equality uses [7, Proposition 5.7.7]. Consequently we get
\[ \{ \iota_{E \times C}^E V_{E' \times \mathcal{C}}(\mathcal{M}(x)) \} \cap \bigcup_{E \times x} \{ \iota_{E \times x}^{E \times \mathcal{C}} V_{E \times \mathcal{C}}(\mathcal{M}(x)) \} = \bigcup_{x \in \text{Ob}\,\mathcal{C}} \{ \iota_{E \times x}^{E \times \mathcal{C}} V_{E' \times \mathcal{C}}(\mathcal{M}(x)) \} \]
for every \(x \in \text{Ob}\,\mathcal{C}\). The last equality is true since \(\mathcal{M}(y) = 0\) if \(y \in \text{Ob}\,(C|E')\) \(-\) \(\text{Ob}\,\mathcal{C}\), by definition. \(\square\)

We set
\[ V_{E \times C}^+ = V_{E \times C} - \bigcup_{E' \subseteq E} \iota_{E \times C}^{E \times \mathcal{C}} V_{E' \times C}(\mathcal{M}) \]
\[ V_{E \times C}(\mathcal{M})^+ = V_{E \times C}^+ \cap V_{E \times C}(\mathcal{M}) \]
\[ V_{E \times C}(\mathcal{M})^+ = \iota_{E \times C}^{G \times \mathcal{P}} V_{E \times \mathcal{C}} \]
\[ V_{E \times C}(\mathcal{M})^+ = \iota_{E \times C}^{G \times \mathcal{P}} V_{E \times \mathcal{C}}(\mathcal{M})^+ \]

The following is established by Quillen for arbitrary \(\mathcal{P}\) and \(\mathcal{M} = \mathcal{M}[23]\) and by Avrunin-Scott for \(\mathcal{P} = \mathcal{M}\) and arbitrary modules [5]. Our generalization is based on both of these special cases.
Theorem 5.1.5 (Stratification). Let $G \ltimes P$ be a finite transporter category, $k$ an algebraically closed field of characteristic $p$ dividing the order of $G$ and $M \in k(G \ltimes P)$-mod. Then
\[
V_{G \ltimes P}(\mathcal{M}) = \bigsqcup_{(E,C)} V_{E \times C}^{G \ltimes P}(\mathcal{M}),
\]
where the index runs over the set of isomorphism classes of Quillen pairs. Moreover we have a homeomorphism
\[
V_{E,C}(\mathcal{M})^+ / W_G(E,C) \simeq V_{E \times C}^{G \ltimes P}(\mathcal{M})^+.
\]

Proof. We know for $M = k$ the Quillen stratification
\[
V_{G \ltimes P}(\mathcal{M}) = \bigsqcup_{(E,C)} V_{E \times C}^{G \ltimes P}(\mathcal{M}),
\]
with
\[
V_{E \times C}^{G \ltimes P}(\mathcal{M})^+ / W_G(E,C) \simeq V_{E \times C}^{G \ltimes P}(\mathcal{M})^+.
\]
From Remark 5.1.3 we have
\[
V_{G \ltimes P}(\mathcal{M}) = \bigsqcup_{(E,C)} W_G(E,C) V_{E \times C}^{G \ltimes P}(\mathcal{M}).
\]
Since Lemma 5.1.4 implies
\[
V_{E \times C}^{G \ltimes P}(\mathcal{M})^+ = V_{E \times C}(\mathcal{M}) - \bigsqcup_{E \subseteq E'} V_{E \times C}(\mathcal{M})^+
\]
we get
\[
V_{G \ltimes P}(\mathcal{M}) = \bigsqcup_{(E,C)} V_{E \times C}^{G \ltimes P}(\mathcal{M})^+.
\]
It follows from Quillen’s original result that, if $(E,C)$ runs over the set of isomorphism classes of Quillen pairs,
\[
V_{G \ltimes P}(\mathcal{M}) = \bigsqcup_{(E,C)} V_{E \times C}^{G \ltimes P}(\mathcal{M})^+,
\]
and that there is a homeomorphism
\[
V_{E,C}(\mathcal{M})^+ / W_G(E,C) \simeq V_{E \times C}^{G \ltimes P}(\mathcal{M})^+.
\]

5.2. Subcategories and tensor products. Here we record a few more expected interesting results as consequences of the Quillen stratification.

Theorem 5.2.1. Suppose $P$ is a $G$-poset and $H$ is a subgroup of $G$. Then, for any $\mathcal{M} \in k(G \ltimes P)$-mod,
\[
\delta_{H \times P}^{-1} V_{H \times P}(\mathcal{M}) = V_{H \times P}(\mathcal{M}).
\]

Proof. By Corollary 4.3.3 $\delta_{H \times P} V_{H \times P}(\mathcal{M}) \subset V_{H \times P}(\mathcal{M})$. Given a Quillen pair $(E,C)$ for $H \ltimes P$, $(E,C)$ is also a Quillen pair for $G \ltimes P$ with $W_H(E,C) \subset W_G(E,C)$. By Theorem 5.1.5, there is a commutative diagram
\[
\begin{CD}
V_{E \times C}(\mathcal{M})^+ / W_H(E,C) @>>> V_{E \times C}(\mathcal{M})^+ / W_G(E,C) \\
@VVV @VVV \\
V_{E \times C}^{H \times P}(\mathcal{M})^+ @>>> V_{E \times C}^{G \ltimes P}(\mathcal{M})^+
\end{CD}
\]
Since the upper horizontal map is surjective and the vertical maps are homeomorphisms, the lower horizontal map is surjective. We are done. 

We could not find a way to generalize the above statement for an arbitrary sub-transporter category of $G \ltimes P$. However we have the following.
Proposition 5.2.2. Suppose $\mathcal{P}$ is a $G$-poset and $\mathcal{Q}$ is a $G$-subposet. Then, for any $M \in kG$-mod,

$$\iota^G_{\mathcal{X} \times \mathcal{P}}^{-1} V_{G \times \mathcal{P}}(\kappa_M) = V_{G \times \mathcal{Q}}(\kappa_M).$$

**Proof.** The proof is similar to the previous one. If $(E, \mathcal{C})$ is a Quillen pair for $G \propto \mathcal{Q}$, then there exists a unique Quillen pair $(E, \mathcal{D})$ for $G \propto \mathcal{P}$. We note that $V_{E \times \mathcal{C}}(\kappa_M)^+ = V_{E \times \mathcal{D}}(\kappa_M)^+$. Also by definition if $g\mathcal{C} = \mathcal{C}$ then $g\mathcal{D} = \mathcal{D}$ as well, for any $g \in G$. It implies that $W_G(E, \mathcal{C}) \subset W_G(E, \mathcal{D})$. \(\square\)

Recall that for any category $\mathcal{D}$ there is a diagonal functor $\Delta: \mathcal{D} \rightarrow \mathcal{D} \times \mathcal{D}$. Then image of $\mathcal{D}$ is written as $\Delta \mathcal{D}$. If $\mathcal{C}$ is a connected poset, then so is $\mathcal{C} \times \mathcal{C}$. Suppose $G \propto \mathcal{P}$ is a transporter category and $(E, \mathcal{C})$ is a Quillen pair for the $G$-poset $\mathcal{P}$. Then $(\Delta E, \mathcal{C} \times \mathcal{C})$ is a Quillen pair for the $(G \times G)$-poset $\mathcal{P} \times \mathcal{P}$. Note that from Remark 2.1.2 $(G \times G) \propto (\mathcal{P} \times \mathcal{P}) \cong (G \propto \mathcal{P}) \times (G \propto \mathcal{P})$.

**Corollary 5.2.3.** Suppose $G$ is a finite group and $\mathcal{P}$ is a finite $G$-poset. If $H$ is a subgroup of $G$ and $\mathcal{Q}$ is a $H$-subposet of $\mathcal{P}$, then $\iota^G_{\mathcal{X} \times \mathcal{P}}^{-1} V_{G \times \mathcal{P}}(\kappa_M) = V_{H \times \mathcal{Q}}(\kappa_M)$. Particularly for any $k(G \times G)$-module $N$

$$\iota^G_{\mathcal{X} \times \mathcal{P}}^{-1} V_{(G \times G) \times \mathcal{P}}(\kappa_N) = V_{G \times \mathcal{P}}(\kappa_N).$$

**Proof.** We apply Theorem 5.2.1 and Proposition 5.2.2 to the following three transporter categories $H \propto \mathcal{Q} \subset H \propto \mathcal{P} \subset G \propto \mathcal{P}$.

For the special case, we consider $G \propto \mathcal{P} \cong \Delta G \propto \Delta \mathcal{P} \subset \Delta G \propto (\mathcal{P} \times \mathcal{P}) \subset (G \times G) \propto (\mathcal{P} \times \mathcal{P})$. \(\square\)

We need a technical result for the last main result.

**Proposition 5.2.4.** Suppose $\mathcal{D}$ and $\mathcal{C}$ are two finite categories with finitely generated ordinary cohomology rings. If $\mathcal{M}, \mathcal{M}' \in k\mathcal{D}$-mod and $\mathcal{N}, \mathcal{N}' \in k\mathcal{C}$-mod, then

$$\text{Ext}^i_{k(\mathcal{D} \times \mathcal{C})}(\mathcal{M} \otimes \mathcal{N}, \mathcal{M}' \otimes k \mathcal{N}') \cong \text{Ext}^i_{k\mathcal{D}}(\mathcal{M}, \mathcal{M}') \otimes_k \text{Ext}^i_{k\mathcal{C}}(\mathcal{N}, \mathcal{N}').$$

**Proof.** Let $\mathcal{P}^\mathcal{D} \rightarrow \mathcal{M} \rightarrow 0$ and $\mathcal{P}^\mathcal{C} \rightarrow \mathcal{N} \rightarrow 0$ be two projective resolutions. Then $\mathcal{P}^\mathcal{D} \otimes_k \mathcal{P}^\mathcal{C} \rightarrow \mathcal{M} \otimes_k \mathcal{N} \rightarrow 0$ is a projective resolution of the $k(\mathcal{D} \times \mathcal{C})$-module $\mathcal{M} \otimes_k \mathcal{N}$. The isomorphism follows from

$$\text{Hom}_{k(\mathcal{D} \times \mathcal{C})}(\mathcal{P}^\mathcal{D} \otimes_k \mathcal{P}^\mathcal{C}, \mathcal{M}' \otimes_k \mathcal{N}') \cong \text{Hom}_{k\mathcal{D}}(\mathcal{P}^\mathcal{D}, \mathcal{M}') \otimes_k \text{Hom}_{k\mathcal{C}}(\mathcal{P}^\mathcal{C}, \mathcal{N}'),$$

and the K"unneth formula. \(\square\)

When we examine transporter categories, the above result has the following consequences.

**Corollary 5.2.5.** Let $G_1 \propto \mathcal{P}_1$ and $G_2 \propto \mathcal{P}_2$ be two transporter categories. Then there is a natural isomorphism

$$V_{(G_1 \times G_2) \times (\mathcal{P}_1 \times \mathcal{P}_2)} = V_{G_1 \times \mathcal{P}_1} \times V_{G_2 \times \mathcal{P}_2}.$$ 

If $M \in kG_1$-mod and $N \in kG_2$-mod, under the above isomorphism we have furthermore

$$V_{(G_1 \times G_2) \times (\mathcal{P}_1 \times \mathcal{P}_2)}(\kappa_M \otimes N) = V_{(G_1 \times \mathcal{P}_1) \times (G_2 \times \mathcal{P}_2)}(\kappa_M \otimes_k \kappa_N) = V_{G_1 \times \mathcal{P}_1}(\kappa_M) \times V_{G_2 \times \mathcal{P}_2}(\kappa_N).$$

**Proof.** From Remark 2.1.2 we have $(G_1 \times G_2) \propto (\mathcal{P}_1 \times \mathcal{P}_2) \cong (G_1 \propto \mathcal{P}_1) \times (G_2 \propto \mathcal{P}_2)$. The statements on varieties are true, because

$$\text{Ext}^i_{k[(G_1 \times G_2) \times (\mathcal{P}_1 \times \mathcal{P}_2)]}(\kappa_M \otimes N, \kappa_M \otimes N) \cong \text{Ext}^i_{k[(G_1 \times \mathcal{P}_1) \times (G_2 \times \mathcal{P}_2)]}(\kappa_M \otimes \kappa_N, \kappa_M \otimes \kappa_N) \cong \text{Ext}^i_{k(G_1 \times \mathcal{P}_1)}(\kappa_M, \kappa_M) \otimes \text{Ext}^i_{k(G_2 \times \mathcal{P}_2)}(\kappa_N, \kappa_N).$$

\(\square\)

Based on the preceding results, we can extend the tensor product formula $V_G(M \otimes_k N) = V_{G}(M) \cap V_{G}(N)$ of Avrunin-Scott [5].

**Theorem 5.2.6.** Let $G$ be a finite group and $\mathcal{P}$ a finite $G$-poset. Suppose $M, N$ are two $kG$-modules. Then $V_{G \times \mathcal{P}}(\kappa_M \otimes \kappa_N) = V_{G \times \mathcal{P}}(\kappa_M) \cap V_{G \times \mathcal{P}}(\kappa_N)$. 
Proof. Let us consider the functor \( \vartriangle : G \rtimes P \to (G \rtimes P) \times (G \rtimes P) \) and the restriction along it. Because

\[ \kappa_M \circ \kappa_N = \text{Res}_{G \rtimes P}^{(G \rtimes P) \times (G \rtimes P)}(\kappa_M \otimes \kappa_N), \]

by Corollary 5.2.3 we have

\[ V_{G \rtimes P}(\kappa_M \circ \kappa_N) = i_{(G \rtimes P) \times (G \rtimes P)}^{-1} V_{G \rtimes P}(\kappa_M \otimes \kappa_N). \]

From \((G \times G) \rtimes (P \times P) \cong (G \rtimes P) \times (G \rtimes P)\) and Corollary 5.2.5 the right-hand-side is

\[ i_{(G \rtimes P) \times (G \rtimes P)}^{-1} \{ V_{G \rtimes P}(\kappa_M) \times V_{G \rtimes P}(\kappa_N) \}, \]

which exactly is \( V_{G \rtimes P}(\kappa_M) \cap V_{G \rtimes P}(\kappa_N) \).

With Theorem 5.2.6, taking Theorem 4.2.1 (5) in account one can see there is some relation between the conditions in (9a) and (9b) in Theorem 4.2.1.

References